An approximation method for pricing continuous barrier options under multi-asset local stochastic volatility models

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Abstract

This paper presents a new approximation method for pricing multi-asset continuous single barrier options under general local stochastic volatility models. The formula applies an asymptotic expansion technique and an approximation for the distribution of the first exit time of diffusion processes. This method focuses on local stochastic volatility models with unknown characteristic function and transition density function. To the best of our knowledge, our approximation formula is the first to achieve analytic approximations for continuous barrier options prices in this environment. In numerical experiments, we confirm the validity of the formula.

Keywords: Approximation Formula, Barrier Options, Local Stochastic Volatility Models, Multi-Asset

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1 Introduction

This paper discusses pricing multi-asset continuous single barrier options. Barrier options are one of the most traded path-dependent exotic derivatives in the OTC (e.g., forex) market, sometimes with basket prices (e.g., index price, basket currency) used as the underlying. These options are broadly divided into two types: continuous and discrete. Although discrete barrier options have been studied at length (e.g., Fang and Oosterlee [12], Feng and Limetsky [13], Fusai et al. [19], Umezawa and Yamazaki [35] and Shiraya et al. [31]), the target of this paper is continuous ones, and we concentrate on the evaluation of the continuous barrier option (CBO) price.

The analytical expressions of CBO prices are known for the one-dimensional Black-Scholes model. However, Black-Scholes is unsuitable for CBO pricing because the model does not account for the volatility smile structure. Instead, many of industry practitioners use local stochastic volatility (LSV) models as to account for the smile structure (see, e.g., Dadachanji [10], Homescu [22], Mercurio [25]).

There is much research which uses explicit transition density functions or characteristic functions for pricing option premiums, but the explicit form of them are not known for general LSV models. One of the typical methods to approximate the option premium under LSV models is an asymptotic expansion, which is used to approximate both the terminal distribution of the underlying asset price and thus option premiums. These methods can be applied to general models with acceptable accuracy and are used in practice, although the precise error is not known. Numerous articles discuss the application of these methods to LSV models, e.g., Hagan et al. [21], Cai et al. [6], Takahashi et al. [33]. This paper extends their application to the domain of CBOs.

Many articles study CBOs with a single underlying. Carr et al. [8] and Carr and Lee [9] used put-call symmetry, Akahori and Imamura [1] developed the property of symmetry in general diffusion models, which also decreased Monte Carlo discretization error. Cai et al. [5], Funahashi and Kijima [18], Alos et al. [2], and Funahashi and Higuchi [17] derived approximation formulas for jump diffusion, local volatility, stochastic volatility model, and local stochastic volatility models respectively. Static hedging can also be applied to CBO barrier by calculating present values of hedging assets. Derman et al. [11] proposed the calendar-spread static hedging method for the Black-Scholes model, and Fink [14] generalized that method for Heston’s stochastic volatility model. Shiraya et al. [30] further proposed an approximation method for general LSV models. Fouque et al. [15], [16] studied approximations of the partial differential equations (PDEs) for fast mean-reverting models. Kato et al. [23], [24] developed a semi-group expansion scheme for the general diffusion process. Brown et al. [4], and Tsuzuki [34] studied CBO pricing boundaries. Their method is based on hedging CBOs and is independent of underlying asset models. Prior work, which derived analytical approximation formulas, cannot be applied to multi-asset cases or the models whose transition density or characteristic functions are unknown.

Numerical methods for solving PDEs in single-asset CBOs are well developed and can be computed in a few seconds. However, PDEs in multi-asset CBOs remain difficult to solve with sufficient accuracy or computational speed for practice.

The Monte Carlo method can price CBOs in both single- and multi-asset cases for LSV models, but its computational cost is high. Moreover, this method needs to discretize the process and judge whether the barrier level was reached or not at the discretized time. It is well known that the discretization error of barrier options is larger than that of plain options, and to overcome this problem, practitioners use the Brownian bridge method to decrease the error. In practice, Monte Carlo method can estimate profit and loss quite accurately (e.g., daily trading P&L). However, computational inefficiency and low speed are the drawbacks of using Monte Carlo for market making in practice, when various prices need to be quoted in a
restricted amount of time.

A number of studies suggest approximation methods for non-barrier basket option prices. Takahashi [32] showed an asymptotic expansion under local volatility (LV) models, Bayer and Laurence [3] used a heat kernel expansion and the Laplace method. Xu and Zheng [36] applied the lower bound technique and asymptotic expansion to European basket call prices in an LV jump-diffusion model, Shiraya and Takahashi [27] applied asymptotic expansion under LSV with jumps. Caldana et al. [7] derived narrow pricing bounds for models whose characteristic function of the vector of log-prices is known. However, these methods cannot be applied directly to pricing CBOs under general LSV models.

Our method is based on the approximated distribution of the first exit time and an asymptotic expansion technique. By applying this method, we derive an explicit formula for multi-asset continuous barrier options prices under general LSV models. To the best of our knowledge, the proposed approximation is the first to achieve an analytic approximation for such options prices. In Numerical Examples, we compare the approximated prices with the values obtained by the Monte Carlo simulations for the one-asset constant elasticity of variance (CEV) and five-asset stochastic alpha, beta, rho (SABR) models. Moreover, by using market-calibrated parameters, we demonstrate the effectiveness of our formula under a quadratic local volatility function with stochastic volatility model.

The organization of this paper is as follows: Section 2 proposes an approximation method for multi-dimensional diffusion processes and derives an approximation formula. In addition, we provide an additional approximation scheme for the distribution of the first exit time. Section 3 presents numerical examples to confirm the validity of our method.

2 Pricing formula of barrier options under local stochastic volatility models

This section shows an approximation method for multi-asset CBOs and shows an explicit formula for barrier options prices. We approximate the barrier option price by the following steps:

1. Approximate the distribution of the terminal payoff function by an asymptotic expansion technique.

2. Approximate the distribution of the first exit time of the underlying processes.

2.1 An approximation method

We suppose a filtered probability space satisfying the usual conditions \((Ω, \mathcal{F}, P, \{\mathcal{F}_t\}_{t≥0})\) where \(P\) is a risk-neutral measure, and \(r\) is a constant risk-free rate. We also assume a model for the underlying asset prices \(S_t = (S^1_t, \cdots, S^d_t)\) and their volatilities \(\sigma_t = (\sigma^1_t, \cdots, \sigma^d_t)\).

For \(ε > 0\), let us assume that \(S^i_T\) and \(σ^i_T\) are given by the solutions of the following stochastic integral equations:

\[
S^i_T = s^i_0 + \int_0^T rS^i_t dt + ε \int_0^T φ^i_t (S^i_t, σ^i_t) dW_t, \tag{1}
\]

\[
σ^i_T = σ^i_0 + \int_0^T λ^i(θ^i - σ^i_t) dt + ε \int_0^T ψ^i_t (σ^i_t) dW_t, \tag{2}
\]

where \(s^i_0\) and \(σ^i_0\) are constant, and \(W = (W^1, \cdots, W^2d)\) are independent Brownian motions. We assume that \(λ^i : \mathbb{R} → \mathbb{R}, φ^i : \mathbb{R} × \mathbb{R} → \mathbb{R}^{2d}\) and \(ψ^i : \mathbb{R} → \mathbb{R}^{2d}\) are some deterministic \(C^∞_b\) functions, and (1) and (2) satisfy the uniform ellipticity condition.
Then, we assume the underlying variate of a barrier option is expressed as
\[ X_t = \sum_{i=1}^{d} w^i S^i_t, \]  
(3)
and the domain of the underlying variate of the single barrier options, whose barrier price is \( B \), is written as \( D = (-\infty, B) \) or \((B, \infty)\) satisfying \( X_0 \in D \). We define a stopping time \( \tau_X \) as
\[ \tau_X = \inf\{t | X_t \notin D\}. \]  
(4)
The knock-in (KI) option becomes active when \( \tau_X \leq T \) (\( T \) is the maturity of the barrier option). On the other hand, the knock-out (KO) option becomes null when \( \tau_X \leq T \). Hereafter, we concentrate on single barrier KI call options. KO options are obtained in a similar manner (see Corollary 2.3).

In this setting, the KI barrier option price \( (V_{KI}) \) is expressed as follows:
\[ V_{KI} = e^{-rT} E[1_{\tau_X < T} f(X_T)], \]  
(5)
where \( f(X_T) \) is the payoff function of the European call option with the strike price \( K \) expressed as
\[ f(X_T) = (X_T - K)^+. \]  
(6)
Here, we can take \( C^\infty_b(\mathbb{R}) \)-function \( \{f_n\} \) such that \( |f_n(x)| \leq C(1+|x|^p) \) and \( \lim_{n \to \infty} f_n(x) = f(x) \) for \( \forall x \), and \( \tau_n X = \inf_{0 \leq t \leq T} \{t | X_t \notin D_n\} \) where \( D_n = [a_n, b_n] \), \( a_n, b_n \in \mathbb{R} \).

Then, there exists \( \exists N \) such that
\[ |E[f_n(X_T)] - E[f(X_T)]| < \hat{\epsilon}, \]  
(7)
\[ |E[1_{\tau_n \leq T} f_n(X_T)] - E[1_{\tau_n < T} f(X_T)]| < \hat{\epsilon}. \]  
(8)
for every \( \hat{\epsilon} > 0 \) and \( n \geq N \). Thus hereafter, we regards \( f \) as \( C^\infty_b(\mathbb{R}) \)-function, \( D \) as a bounded interval, and assume \( d(\text{supp}(f), \partial D) \geq \bar{\epsilon} > 0 \).

Firstly, in order to approximate the distribution of the terminal payoff function \( f(X_T) \), we assume the asymptotic expansions of \( S_T^i \) and \( \sigma_T^i \) around \( \epsilon = 0 \) as follows:
\[ S_T^i = S_T^{i,0} + \epsilon S_T^{i,1} + \frac{\epsilon^2}{2!} S_T^{i,2} + \cdots, \]  
(9)
\[ \sigma_T^i = \sigma_T^{i,0} + \epsilon \sigma_T^{i,1} + \frac{\epsilon^2}{2!} \sigma_T^{i,2} + \cdots, \]  
(10)
where \( S_t^{i,j} := \frac{\partial S_t^i}{\partial \epsilon} \big|_{\epsilon=0} \), \( \sigma_t^{i,j} := \frac{\partial^2 S_t^i}{\partial \epsilon^2} \big|_{\epsilon=0} \). For example, \( S_T^{i,0} = e^{rT} s^i_0 \) and \( \sigma_T^{i,0} = \theta^i + (\sigma_0^i - \theta^i)e^{-\lambda^i(T-t)} \), and \( S_T^{i,1} \) is calculated as follows:
\[ S_T^{i,1} = \frac{\partial S_T^i}{\partial \epsilon} \big|_{\epsilon=0} = \int_0^T r S_t^{i,1} dt + \int_0^T \phi_i (\sigma_0^i, s_0^i) dW_t, \]  
(11)
and this integral equation can be solved as
\[ S_T^{i,1} = \int_0^T e^{r(T-t)} \phi_i (\sigma_0^i, s_0^i) dW_t. \]  
(12)
\( S_T^{i,2}, S_T^{i,3}, \sigma_T^{i,1} \) and \( \sigma_T^{i,2} \) are calculated as in Appendix A, and the higher order expansion is obtained in a similar way.
Here, we define $X_T^{[j]}$ ($j = 1, 2, \cdots$) and $y$ as

$$X_T^{[j]} = \sum_{i=1}^{d} w^i S_T^{[j]},$$

$$y = \frac{X_T^{[0]} - K}{\epsilon}. \quad (13)$$

Then, $X_T$ and $f(X_T)$ are expressed as follows:

$$X_T = \sum_{j=0}^{m} \frac{\epsilon^j}{j!} X_T^{[j]} + o(\epsilon^m),$$

$$f(X_T) = \epsilon \left( \sum_{j=1}^{m} \frac{\epsilon^{j-1}}{j!} X_T^{[j]} + o(\epsilon^{m-1}) + y \right)^+$$

$$= \epsilon f \left( X_T^{[1]} + y \right) + \sum_{m=1}^{M-1} \sum_{j=1}^{m} \frac{\epsilon^{m+1}}{j!} f^{(j)} \left( X_T^{[1]} + y \right) \sum_{n_1 + \cdots + n_j - j = m} \frac{X_T^{[n_1]}}{n_1!} \cdots \frac{X_T^{[n_j]}}{n_j!} + o(\epsilon^M)$$

$$= \sum_{m=1}^{M} \epsilon^m f_{m,T} + o(\epsilon^M), \quad (16)$$

where

$$f_{1,T} = f \left( X_T^{[1]} + y \right),$$

$$f_{m,T} = \sum_{j=1}^{m-1} \frac{1}{j!} f^{(j)} \left( X_T^{[1]} + y \right) \sum_{n_1 + \cdots + n_j - j = m-1} \frac{X_T^{[n_1]}}{n_1!} \cdots \frac{X_T^{[n_j]}}{n_j!}.$$

Next, we approximate the distribution of the first exit time. Let

$$\hat{X}_t = X_0 + \frac{X_T^{[0]} - X_T^0}{T} t + \epsilon \sum_{i=1}^{d} w^i \int_0^t \sqrt{\Sigma_T} dW_i,$$

$$\Sigma_T^{(i,j)} = \int_0^T \left( w^i e^{r(T-t)} \phi^j_i \left( \sigma_t^{[0]}, S_t^{[0]} \right) \right)^2 dt, \quad (20)$$

where $\phi_i^j$ is the $j$-th element of $\phi_i$, $\Sigma_T := \left( \Sigma_T^{1,1}, \cdots, \Sigma_T^{1,2d} \right)$, and $\sqrt{\Sigma_T} := \left( \sqrt{\Sigma_T^{1,1}}, \cdots, \sqrt{\Sigma_T^{1,2d}} \right)$.

Then, $\hat{X}_T = X_T^{[0]} + \epsilon X_T^{[1]}$. Lemma 2.1 shows the probability of the first exit time of $\hat{X}_t$ which is well known as an application of the Brownian bridge.

**Lemma 2.1.** When $D = (-\infty, B)$ or $(B, \infty)$ with $\bar{X}_0 = x$, and $\bar{X}_T = w$, $x, w \in D$, the distribution of the first exit time of $\hat{X}_t$ is expressed as follows:

$$p_X(x, w, B, T) := \exp \left( -\frac{2(B-x)(B-w)}{\Sigma_T} \right), \quad (21)$$

where

$$\Sigma_T := \sum_{i=1}^{d} \int_0^T \left( w^i e^{r(T-t)} \phi_i \left( \sigma_t^{[0]}, S_t^{[0]} \right) \right)^2 dt. \quad (22)$$
KO option premiums are approximated by applying

\[ V_{KI}(M) = e^{-rT}E \left[ p_X(X_0, \hat{X}_T, B, T) \sum_{m=1}^{M} e^m f_{m,T} \right]. \tag{23} \]

To obtain the explicit expression of the right-hand side of (23), we calculate it in Section 2.2.

First, we show \( V_{KI}(M) \) is an approximation of the KI barrier option price \( V_{KI} \).

**Proposition 2.2.** The difference between the approximated KI call option price \( V_{KI}(M) \) and the true price \( V_{KI} \) is estimated as follows:

\[ |V_{KI} - V_{KI}(M)| < c(T, \epsilon), \tag{24} \]

where \( c(T, \epsilon) \) is a deterministic function satisfying \( \lim_{T, \epsilon \downarrow 0} c(T, \epsilon) = 0 \).

Proof.

\[ |V_{KI} - V_{KI}(M)| < e^{-rT} \left| E \left[ 1_{T_X < T} f(X_T) \right] - E \left[ p_X(X_0, \hat{X}_T, B, T) f(\hat{X}_T) \right] \right| + e^{-rT} \left| E \left[ p_X(X_0, \hat{X}_T, B, T) f(\hat{X}_T) \right] - E \left[ p_X(X_0, \hat{X}_T, B, T) \sum_{m=1}^{M} e^m f_{m,T} \right] \right|. \tag{25} \]

Here, Theorem 2.1 in Gobet [20] derives

\[ e^{-rT} \left| E \left[ 1_{T_X < T} f(X_T) \right] - E \left[ p_X(X_0, \hat{X}_T, B, T) f(\hat{X}_T) \right] \right| < c_a(T, \epsilon). \tag{26} \]

where \( c_a \) is a function corresponds to the right-hand side of (14) in [20]. We note that the function \( c_a \) depends on \( \epsilon \) in our notations because the estimate in [20] depends on the size of the coefficients (see, e.g., Lemma 3.3 of p.178 in [20]).

On the other hand, for a increasing function \( c_b \) satisfying \( \lim_{T \rightarrow 0} c_b(T) = 0 \), the next estimate

\[ e^{-rT} \left| E \left[ p_X(X_0, \hat{X}_T, B, T) f(\hat{X}_T) \right] - E \left[ p_X(X_0, \hat{X}_T, B, T) \sum_{m=1}^{M} e^m f_{m,T} \right] \right| = e^{-rT} \left| E \left[ p_X(X_0, \hat{X}_T, B, T) \epsilon f_{1,T} \right] - E \left[ p_X(X_0, \hat{X}_T, B, T) \sum_{m=1}^{M} e^m f_{m,T} \right] \right| < c_b(T) \epsilon^2, \tag{27} \]

is obtained in a similar way to Theorem 2.1 in [28] and Appendix B in [29].

Thus, we obtain (24) by setting \( c(T, \epsilon) = c_a(T, \epsilon) + c_b(T) \epsilon^2 \). \qed

**Corollary 2.3.** KO option premiums are approximated by applying \( q_X(x, w, B, T) = 1 - p_X(x, w, B, T) \) instead of \( p_X(x, w, B, T) \) in (21).

### 2.2 An approximation formula

This subsection gives the explicit expression of \( V_{KI}(M) \). In this subsection, we apply \( M = 3 \) to obtaining the approximated option price (i.e. \( V_{KI}(3) \)) because the calculation becomes complicated in the higher cases. To obtain the higher values, we can calculate them by the method in [33].

In the case of an European option with a strike price \( K \), the third order expansion of a payoff function is expressed as follows:

\[ f(X_T) = (X_T - K)^+ \]
Here, we set
\[ \int \exp \left( z \right) = K_I \quad \text{for} \quad z < 0 \]
and
\[ \int \exp \left( z \right) = 0 \quad \text{for} \quad z > 0 \]
Then,
\[ V_{KL}[3] = e^{-rT} \mathbb{E} \left[ \left( \epsilon \left( X_T^1 + y + o(\epsilon^3) \right) + e^2 \mathbf{1}_{\{X_T^3 > y\}} \frac{1}{2} X_T^2 \right) \right] \]
\[ + e^3 \left( \frac{1}{6} \mathbf{1}_{\{X_T^3 > y\}} X_T^3 + \frac{1}{8} \delta_{\{X_T^3 = y\}} \left( X_T^2 \right)^2 \right) + o(\epsilon^3). \]  
(28)

where \( \bar{X}_T = X_T^1 \), \( z = \frac{1}{2} \left( \sum_{i=1}^{d} w_i s_i + \alpha - \bar{K} \right) \), \( \alpha = \sum_{i=1}^{d} w_i X_T^1 - \sum_{i=1}^{d} w_i s_i^0 \), \( b := \frac{1}{\epsilon} \left( B - \sum_{i=1}^{d} w_i s_i^0 \right) \), and \( n(x; 0, \Sigma) = \frac{1}{\sqrt{2\pi\Sigma_T}} \exp \left( \frac{-x^2}{2\Sigma_T} \right) \).

Since \( n(x; 0, \Sigma) = \frac{1}{\sqrt{2\pi\Sigma_T}} \exp \left( \frac{-x^2}{2\Sigma_T} \right) \), we obtain
\[ \exp \left( \frac{-2b(b - \alpha - x)}{\Sigma_T} \right) n(x; 0, \Sigma_T) = \frac{1}{\sqrt{2\pi\Sigma_T}} \exp \left( \frac{-x^2 - 4bx - 4\alpha x + 4b^2}{2\Sigma_T} \right) \]
\[ = \frac{1}{\sqrt{2\pi\Sigma_T}} \exp \left( \frac{-(x - 2b)^2}{2\Sigma_T} \right) \exp \left( \frac{2b\alpha}{\Sigma_T} \right). \]  
(30)

Here, we set \( \gamma_T = \exp \left( \frac{2b\alpha}{\Sigma_T} \right) \), then
\[ V_{KL}[3] = e^{-rT} \mathbb{E} \left[ \epsilon \int_{-\hat{z}}^{\infty} (x + \hat{z}) n(x; 0, \Sigma_T) dx \right] \]
\[ + e^2 \int_{-\hat{z}}^{\infty} \mathbb{E} \left[ X_T^2 | X_T = x + 2b \right] n(x; 0, \Sigma_T) dx \]
\[ + e^3 \left( \frac{1}{6} \mathbb{E} \left[ X_T^3 | X_T = x + 2b \right] n(x; 0, \Sigma_T) dx \right) \]
\[ + \frac{1}{8} \mathbb{E} \left[ \left( X_T^2 \right)^2 | X_T = -\hat{z} \right] n(-\hat{z}; 0, \Sigma_T) \right] \gamma_T, \]  
(31)

where \( \hat{z} = 2b + z \).

The conditional expectations in (31) are obtained by using formulas in Appendix B and the next integral formulas.
\[ \int_{-z}^{\infty} H_k(x, \Sigma) n(x; 0, \Sigma) dx = \Sigma H_{k-1}(-z, \Sigma) n(z; 0, \Sigma), \]  
(32)
The approximation formula for the initial value of down and in call option calculate the drift and variance adjusted processes

calculate non-discounted approximated digital option prices

\[ \text{calculate } \exp \text{ similar way with combining European option.} \]

Up and in options and KO options premiums are obtained in a similar way with combining European option.

**Theorem 2.4.** The approximation formula for the initial value of down and in call option price with a maturity \( T \), a barrier price \( B \), and a strike price \( K \) is given by the formula:

\[
V_{KI}[3] = e^{-rT} \left( \epsilon \left\{ \hat{z}N \left( \frac{\hat{z}}{\sqrt{\Sigma_T}} \right) + \Sigma_T \tilde{n}(\hat{z}; 0, \Sigma_T) \right\} \right.
+ \epsilon^2 \left\{ \frac{C_{2.1}}{2} \left( \frac{4 \beta^2}{\Sigma_T^2} N \left( \frac{\hat{z}}{\sqrt{\Sigma_T}} \right) + \frac{4 b n(\hat{z}; 0, \Sigma_T)}{\Sigma_T} \right) + \frac{N \left( \frac{\hat{z}}{\Sigma_T} \right) - \hat{z} n(\hat{z}; 0, \Sigma_T)}{\Sigma_T} \right\} \right.
+ \epsilon^3 \left\{ \frac{C_{3.1}}{6} \left( \frac{8 \beta^3}{\Sigma_T^3} N \left( \frac{\hat{z}}{\sqrt{\Sigma_T}} \right) + \frac{12 b n(\hat{z}; 0, \Sigma_T)}{\Sigma_T^2} \right) + \frac{6 b N \left( \frac{\hat{z}}{\Sigma_T} \right) \hat{z} n(\hat{z}; 0, \Sigma_T)}{\Sigma_T^2} \right\} \right.
\left. + \left( \frac{2}{\Sigma_T} + \frac{\hat{z}^2}{\Sigma_T} \right) n(\hat{z}; 0, \Sigma_T) \right) + \frac{C_{3.2}}{6} \left( n(\hat{z}; 0, \Sigma_T) + \frac{2 b}{\Sigma_T} N \left( \frac{\hat{z}}{\Sigma_T} \right) \right) \right.
\left. + \frac{C_{3.3} \left( \frac{\hat{z}^4}{\Sigma_T^4} - \frac{6 \hat{z}^2}{\Sigma_T^3} + \frac{3}{\Sigma_T} \right) + \frac{C_{3.4} \left( \frac{\hat{z}^2}{\Sigma_T^2} - \frac{1}{\Sigma_T} \right)}{\Sigma_T} n(\hat{z}; 0, \Sigma_T) \right\} \right) \gamma_T, \tag{34}
\]

where \( z = \frac{1}{\epsilon} (\sum_{i=1}^{d} w_i s_i^t + \alpha - K) \), \( \alpha = \epsilon r \sum_{i=1}^{d} w_i s_0^{i} T \), \( b := \frac{1}{\epsilon} \left( B - \sum_{i=1}^{d} w_i s_0^{i} \right) \), \( \gamma_T = \exp \left( \frac{2 \alpha}{\Sigma_T} \right) \), and \( \hat{z} = 2 b + z \). \( r \) is a risk-free rate, \( N(x) \) denotes the standard normal distribution function, and \( \Sigma_T \) is defined as (22). The coefficients \( C_{1.1}, \ldots, C_{3.4} \) are some constants.

### 2.3 Adjustment of the first exit probability

Since the distribution of the first exit time is approximated by that of \( \hat{X}_t \) with formula (21), it still has room for improvement. In this subsection, we provide an adjustment method of this probability. Note that we explain the adjustment method in the one asset with the domain \( D = (B, \infty) \) case. Thus, \( X_t \) is expressed as follows:

\[
X_t = X_0 + \int_0^t r X_s ds + \epsilon \int_0^t \phi(\sigma_s, X_s) dW_s. \tag{35}
\]

The multi-asset and \( D = (-\infty, B) \) cases are obtained in a similar way.

The error of the probability of the first exit time is caused by the difference of the distributions between \( \hat{X}_t \) and \( X_t \). We will adjust this probability in the following steps:

i calculate non-discounted approximated digital option prices \( p_h \) whose maturity and strike are \( t_h, h = 1, 2 \) and \( K = B \), respectively.

ii calculate the drift and variance adjusted processes \( \chi_t \) by using the inverse function of the cumulative distribution function (CDF) of the standard normal distribution.

iii calculate \( p_{\chi}(x, w, B, T) \) for \( \chi_t \), and apply it to the formula (34).
Firstly, we calculate the approximated non-discounted digital option prices \( p_h \) with the maturity \( t_h (h = 1, 2) \) and the strike price \( K = B \) (the barrier price) by using an asymptotic expansion scheme (see Theorem 2.4) or a numerical difference method. \( p_h \) is an approximated probability that the underlying asset price does not exist in the domain at time \( t_h \).

Next, in Step ii, we define the adjusted process \( \chi_t \) as

\[
\chi_t = X_0 + \int_0^t \mu ds + \epsilon \int_0^t \sqrt{\Sigma} dW_s, \tag{36}
\]

where

\[
\sqrt{\Sigma} := \frac{\left(1 - \frac{t_2}{t_1}\right) (B - X_0)}{\sqrt{t_2 N^{-1}(p_2) - \frac{t_2}{t_1} N^{-1}(p_1)}}, \tag{37}
\]

\[
\mu := \left(B - X_0 - N^{-1}(p_2) \sqrt{\Sigma t_2}\right) \frac{1}{t_2}. \tag{38}
\]

The function \( N^{-1} \) is the inverse function of CDF of the standard normal distribution, and the value is obtained easily (e.g., Moro \[26\]). Then, \( \chi_t \) satisfies \( p_1 = P(\chi_{t_1} < B) \) and \( p_2 = P(\chi_{t_2} < B) \).

Finally, in Step iii, the adjusted approximated KI option’s value

\[
\hat{V}_{KI}(M) := e^{-rT} E \left[p_\chi(X_0, \hat{X}_T, B, T) \sum_{m=1}^{M} e^{m f_{m,T}} \right], \tag{39}
\]

is obtained by substituting \( \bar{\alpha} = \mu T, \hat{\Sigma} \) for \( \alpha, \Sigma_T \) in \( p_\chi(X_0, \hat{X}_T, B, T) \) of (29), respectively.

**Remark 2.5.** The difference between \( p_\chi(X_0, x, B, T) \) and \( p_\chi(X_0, x, B, T) \) is estimated as

\[
|p_\chi(X_0, x, B, T) - p_\chi(X_0, x, B, T)| < c_c(T, \epsilon), \tag{40}
\]

where \( c_c(T, \epsilon) \) is an increasing function of \( T \) satisfying \( \lim_{T, \epsilon \to 0} c_c(T, \epsilon) = 0 \) since both of the values are expressed in the form of (21). Thus, this error is absorbed in the function \( c(T, \epsilon) \) of (24), and we can show the convergence \( \lim_{T, \epsilon \to 0} V_{KI}[3] = V_{KI} \). Although \( \hat{V}_{KI}[3] \) is not always improved from \( V_{KI}[3] \), this approximation works well. We show the effect of this method by numerical examples in Section 3.

**Remark 2.6.** In the above argument, we used two points to adjust the hitting probability, though it can be adjusted from one point. To adjust the hitting probability with one point, we set \( \hat{\Sigma} T = \Sigma_T \) in (37) and (38), and the option premium (define it as \( \hat{V}_{KI}(M) \)) is obtained in a similar way to the above calculations.

However, the accuracy of two points adjustment is better than that of one point adjustment in many cases. We also see the difference between them in Section 3.

### 3 Numerical Examples

Here, we show some numerical examples by applying the formula in Theorem 2.4.

In Section 3.1, we compare the accuracy of our results with that of Funahashi and Kijima [18] under CEV model case, and then show the result of the SABR model with five assets case.

In Section 3.2, we apply the method to the quadratic local volatility with stochastic volatility model and derive prices after calibrating the parameters with USDJPY currency options market data.
The program is implemented in C++, and the computational time is calculated with one core of Intel Core(TM) i7-3960X CPU @ 3.30GHz 32GB RAM. In this section, we set $\epsilon = 1$, and in order to adjust the probability of the first exit time introduced in Section 2.3, the maturities of the digital options are set at $\frac{T}{2}$ and $T$ for $\hat{V}_{KI}[3]$ and at $T$ for $V_{KI}[3]$. ($\hat{V}_{KI}[3]$ is adjusted with only the drift, and $V_{KI}[3]$ is adjusted with the drift and volatility of the process $\chi_t$ in Section 2.3)

### 3.1 CEV and SABR models

In this subsection, we assume the underlying assets prices follow CEV local volatility model or SABR local stochastic volatility model, which is expressed as follows:

\[ S^i_t = S^i_0 + \epsilon \int_0^t r S^i_t dt + \epsilon \int_0^t \sigma^i_t (S^i_t)^\beta dZ^i_t, \quad (41) \]

\[ \sigma^i_t = \sigma^i_0 + \epsilon \int_0^t \nu_i \sigma^i_t dZ^i_t, \quad (42) \]

CEV model is the special case of SABR model ($\nu = 0$ in (42)).

Firstly, we evaluate one asset CEV model $KI$ options and compare with the results of Funahashi and Kijima [18]. We apply the parameters in Table 1 which are the same as those of Table 3 in [18].

<table>
<thead>
<tr>
<th></th>
<th>$S_0$</th>
<th>$\sigma_0$</th>
<th>$r$</th>
<th>$\beta$</th>
<th>$\nu$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case (A)</td>
<td>100</td>
<td>150%</td>
<td>0</td>
<td>0.5</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>Case (B)</td>
<td>100</td>
<td>150%</td>
<td>0</td>
<td>0.5</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Case (C)</td>
<td>100</td>
<td>300%</td>
<td>0</td>
<td>0.5</td>
<td>0.5</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Parameters of the CEV model

By using these parameters, we evaluate the down and in barrier option whose barrier price is 95, and the strikes are from 95 to 100. “MC” shows the price with the Monte Carlo simulation, “FK(3.19)” shows the price written in Funahashi and Kijima [18]. “$\hat{V}_{KI}(3)$”, “$V_{KI}(3)$” and “$\hat{V}_{KI}(3)$” are our approximated values defined in Section 2. The values of “MC” and “FK(3.19)” in Table 3 are used those in Table 3 of [18]. The rows of Diff show the difference between the approximated prices and the prices calculated with Monte Carlo simulations.

Next, we examine the SABR with five assets cases. The parameters of the local volatility are the same as those in Table 1 for all underlying assets, and we use the values of the stochastic volatilities’ parameters as in Table 2.

<table>
<thead>
<tr>
<th></th>
<th>$S_0$</th>
<th>$\sigma_0$</th>
<th>$r$</th>
<th>$\beta$</th>
<th>$\nu^i$</th>
<th>$\rho_{Z^i;Z^j}$</th>
<th>$\rho_{Z^i;Z^j}$</th>
<th>$\rho_{Z^i;Z^j}$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case (A)</td>
<td>100</td>
<td>150%</td>
<td>0</td>
<td>0.5</td>
<td>15%</td>
<td>-0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.5</td>
</tr>
<tr>
<td>Case (B)</td>
<td>100</td>
<td>150%</td>
<td>0</td>
<td>0.5</td>
<td>15%</td>
<td>-0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>1</td>
</tr>
<tr>
<td>Case (C)</td>
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<td>300%</td>
<td>0</td>
<td>0.5</td>
<td>30%</td>
<td>-0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table 2: Parameters of the SABR volatilities

Here, $i = 1, \cdots, 5$, $\langle X, Y \rangle_t = \rho_{X,Y} t$, and the weight of each asset is set as $w^i = 0.2$. The strikes, the barrier prices and the maturities are the same as those of the CEV model cases. As for the LSV model, we compare the approximated values with the results of a Monte Carlo simulation whose time partition is 1,024/year, and the number of simulations is 1,000,000. (Then, the standard error of Monte Carlo method is less than 0.01 in all cases.)

The results of the CEV model are in Table 3, and those of the multi-asset SABR model are in Table 4.
<table>
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<th>100</th>
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<th>102</th>
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<th>104</th>
<th>105</th>
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<td>1.85</td>
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<td>0.81</td>
<td>0.67</td>
<td>0.56</td>
<td>0.46</td>
</tr>
<tr>
<td></td>
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<td>1.85</td>
<td>1.58</td>
<td>1.35</td>
<td>1.15</td>
<td>0.96</td>
<td>0.80</td>
<td>0.67</td>
<td>0.54</td>
<td>0.44</td>
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<td>1.57</td>
<td>1.34</td>
<td>1.14</td>
<td>0.96</td>
<td>0.81</td>
<td>0.67</td>
<td>0.56</td>
<td>0.46</td>
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<td>1.57</td>
<td>1.33</td>
<td>1.13</td>
<td>0.95</td>
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<td>0.55</td>
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<tr>
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<td>0.00</td>
<td>0.00</td>
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<td>-0.02</td>
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</tr>
<tr>
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</tr>
<tr>
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</tr>
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<td>-0.01</td>
<td>-0.02</td>
<td>-0.01</td>
<td>-0.01</td>
<td>-0.01</td>
<td>-0.01</td>
<td>-0.01</td>
<td>-0.01</td>
</tr>
<tr>
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<td>3.07</td>
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<td>2.49</td>
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<td>3.03</td>
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<td>1.77</td>
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<td>3.04</td>
<td>2.74</td>
<td>2.47</td>
<td>2.21</td>
<td>1.98</td>
<td>1.77</td>
<td>1.58</td>
<td>1.40</td>
</tr>
<tr>
<td>Diff</td>
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<td>-0.01</td>
<td>-0.03</td>
<td>-0.04</td>
<td>-0.05</td>
<td>-0.06</td>
<td>-0.07</td>
<td>-0.08</td>
<td>-0.10</td>
<td>-0.11</td>
</tr>
<tr>
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<td>-0.04</td>
<td>-0.04</td>
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<td>-0.02</td>
<td>-0.02</td>
</tr>
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</tr>
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<td>-0.03</td>
<td>-0.02</td>
<td>-0.02</td>
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<td>5.33</td>
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<td>4.65</td>
<td>4.34</td>
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<td>5.61</td>
<td>5.25</td>
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<td>4.27</td>
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<td>5.38</td>
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</tr>
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<td>0.04</td>
<td>0.04</td>
<td>0.04</td>
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</tr>
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<td>V_{KI}(3)^3</td>
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<td>-0.02</td>
<td>-0.02</td>
<td>-0.02</td>
<td>-0.03</td>
<td>-0.03</td>
</tr>
</tbody>
</table>

Table 3: Numerical Results of knock-in options under CEV model with $S_0 = 100$, $\beta = 0.5$, $B = 95$
<table>
<thead>
<tr>
<th>Case</th>
<th>Premium</th>
<th>Strike</th>
<th>95</th>
<th>96</th>
<th>97</th>
<th>98</th>
<th>99</th>
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<th>101</th>
<th>102</th>
<th>103</th>
<th>104</th>
<th>105</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>MC</td>
<td>1.74</td>
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<td>1.22</td>
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<td>0.82</td>
<td>0.67</td>
<td>0.54</td>
<td>0.43</td>
<td>0.34</td>
<td>0.26</td>
<td>0.20</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$V_{KI}(3)$</td>
<td>1.75</td>
<td>1.47</td>
<td>1.23</td>
<td>1.01</td>
<td>0.83</td>
<td>0.67</td>
<td>0.54</td>
<td>0.43</td>
<td>0.34</td>
<td>0.26</td>
<td>0.20</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\dot{V}_{KI}(3)$</td>
<td>1.73</td>
<td>1.45</td>
<td>1.21</td>
<td>1.00</td>
<td>0.82</td>
<td>0.66</td>
<td>0.53</td>
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<td>0.26</td>
<td>0.20</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\ddot{V}_{KI}(3)$</td>
<td>1.76</td>
<td>1.48</td>
<td>1.23</td>
<td>1.02</td>
<td>0.84</td>
<td>0.68</td>
<td>0.55</td>
<td>0.43</td>
<td>0.34</td>
<td>0.27</td>
<td>0.21</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>$V_{KI}(3)$</td>
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<td>0.01</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
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<td>-0.01</td>
<td>-0.01</td>
<td>-0.01</td>
<td>-0.01</td>
<td>-0.01</td>
<td>-0.00</td>
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<td>-0.00</td>
<td>-0.00</td>
<td>-0.00</td>
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<td>0.01</td>
<td>0.01</td>
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<td></td>
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<tr>
<td>C</td>
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<td>2.17</td>
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<td>0.02</td>
<td>0.01</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\dot{V}_{KI}(3)$</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>-0.00</td>
<td>-0.00</td>
<td>-0.00</td>
<td>-0.00</td>
<td>-0.00</td>
<td>-0.00</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\dddot{V}_{KI}(3)$</td>
<td>0.04</td>
<td>0.04</td>
<td>0.04</td>
<td>0.04</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>$\dddot{V}_{KI}(3)$</td>
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<td>0.03</td>
<td>0.03</td>
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<td>0.02</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\dddot{V}_{KI}(3)$</td>
<td>-0.03</td>
<td>-0.03</td>
<td>-0.02</td>
<td>-0.02</td>
<td>-0.01</td>
<td>-0.00</td>
<td>0.00</td>
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<td>0.03</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\dddot{V}_{KI}(3)$</td>
<td>-0.03</td>
<td>-0.03</td>
<td>-0.02</td>
<td>-0.01</td>
<td>-0.01</td>
<td>0.00</td>
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<td>0.02</td>
<td>0.02</td>
<td>0.03</td>
<td></td>
</tr>
</tbody>
</table>

Table 4: Numerical Results of knock-in options under five assets SABR model with $S_0 = 100$, $\beta = 0.5$, $B = 95$
Regarding the CEV local volatility cases, Table 3 shows that the results of both [18] and our method are accurate in short maturity with small volatility case (Case (A)). In addition, our new method is much stable and more accurate than that of [18], especially for the long maturity and high volatility cases (Case (B) and (C)). The size of the error in $\tilde{V}$ is stable compared to that of $V$.

Table 4 shows that our method is stable even in the multi-asset LSV model cases. Since the hitting probability is adjusted, the accuracy of $\tilde{V}$ and $\hat{V}$ are better than that of $V$. Calculations of Case (B) in Table 4 take about 6,800 ~ 7,000 seconds by the Monte Carlo simulations and about 0.01 seconds by our approximation formula.

As expected from the Theorem 2.2, the errors with large $\sigma_0$, $\nu$ and $T$ are larger than those of small ones. This is because the larger $\sigma_0^2$ and/or $\nu^2$ mean larger $\epsilon$ which along with larger $T$, and it makes the error to high (see Theorem 2.2).

### 3.2 Currency barrier options

In this subsection, we confirm the practical validity by using the parameters calibrated to the real market data. The model of this section is a quadratic local volatility function with a stochastic volatility model which is expressed as follows:

\[
S_T = S_0 + \int_0^T \mu S_t dt + \epsilon \int_0^T \sigma_t (\alpha + \beta S_t + \gamma S_t^2) dZ^S_t, \tag{43}
\]

\[
\sigma_T = \sigma_0 + \epsilon \int_0^T \nu \sigma_t dZ^\nu_t. \tag{44}
\]

The quadratic local volatility function with a stochastic volatility model is one of the most popular models in the forex market.

The parameters are calibrated to the data of USDJPY 3M maturity on August 1, 2017 downloaded from Bloomberg. The details of market data are that the spot price: 110.36, the forward price: 109.88, ATM volatility: 8.95, 25 delta risk reversal: -1.32, 10 delta risk reversal: -2.47, 25 delta butterfly: 0.30, 10 delta butterfly: 0.84*, and the calibrated parameters are in Table 5.

<table>
<thead>
<tr>
<th></th>
<th>$S_0$</th>
<th>$\sigma_0$</th>
<th>$\mu$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\nu$</th>
<th>$\rho$</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>110.36</td>
<td>1.7745</td>
<td>-0.0173</td>
<td>191.6225</td>
<td>-3.2894</td>
<td>0.0145</td>
<td>0.8167</td>
<td>-0.1839</td>
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</tbody>
</table>

Table 5: Calibrated parameters

We examine the down and out call options as the forward premium whose strikes are from 105 to 115, the barrier prices are 105, 107 and 109, and the maturity is three months. The results are in Table 6.

*Please see, e.g., Bloomberg for the details of the conventions in Forex options market.
<table>
<thead>
<tr>
<th>Barrier</th>
<th>Strike</th>
<th>105</th>
<th>106</th>
<th>107</th>
<th>108</th>
<th>109</th>
<th>110</th>
<th>111</th>
<th>112</th>
<th>113</th>
<th>114</th>
<th>115</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B = 105$</td>
<td>Premium</td>
<td>MC</td>
<td>4.98</td>
<td>4.28</td>
<td>3.61</td>
<td>2.96</td>
<td>2.37</td>
<td>1.84</td>
<td>1.39</td>
<td>1.02</td>
<td>0.73</td>
<td>0.51</td>
</tr>
<tr>
<td>$V_{KI}(3)$</td>
<td>5.03</td>
<td>4.35</td>
<td>3.67</td>
<td>3.02</td>
<td>2.41</td>
<td>1.88</td>
<td>1.42</td>
<td>1.04</td>
<td>0.74</td>
<td>0.51</td>
<td>0.35</td>
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</tr>
<tr>
<td>$V_{KI}(3)$</td>
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<td>4.37</td>
<td>3.68</td>
<td>3.03</td>
<td>2.42</td>
<td>1.88</td>
<td>1.42</td>
<td>1.04</td>
<td>0.74</td>
<td>0.51</td>
<td>0.35</td>
<td></td>
</tr>
<tr>
<td>$V_{KI}(3)$</td>
<td>4.98</td>
<td>4.29</td>
<td>3.62</td>
<td>2.97</td>
<td>2.38</td>
<td>1.85</td>
<td>1.40</td>
<td>1.03</td>
<td>0.73</td>
<td>0.51</td>
<td>0.35</td>
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</tr>
<tr>
<td>Diff</td>
<td>$V_{KI}(3)$</td>
<td>0.05</td>
<td>0.06</td>
<td>0.06</td>
<td>0.06</td>
<td>0.05</td>
<td>0.04</td>
<td>0.03</td>
<td>0.02</td>
<td>0.01</td>
<td>0.01</td>
<td></td>
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<td>0.08</td>
<td>0.06</td>
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<td>0.02</td>
<td>0.01</td>
<td>0.01</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V_{KI}(3)$</td>
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<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
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<tr>
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<td>2.27</td>
<td>1.77</td>
<td>1.34</td>
<td>0.99</td>
<td>0.71</td>
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<tr>
<td>$V_{KI}(3)$</td>
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<td>3.45</td>
<td>2.87</td>
<td>2.32</td>
<td>1.82</td>
<td>1.38</td>
<td>1.02</td>
<td>0.73</td>
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<tr>
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<td>0.73</td>
<td>0.51</td>
<td>0.35</td>
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</tr>
<tr>
<td>$V_{KI}(3)$</td>
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<td>3.41</td>
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<td>2.28</td>
<td>1.79</td>
<td>1.36</td>
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<tr>
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<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
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<td>1.08</td>
<td>0.81</td>
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<tr>
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<td>$V_{KI}(3)$</td>
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<td>-0.04</td>
<td>-0.01</td>
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<tr>
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<td>0.02</td>
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</tr>
</tbody>
</table>

Table 6: USDJPY 3M knock-out barrier option prices with $S_0 = 110.36$
It is observed that our approximation works well, and the adjustment method introduced in Section 2.3 makes the error small. These results show our approximation is reasonable when the premiums need to be evaluated in a short time.

4 Acknowledgments

This research is supported by CARF (Center for Advanced Research in Finance).

References


A Expansion of Underlying Processes

In this section, we show the terms $S_T^{[2]}$, $S_T^{[3]}$, $\sigma_T^{[1]}$ and $\sigma_T^{[2]}$ which are in Section 2 with one-dimensional case. The multi-dimensional cases are obtained in a similar way.

\[
\sigma_T^{[1]} = \int_0^T e^{-\lambda(T-t)} \psi(\sigma_0) \, dW_t,
\]

\[
S_T^{[2]} = 2 \int_0^T e^{rT} \int_0^t e^{-rs} \phi(\sigma_0, s_0) \, dW_u \partial S \phi(\sigma_0, s_0) \, dW_t
+ 2 \int_0^T e^{r(T-t)} \int_0^t e^{-\lambda(t-s)} \psi(\sigma_0) \, dW_u \partial \psi(\sigma_0, s_0) \, dW_t,
\]

\[
\sigma_T^{[2]} = 2 \int_0^T e^{-\lambda T} \partial \psi(\sigma_0) \int_0^t e^{\lambda s} \psi(\sigma_0) \, dW_s \, dW_t,
\]

\[
S_T^{[3]} = 3 \int_0^T e^{rT} \partial S \phi(\sigma_0, s_0) \left\{ 2 \int_0^t \int_0^s e^{-ru} \phi(\sigma_0, s_0) \, dW_u \partial S \phi(\sigma_0, s_0) \, dW_s \right\} \, dW_t
+ 2 \int_0^T \int_0^s e^{-\lambda(s-u)} \psi(\sigma_0) \, dW_u e^{-rs} \partial \phi(\sigma_0, s_0) \, dW_s \right\} \, dW_t
+ 3 \int_0^T e^{r(T-t)} \partial S \phi(\sigma_0, s_0) \left( \int_0^t e^{r(t-s)} \phi(\sigma_0, s_0) \, dW_s \right)^2 \, dW_s
+ 3 \int_0^T e^{r(T-t)} \partial \phi(\sigma_0, s_0) 2 \int_0^t \int_0^s e^{-\lambda t} \partial \psi(\sigma_0) \int_0^s e^{\lambda u} \psi(\sigma_0) \, dW_u \partial S \phi(\sigma_0, s_0) \, dW_s \, dW_t
+ 3 \int_0^T e^{r(T-t)} \partial \phi(\sigma_0, s_0) 2 \left( \int_0^t e^{-rs} \phi(\sigma_0, s_0) \, dW_s \right) \left( \int_0^t e^{-\lambda(t-s)} \psi(\sigma_0) \, dW_s \right) \, dW_t
+ 3 \int_0^T e^{r(T-t)} \partial S \phi(\sigma_0, s_0) \left( \int_0^T e^{-\lambda(t-s)} \psi(\sigma_0) \, dW_s \right)^2 \, dW_t.
\]

B Conditional Expectations Formulas of the Wiener-Itô Integrals

This section summarizes conditional expectation formulas derived by Takahashi [32] for explicit computation of the asymptotic expansions up to the third order.
In the following, $W$ is a $d$-dimensional Brownian motion and $q_i = (q_{i1}, \ldots, q_{id})'$ where $\hat{q}_i \in L^2[0, T]$, $i = 1, 2, \ldots, 5$ and $x'$ denotes the transpose of $x$. $H_n(x; \Sigma)$ denotes the Hermite polynomial of degree $n$ and $\Sigma = \int_0^T |q_{i1}|^2 dt$. For the derivation and more general results, see Section 3 in Takahashi, Takehara and Toda [33].

1. 

\[
E \left[ \int_0^T q_{i2} dW_t \int_0^T q_{i1} dW_v \right] = \left( \int_0^T q_{i2} q_{i1} dt \right) \frac{H_1(x; \Sigma)}{\Sigma}. \tag{49}
\]

2. 

\[
E \left[ \int_0^T \int_0^t q_{i2} dW_u q_{i3} dW_t \int_0^T q_{i1} dW_v = x \right] = \left( \int_0^T \int_0^t q_{i2} q_{i1} u q_{i3} q_{i1} dt \right) \frac{H_2(x; \Sigma)}{\Sigma^2}. \tag{50}
\]

3. 

\[
E \left[ \left( \int_0^T q_{i2} dW_u \right) \left( \int_0^T q_{i3} dW_s \right) \int_0^T q_{i1} dW_v = x \right] = \left( \int_0^T q_{i2} q_{i1} u \int_0^T q_{i3} q_{i1} ds \right) \frac{H_3(x; \Sigma)}{\Sigma^3}. \tag{51}
\]

4. 

\[
E \left[ \int_0^T \int_0^s q_{i2} dW_u q_{i3} dW_s q_{i4} dW_t \int_0^T q_{i1} dW_v = x \right] = \left( \int_0^T \int_0^s q_{i2} q_{i1} u \int_0^s q_{i3} q_{i1} dsdtdt \right) \frac{H_4(x; \Sigma)}{\Sigma^4}. \tag{52}
\]

5. 

\[
E \left[ \left( \int_0^T q_{i2} dW_u \right) \left( \int_0^T q_{i3} dW_s \right) \left( \int_0^T q_{i4} dW_t \right) \int_0^T q_{i1} dW_v = x \right] = \int_0^T \left( \int_0^T q_{i2} q_{i1} u \left( \int_0^T q_{i3} q_{i1} ds \right) \frac{H_5(x; \Sigma)}{\Sigma^5} \right)
+ \left( \int_0^T \int_0^t q_{i2} q_{i1} u q_{i4} q_{i1} dt \right) \frac{H_5(x; \Sigma)}{\Sigma^5}. \tag{53}
\]

6. 

\[
E \left[ \left( \int_0^T q_{i2} dW_s q_{i3} dW_t \right) \left( \int_0^T q_{i4} dW_u \right) \int_0^T q_{i1} dW_v = x \right] = \left( \int_0^T q_{i3} q_{i1} \left( \int_0^T q_{i2} q_{i1} dsdtdt \right) \frac{H_5(x; \Sigma)}{\Sigma^5} \right)
+ \left( \int_0^T q_{i3} q_{i1} \int_0^t q_{i2} q_{i4} q_{i1} dsdtdt \right) \frac{H_5(x; \Sigma)}{\Sigma^5}. \tag{54}
\]

7. 

\[
E \left[ \left( \int_0^T q_{i2} dW_s q_{i3} dW_t \right) \left( \int_0^T q_{i4} dW_u q_{i5} dW_r \right) \int_0^T q_{i1} dW_v = x \right] = \left( \int_0^T q_{i3} q_{i1} \left( \int_0^T q_{i2} q_{i1} dsdtdt \right) \frac{H_5(x; \Sigma)}{\Sigma^5} \right)
+ \left( \int_0^T q_{i3} q_{i1} \int_0^t q_{i2} q_{i4} q_{i1} dsdtdt \right) \frac{H_5(x; \Sigma)}{\Sigma^5}
+ \left( \int_0^T q_{i5} q_{i1} \int_0^t q_{i2} q_{i5} q_{i1} dsdtdt \right) \frac{H_5(x; \Sigma)}{\Sigma^5}
+ \left( \int_0^T q_{i5} q_{i1} \int_0^t q_{i2} q_{i4} q_{i5} q_{i1} dsdtdt \right) \frac{H_6(x; \Sigma)}{\Sigma^6}. \tag{55}
\]
Remark B.1. When the above multiple integrals are obtained as closed-forms, we have obviously no problems in terms of computational complexity and speed. Thus, let us discuss the cases that their closed-forms are not available, and numerical integrations are necessary. All the multiple integrals are computed by the program code with only one loop against the time parameter. For instance, multiple integrals are approximated for the numerical integration as follows:

\[
\int_0^T f(s) \int_0^t g(u) du ds \approx \sum_{i=1}^{I} \Delta t_i f(t_i) \sum_{j=1}^{J} \Delta t_j g(t_j)
= \sum_{i=1}^{I} \Delta t_i f(t_i) \left( G(t_{i-1}) + \Delta t_i g(t_i) \right),
\]

where \(\Delta t_i = (t_i - t_{i-1})\), \(G(t_i) = G(t_{i-1}) + \Delta t_i g(t_i)\) and \(G(t_0) = 0\).

Hence, the order of the computational effort is at most \(M\), where \(M\) is the number of time-steps for the discretization in the numerical integral. Note that we have no problems in terms of computational complexity and speed since various fast numerical integration methods are available such as the extrapolation method.