An approximation method for pricing continuous barrier options under multi-asset local stochastic volatility models

Kenichiro Shiraya
Graduate School of Economics, the University of Tokyo

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Kenichiro Shiraya
Graduate School of Economics, the University of Tokyo
7-3-1, Hongo, Bunkyo-ku, Tokyo, Japan.
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Abstract

This paper presents a new approximation method for pricing multi-asset continuous single barrier options under general local stochastic volatility models. The formula applies an asymptotic expansion technique and an approximation for the hitting probability. This method focuses on local stochastic volatility models with unknown characteristic function and transition density function. To the best of our knowledge, our approximation formula is the first to achieve analytic approximations for continuous barrier options prices in this environment. In numerical experiments, we examine the validity of the formula.

Keywords: Approximation Formula, Barrier Options, Local Stochastic Volatility Models, Multi-Asset
1 Introduction

This paper discusses pricing multi-asset continuous single barrier options. Barrier options are one of the most traded path-dependent exotic derivatives, sometimes with basket prices (e.g., index price) used as the underlying. These options are broadly divided into two types: continuous and discrete. Although discrete barrier options have been studied at length (e.g., Fang and Oosterlee [13], Feng and Linetsky [14], Fusai et al. [20], Umezawa and Yamazaki [37] and Shiraya et al. [32]), the target of this paper is continuous ones, and we concentrate on the evaluation of the continuous barrier option (CBO) price.

The analytical expressions of CBO prices are known for the one-dimensional Black-Scholes model. However, Black-Scholes is unsuitable for CBO pricing because the model does not account for the volatility smile structure. Instead, many of industry practitioners use local stochastic volatility (LSV) models as to account for the smile structure (see, e.g., Dadachanji [11], Homescu [22], Mercurio [25]).

There is much research which uses explicit transition density functions or characteristic functions for pricing option prices, but the explicit form of them are not known for general LSV models. One of the typical methods to approximate the option premium under LSV models is an asymptotic expansion, which is used to approximate both the terminal distribution of the underlying asset price and thus option prices. These methods can be applied to general models with acceptable accuracy and are used in practice, although the precise error is not known. Numerous articles discuss the application of these methods to LSV models, e.g., Hagan et al. [21], Cai et al. [7], Takahashi et al. [35]. This paper extends their application to the domain of CBOs.

Many articles study CBOs with a single underlying. Carr et al. [9] and Carr and Lee [10] used put-call symmetry, Akahori and Imamura [1] developed the property of symmetry in general diffusion models. Cai et al. [6], Funahashi and Kijima [19], Alós et al. [2], and Funahashi and Higuchi [18] derived approximation formulas for jump diffusion, local volatility, stochastic volatility model, and local stochastic volatility models respectively. Static hedging can also be applied to CBO barrier by calculating present values of hedging assets. Derman et al. [12] provided a calendar-spread static hedging method for the Black-Scholes model, and Fink [15] extended it to a stochastic volatility model. Shiraya et al. [31] further proposed an approximation method for general LSV models. Fouque et al. [16], [17] studied approximations of the partial differential equations (PDEs) for fast mean-reverting models. Kato et al. [23], [24] developed a semi-group expansion scheme for the general diffusion process. Brown et al. [5], and Tsuzuki [36] studied CBO pricing boundaries. Their method is based on hedging CBOs and is independent of underlying asset models. Prior work, which derived analytical approximation formulas, cannot be applied to multi-asset cases or the models whose transition density or characteristic functions are unknown.

Numerical methods for solving PDEs in single-asset CBOs are well developed and can be computed in a few seconds. However, PDEs in multi-asset CBOs remain difficult to solve with sufficient accuracy or computational speed for practice.

The Monte Carlo method can evaluate CBO prices in both single- and multi-asset cases for LSV models, but its computational cost is high. Moreover, this method needs to discretize the process and judge whether the barrier level was reached or not at the discretized time. It is well known that the discretization error of barrier options is larger than that of plain options, and to overcome this problem, practitioners use the Brownian bridge method to decrease the error. In practice, Monte Carlo method can estimate profit and loss quite accurately (e.g., daily trading P&L). However, computational inefficiency and low speed are the drawbacks of using Monte Carlo for market making in practice, when various prices need to be quoted in a restricted amount of time.

A number of studies suggest approximation methods for non-barrier basket option prices.
Takahashi [34] showed an asymptotic expansion under local volatility (LV) models. Bayer and Laurence [3] used a heat kernel expansion and the Laplace method. Xu and Zheng [38] applied the lower bound technique and asymptotic expansion to European basket call prices in an LV jump-diffusion model. Shiraya and Takahashi [28] applied asymptotic expansion under LSV with jumps. Caldana et al. [8] derived narrow pricing bounds for models whose characteristic function of the vector of log-prices is known. However, these methods cannot be applied directly to pricing CBOs under general LSV models.

Our method is based on the approximated barrier hitting probability and an asymptotic expansion technique. By applying this method, we derive an explicit formula for multi-asset continuous barrier options prices under general LSV models. To the best of our knowledge, the proposed approximation is the first to achieve an analytic approximation for such options prices. In Numerical Examples, we compare the approximated prices with the values obtained by the Monte Carlo simulations for the constant elasticity of variance (CEV) and stochastic alpha, beta, rho (SABR) models (see [21]), and demonstrate the effectiveness of our formula.

The organization of this paper is as follows: Section 2 proposes an approximation method for barrier option prices by combining an expansion method introduced by Takahashi [34] with the Brownian bridge technique. In addition, we provide an additional approximation scheme for the barrier hitting probability. Section 3 presents numerical examples to examine the validity of our method.

2 Pricing formula of barrier options under local stochastic volatility models

This section shows an approximation method for multi-asset CBOs and shows an explicit formula for barrier option prices. We approximate the barrier option price by the following steps:

1. Approximate the distribution of the terminal payoff function by an asymptotic expansion technique.

2. Approximate the barrier hitting probability of the underlying processes.

2.1 An approximation method

We suppose a filtered probability space satisfying the usual conditions \((\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\}_{t \geq 0})\) where \(\mathbf{P}\) is a risk-neutral measure. We also assume a model for the underlying asset prices \(S_t = (S^1_t, \ldots, S^d_t)\) and their volatilities \(\sigma_t = (\sigma^1_t, \ldots, \sigma^d_t)\), and set \(Y_t = (Y^1_t, \ldots, Y^{2d}_t) \coloneqq (S^1_t, \ldots, S^d_t, \sigma^1_t, \ldots, \sigma^d_t)\).

For \(\epsilon > 0\), let us assume that \(Y_T\) is given by the solutions of the following stochastic integral equation. (For the sake of simplicity, we abbreviate the dependence of \(\epsilon\) from notations.)

\[
Y_T = y_0 + \int_0^T \kappa(Y_t) dt + \epsilon \int_0^T \phi(Y_t) dW_t, \tag{1}
\]

where \(y_0 = (y^1_0, \ldots, y^{2d}_0)\) is a constant vector, \(W = (W^1, \ldots, W^{2d})\) are independent Brownian motions, \(\kappa : \mathbb{R}^{2d} \to \mathbb{R}^{2d}, \phi : \mathbb{R}^{2d} \to \mathbb{R}^{2d \times 2d}\) are deterministic bounded \(C^\infty\) functions, and (1) satisfies the uniform ellipticity condition.

The underlying variate of a barrier option is expressed as

\[
X_t := \sum_{i=1}^d w^i Y^i_t, \tag{2}
\]
where \( w^j \) are constant weights. We assume \( X_t \) has a smooth density.

Then, we think about single barrier options whose domain of the underlying variate is written as \( D := (-\infty, B) \) for an upper barrier case, \( D := (B, \infty) \) for a lower barrier case, and define a stopping time \( \tau_X \) as

\[
\tau_X := \inf\{ t > 0 | X_t \notin D \}. \tag{3}
\]

In this setting, a single barrier knock-in (KI) option price \((V_{KI})\) and a single barrier knock-out (KO) option price \((V_{KO})\) are expressed as follows:

\[
V_{KI} := e^{-rT} \mathbb{E}[1_{\tau_X \leq T} f_K(X_T)], \tag{4}
\]

\[
V_{KO} := e^{-rT} \mathbb{E}[1_{\tau_X > T} f_K(X_T)] = e^{-rT} \mathbb{E}[f_K(X_T)] - e^{-rT} \mathbb{E}[1_{\tau_X \leq T} f_K(X_T)], \tag{5}
\]

where \( f_K(X_T) \) is the payoff function of the European call option with the strike price \( K \) expressed as

\[
f_K(X_T) := (X_T - K)^+. \tag{6}
\]

Since we need to differentiate the payoff function to use an asymptotic expansion, we approximate \( f_K \) with a smooth function.

**Lemma 2.1.** For \( \forall \epsilon > 0 \), there exist \( \tilde{f}_K \in C^\infty(\mathbb{R}) \) which satisfies \( |\tilde{f}_K(x)| \leq C(1 + |x|^p) \), and

\[
|\mathbb{E}[\tilde{f}_K(X_T)] - \mathbb{E}[f_K(X_T)]| < \epsilon, \tag{7}
\]

\[
|\mathbb{E}[1_{\tau_X > T} \tilde{f}_K(X_T)] - \mathbb{E}[1_{\tau_X > T} f_K(X_T)]| < \epsilon. \tag{8}
\]

**Proof.** Since

\[
|\mathbb{E}[1_{\tau_X > T} \tilde{f}_K(X_T)] - \mathbb{E}[1_{\tau_X > T} f_K(X_T)]| \\
\leq |\mathbb{E}[1_{\tau_X > T} \tilde{f}_K(X_T) - f_K(X_T)]| \\
\leq |\mathbb{E}[\tilde{f}_K(X_T) - f_K(X_T)]|,
\]

it is enough to show (7). Here, since \( C^\infty(\mathbb{R}) \) is dense in \( C(\mathbb{R}) \), there exist \( \tilde{f}_K \in C^\infty(\mathbb{R}) \) satisfying

\[
|\mathbb{E}[\tilde{f}_K(X_T)] - \mathbb{E}[f_K(X_T)]| < \epsilon. \tag{10}
\]

\[\square\]

### 2.1.1 An asymptotic expansion for \( f_K(X_T) \)

In this subsection, we summarize an expansion of \( f_K(X_T) \) developed by Takahashi [34]. It is sufficient to calculate an asymptotic expansion for a smooth function \( \tilde{f}_K \) in Lemma 2.1. Thus hereafter, we assume \( f_K \in C^\infty(\mathbb{R}) \).

In order to approximate the distribution of the terminal payoff function \( f_K(X_T) \), we assume the asymptotic expansions of \( Y_T \) around \( \epsilon = 0 \) as follows:

\[
Y_T = Y_T^{[0]} + \epsilon Y_T^{[1]} + \frac{\epsilon^2}{2!} Y_T^{[2]} + \cdots, \tag{11}
\]

where \( Y_t^{[j]} = (Y_t^{1, [j]}, \ldots, Y_t^{2d, [j]}) := \frac{\partial^j Y_t}{\partial \epsilon^j} \big|_{\epsilon = 0} \).

The zeroth order term \( Y_T^{[0]} \) is the solution to the following integral equation:

\[
Y_T^{[0]} = y_0 + \int_0^T \kappa \left( Y_t^{[0]} \right) dt. \tag{12}
\]
$Y_{T}^{[1]}$ is also the solution of

$$
Y_{T}^{[1]} = \frac{\partial Y(\epsilon, T)}{\partial \epsilon} \bigg|_{\epsilon=0} = \int_{0}^{T} \kappa(Y_{t}^{[1]}) \, dt + \int_{0}^{T} \phi(Y_{t}^{[0]}) \, dW_{t},
$$

(13)

$Y^{[2]}$, $Y^{[3]}$, $\cdots$ are obtained in a similar way.

Then, $X_{T}$ and $f(X_{T})$ are expressed as follows:

$$
X_{T} = \sum_{j=0}^{m} \epsilon^{j} X_{T}^{[j]} + o(\epsilon^{m}),
$$

(14)

$$
f_{K}(X_{T}) = \sum_{m=1}^{M} \epsilon^{m} f_{m,T} + o(\epsilon^{M}),
$$

(15)

where $X_{T}^{[j]} := \sum_{i=1}^{d} \psi Y_{T}^{(i)[j]}$, $z := \frac{X_{T}^{[i]} - K}{\epsilon}$, and

$$
f_{i,T} := f_z \left( X_{T}^{[i]} \right),
$$

(16)

$$
f_{m,T} := \sum_{j=1}^{m-1} \frac{1}{j!} f_{z}^{(j)} \left( X_{T}^{[i]} \right) \sum_{n_{1}+\cdots+n_{j}=m-1} \frac{X_{T}^{[n_{1}]} \cdots X_{T}^{[n_{j}]}}{n_{1}! \cdots n_{j}!}.
$$

(17)

### 2.1.2 An application of the barrier hitting probability of the Brownian bridge

Next, we approximate the hitting probability by using the distribution of the maximum of the Brownian bridge.

Hereafter, we concentrate on single barrier down and in call options whose domain is $D = (B, \infty)$ with $x \in D$ and $B < K$. Other options are obtained in a similar way. In addition, we assume $\kappa$ satisfies that $Y_{T}^{[i],[1]}$, $i = 1, \cdots, d$ have Gaussian distributions (e.g., $\kappa(x) = ax + b$ for $a, b \in \mathbb{R}^{d}$), and $Y_{T}^{[j],[j]}$, $j = 2, 3, \cdots, i = 1, \cdots, d$ are expressed as multiple Ito-integrals. Hereafter, we assume the mean of $X_{T}^{[1]}$ is 0 for simplicity. The option prices in $\mathbb{E}[X_{T}^{[1]}] \neq 0$ cases are obtained in a similar way.

While up to the first order expansion $X_{T}^{[0]} + \epsilon X_{T}^{[1]}$ is a Gaussian process, the coefficient of the drift term is not constant. Thus, we make an alternative Gaussian process which has a constant coefficient in the drift term and the same distribution to that of $X_{T}^{[0]} + \epsilon X_{T}^{[1]}$ at the maturity in order to use an explicit formula for the barrier hitting probability of the Brownian bridge (see e.g., [4]).

Since $Y_{T}^{[i],[1]}$ (the $i$-th element of $Y_{T}^{[1]}$) has a Gaussian distribution, the sum of them also has a Gaussian distribution. Thus for a Brownian motion $Z_{t}$, $X_{T}^{[0]} + \epsilon X_{T}^{[1]}$ is expressed as:

$$
X_{T}^{[0]} + \epsilon X_{T}^{[1]} = \sum_{i=1}^{d} \psi^{i} \left( Y_{T}^{[i],[0]} + \epsilon Y_{T}^{[i],[1]} \right)
$$

$$
= \sum_{i=1}^{d} \psi^{i} Y_{T}^{[i],[0]} + \epsilon \sqrt{\Sigma} Z_{t},
$$

(18)

where

$$
\Sigma := \text{Var}(X_{T}^{[1]}).
$$

(19)
Since our expansion method uses only the distribution at the maturity, we make an alternative Gaussian process \( \hat{X}_t \) defined as:
\[
\hat{X}_t := X_0 + \frac{X_T^0 - X_0}{T} t + \sqrt{\Sigma} Z_t. \tag{20}
\]

Then, at the maturity,
\[
\hat{X}_T = X_0 + X_T^0 - X_0 + \sqrt{\Sigma} Z_T
\]
\[
= X_T^0 + \sqrt{\Sigma} \varepsilon_{X_t}. \tag{21}
\]

Thus, we use \( \hat{X}_t \) instead of \( X_t^0 + \varepsilon_{X_t} \) to approximate the barrier hitting probability.

The following Lemma 2.2 shows the barrier hitting probability of the Brownian bridge on \( \hat{X}_t \) which is defined as
\[
p_{\hat{X}}(x, w, B) := \mathbb{P}(\tau_{\hat{X}} \leq T | \hat{X}_0 = x, \hat{X}_T = w), \tag{22}
\]
\[
\tau_{\hat{X}} := \inf\{ t > 0 | \hat{X}_t \notin D \}, \tag{23}
\]
and the proof is shown in e.g., Section 3 of [4].

**Lemma 2.2.** When \( x, w \geq B \), the barrier hitting probability of the Brownian bridge on \( \hat{X}_t \) is expressed as follows:
\[
p_{\hat{X}}(x, w, B) = \exp\left( - \frac{2(B - x)(B - w)}{\varepsilon^2 \Sigma T} \right), \tag{24}
\]

We define an approximated down and in call option price \( V_{KI}(M) \) as
\[
V_{KI}(M) := e^{-rT} \mathbb{E} \left\{ p_{\hat{X}}(X_0, \hat{X}_T, B) \sum_{m=1}^{M} \varepsilon^m f_{m,T} \right\}. \tag{25}
\]

To obtain the explicit expression of the right-hand side of (25), we calculate it in Section 2.2.

**Corollary 2.3.** KO option prices are approximated by applying \( q_{\hat{X}}(x, w, B) = 1 - p_{\hat{X}}(x, w, B) \) instead of \( p_{\hat{X}}(x, w, B) \) in (25).

Next, we show \( V_{KI}(M) \) is an approximation of the down and in barrier option price \( V_{KI} \).

**Proposition 2.4.** For \( D = (B, \infty) \), \( \tau_X = \inf\{ t > 0 | X_t \notin D \} \), and \( K, x \in D \), the difference between the approximated down and in call option price \( V_{KI}(M) \) and the true price \( V_{KI} = e^{-rT} \mathbb{E} \left[ 1_{\tau_X \leq T} f_K(X_T) \right] \) is estimated as follows:
\[
|V_{KI} - V_{KI}(M)| < c_1(T) \varepsilon^{M+1} + c_2(T), \tag{26}
\]
where \( c_1(T) \) and \( c_2(T) \) are deterministic functions satisfying \( \lim_{T \downarrow 0} c_1(T) = 0 \) and \( \lim_{T \downarrow 0} c_2(T) = 0 \).

**Proof.** The difference between \( V_{KI} \) and \( V_{KI}(M) \) is estimated as follows:
\[
|V_{KI} - V_{KI}(M)|
\]
\[
= e^{-rT} \left\{ \mathbb{E} \left[ 1_{\tau_X \leq T} f_K(X_T) \right] - \mathbb{E} \left[ p_{\hat{X}}(X_0, \hat{X}_T, B) \sum_{m=1}^{M} \varepsilon^m f_{m,T} \right] \right\}
\]
\[
= e^{-rT} \left\{ \mathbb{E} \left[ 1_{\tau_X \leq T} 1 f_K(X_T) - \sum_{m=1}^{M} \varepsilon^m f_{m,T} + \sum_{m=1}^{M} \varepsilon^m f_{m,T} \right] - \mathbb{E} \left[ p_{\hat{X}}(X_0, \hat{X}_T, B) \sum_{m=1}^{M} \varepsilon^m f_{m,T} \right] \right\}
\]
\[ \leq e^{-rT}\left\{ \left| \mathbb{E}\left[ f_K(X_T) - \sum_{m=1}^{M} e^m f_{m,T}\right]\right| + \left| \mathbb{E}\left[ \left( 1_{\tau_X \leq T} + p_{X}(X_0, \hat{X}_T, B) \right) \sum_{m=1}^{M} e^m f_{m,T}\right]\right| \right\} =: V_1 + V_2. \] 

First, the approximated smooth payoff function \( f \) satisfies Condition (A)-iii in [29], and the coefficients of (1) satisfy Condition (A)-i, ii in [29]. Thus, we obtain

\[ V_1 < c_1(T) e^{M+1}. \]

for a deterministic function \( c_2 \) satisfying \( \lim_{T \to 0} c_2(T) = 0 \).

Next, for a deterministic function \( c_1(T) \), Hölder’s inequality yields

\[ V_2 \leq \left( \mathbb{P}(\tau_X \leq T) + 2\mathbb{P}(\tau_X \leq T)^{1/2}\mathbb{E}[p_{X}(X_0, \hat{X}_T, B)^4]^{1/2} + \mathbb{E}[p_{X}(X_0, \hat{X}_T, B)^2]\right)^{1/2} \times \left( \mathbb{E}\left[ \left( \sum_{m=1}^{M} e^m f_{m,T}\right)^2\right]\right)^{1/2} < c_2(T). \]

Since \( \lim_{T \downarrow 0} \mathbb{E}[p_{X}(x, \hat{X}, B)] = 0 \) and \( \lim_{T \downarrow 0} \mathbb{P}(\tau_X \leq T) = 0 \), \( c_2(T) \) satisfies \( \lim_{T \downarrow 0} c_2(T) = 0 \).

\[ 2.2 \text{ An approximation formula} \]

This subsection gives an explicit expression of \( V_{K1}(M) \). We apply \( M = 3 \) to obtain the approximated option price (i.e. \( V_{K1}(3) \)) because the calculation becomes complicated in the higher cases. To obtain the higher values, we can calculate them by the method in [35]. Moreover, until the order \( M = 3 \), we can apply the derivative in the sense of distribution, and the result is the same as that of the limit of the smooth function.

In the case of a European option with a strike price \( K \), the third order expansion of a payoff function is expressed as follows:

\[ (X_T - K)^+ = \epsilon \left( X_T^{[1]} + \frac{\epsilon}{2} X_T^{[2]} + \frac{\epsilon^2}{6} X_T^{[3]} + z + o(\epsilon^3) \right)^+ \]

\[ = \epsilon \left( X_T^{[1]} + z \right)^+ + \epsilon^2 1_{\{X_T^{[1]} > -z \}} \frac{1}{2} X_T^{[2]} \]

\[ + \epsilon^3 \left( \frac{1}{6} 1_{\{X_T^{[1]} > -z \}} X_T^{[3]} + \frac{1}{8} \delta_{\{X_T^{[1]} = -z \}} \left( X_T^{[2]} \right)^2 \right) + o(\epsilon^3). \]

Then,

\[ V_{K1}(3) = e^{-rT} \mathbb{E}\left[ p_{X}(X_0, \hat{X}_T, B) \left( \epsilon \left( X_T^{[1]} + z \right)^+ + \epsilon^2 1_{\{X_T^{[1]} > -z \}} \frac{1}{2} X_T^{[2]} \right) \right. \]

\[ + \epsilon^3 \left( \frac{1}{6} 1_{\{X_T^{[1]} > -z \}} X_T^{[3]} + \frac{1}{8} \delta_{\{X_T^{[1]} = -z \}} \left( X_T^{[2]} \right)^2 \right) \left| X_T^{[1]} = x \right. \]

\[ = e^{-rT} \left( \epsilon \int_{-z}^{\infty} (x + z) \exp \left( -\frac{2b(\epsilon^2 b - \alpha - \epsilon x)}{\Sigma_T} \right) n(x; 0, \Sigma_T) dx \right. \]

\[ + \epsilon^2 \frac{1}{2} \int_{-z}^{\infty} \exp \left( -\frac{2b(\epsilon^2 b - \alpha - \epsilon x)}{\Sigma_T} \right) \mathbb{E}\left[ X_T^{[2]} | X_T^{[1]} = x \right] n(x; 0, \Sigma_T) dx \]

\[ + \epsilon^3 \left( \frac{1}{6} \int_{-z}^{\infty} \exp \left( -\frac{2b(\epsilon^2 b - \alpha - \epsilon x)}{\Sigma_T} \right) \mathbb{E}\left[ X_T^{[3]} | X_T^{[1]} = x \right] n(x; 0, \Sigma_T) dx \right). \]
where \( \alpha := X^{[0]} - X_0, b := \frac{1}{2}(B - X_0), \Sigma_T := \text{Var}(X_T) = \Sigma T, \) \( z = \frac{X^{[0]} - K}{e} \) and \( n(x; y, \Sigma) := \frac{1}{\sqrt{2\pi\Sigma}} \exp \left( -\frac{(x-y)^2}{2\Sigma} \right). \) Then, we obtain

\[
\exp \left( -\frac{2b(e^2b - \alpha - cx)}{\Sigma_T} \right) n(x; 0, \Sigma_T) = \frac{1}{\sqrt{2\pi\Sigma_T}} \exp \left( -\frac{x^2 - 4ebx - 4ba + 4e^2b^2}{2\Sigma_T} \right) = n(x ; 2eb, \Sigma_T) \exp \left( \frac{2ba}{\Sigma_T} \right).
\]

Here, we set \( \gamma_T := \exp \left( \frac{2ba}{\Sigma_T} \right), \)

\[
V_{KI}(3) = e^{-rT} \left[ e^{\int_{-\hat{z}}^{\hat{z}} (x + \hat{z}) n(x; 0, \Sigma_T) dx} + \frac{1}{2} \int_{-\hat{z}}^{\hat{z}} \mathbb{E} \left[ X^{[2]}_T | X^{[1]}_T = x + 2eb \right] n(x; 0, \Sigma_T) dx \right. \\
+ e^3 \left( \frac{1}{6} \int_{-\hat{z}}^{\hat{z}} \mathbb{E} \left[ X^{[3]}_T | X^{[1]}_T = x + 2eb \right] n(x; 0, \Sigma_T) dx \right. \\
+ \left. \left. \frac{1}{8} \mathbb{E} \left[ (X^{[2]}_T)^2 | X^{[1]}_T = -\hat{z} \right] n(-\hat{z}; 0, \Sigma_T) \right) \right] \gamma_T,
\]

where \( \hat{z} := 2eb + z. \)

The conditional expectations in (33) are obtained by using formulas introduced by Takanashi [34] and the next integral formulas.

\[
\int_{-\hat{z}}^{\hat{z}} H_k(x, \Sigma) n(x; 0, \Sigma) dx = \Sigma H_{k-1}(-\hat{z}, \Sigma) n(z; 0, \Sigma),
\]

where

\[
H_k(x, \Sigma) := (-\Sigma)^k e^{\frac{x^2}{2\Sigma}} \frac{d^k}{dx^k} e^{-\frac{x^2}{2\Sigma}}.
\]

From these calculations, we obtain the approximation formula of the down and in option price as the next theorem. Up and in options and KO options prices are obtained in a similar way with combining European type options.

**Theorem 2.5.** The approximation formula for the initial value of down and in call option price with a maturity \( T, \) a barrier price \( B, \) and a strike price \( K \) is given by the formula:

\[
V_{KI}(3) = e^{-rT} \left[ e^{\frac{\hat{z}N \left( \frac{\hat{z}}{\sqrt{\Sigma_T}} \right)}{\Sigma_T}} + \Sigma_T n(\hat{z}; 0, \Sigma_T) \right] \\
+ e^2 \left\{ C_{2.1} \frac{4e^2b^2}{\Sigma_T^2} N \left( \frac{\hat{z}}{\sqrt{\Sigma_T}} \right) + \frac{4ebn(\hat{z}; 0, \Sigma_T)}{\Sigma_T} + \frac{N(\frac{\hat{z}}{\Sigma_T}) - \hat{z}n(\hat{z}; 0, \Sigma_T)}{\Sigma_T} \right\} \\
+ e^3 \left\{ C_{3.1} \frac{8e^3b^3}{\Sigma_T^3} N \left( \frac{\hat{z}}{\sqrt{\Sigma_T}} \right) + \frac{12e^2b^2n(\hat{z}; 0, \Sigma_T)}{\Sigma_T^2} + \frac{6ebN(\frac{\hat{z}}{\Sigma_T}) n(\hat{z}; 0, \Sigma_T)}{\Sigma_T^3} \right\} \\
+ \left( \frac{2}{\Sigma_T} + \frac{\hat{z}^2}{\Sigma_T^2} \right) n(\hat{z}; 0, \Sigma_T) + \frac{C_{3.2}}{6} \left( n(\hat{z}; 0, \Sigma_T) + \frac{2eb}{\Sigma_T} N(\frac{\hat{z}}{\Sigma_T}) \right)
\]
In order to compute the constants $C_{i,j}$, we can use a numerical integration method. Please see e.g., Appendix in [33].

### 2.3 Adjustment of the barrier hitting probability

Since the barrier hitting probability is approximated by that of $\tilde{X}_t$ with the formula (24), it still has room for improvement. In this subsection, we provide an adjustment method of this probability. Note that we explain the adjustment method in the one asset with the domain $D = (B, \infty)$ case. Thus, $X_t$ is expressed as follows:

$$X_t = X_0 + \int_0^t \kappa(X_s)ds + \epsilon \int_0^t \phi(X_s)dW_s. \tag{37}$$

The multi-asset and $D = (-\infty, B)$ cases are obtained in a similar way.

The error of the barrier hitting probability is caused by the difference of the distributions between $\tilde{X}_t$ defined in (20) and the original process $X_t$. In order to reduce the error of the barrier hitting probability, we will make a new process instead of $\tilde{X}_t$ in the following steps:

i. Firstly, we calculate approximated barrier hitting probabilities $P_h(M)$ at $t_h$, $h = 1, 2$ by an asymptotic expansion scheme. That is, $P_h(M) := P \left( \sum_{m=0}^{M} \frac{m}{m!} X_t^{[m]} \leq B \right)$. In our numerical example, we calculate it by the difference of European option prices with $M = 3$.

ii. Next, in order to reduce the error of the barrier hitting probability, we make an alternative Gaussian process $\chi_t$ defined as

$$\chi_t := X_0 + \mu t + \epsilon \sqrt{\hat{\Sigma}}dW_t, \tag{38}$$

where $\mu$ and $\hat{\Sigma}$ are chosen by satisfying $P (\chi_{t_h} \leq B) = P_h(M)$ ($h = 1, 2$). That is, $\mu$ and $\hat{\Sigma}$ are the solutions of the next simultaneous equations

$$\begin{cases} X_0 + \mu t_1 + \epsilon \sqrt{\hat{\Sigma}} \sqrt{t_1} N^{-1}(P_1(M)) = B, \\
X_0 + \mu t_2 + \epsilon \sqrt{\hat{\Sigma}} \sqrt{t_2} N^{-1}(P_2(M)) = B, \tag{39} \end{cases}$$

where the function $N^{-1}$ is the inverse function of cumulative distribution function of the standard normal distribution, and the value is obtained easily (see e.g., Moro [26]).

By solving the equations, we obtain

$$\epsilon \sqrt{\hat{\Sigma}} = \frac{\left(1 - \frac{t_2}{t_1}\right) B - X_0 \left(1 - \frac{t_2}{t_1}\right)}{\sqrt{t_2} N^{-1}(P_2(M)) - \frac{t_2}{t_1} N^{-1}(P_1(M))}, \tag{40}$$

$$\mu = \left(B - X_0 - \epsilon \sqrt{\hat{\Sigma}} \sqrt{t_1} N^{-1}(P_1(M)) \right) \frac{1}{t_1}. \tag{41}$$
Here, the barrier hitting probability of $\hat{X}$ coincides with that of the first order value $X[0] + \epsilon X[1]$ only at the maturity $T$. On the other hand, if we use the third order expansion to obtain the probability $P_h(3)$, the barrier hitting probability of the adjusted process $\chi$ coincides with that of the third order value $X[0] + \epsilon X[1] + \frac{\epsilon^2}{2} X[2] + \frac{\epsilon^3}{6} X[3]$ at two points $t_1, t_2 \in (0, T]$.

Thus, while the both processes used for approximating the barrier hitting probability are assumed Gaussian, we expect that the option premium with the adjustment is more accurate than the one using $\hat{X}$.

iii Finally, in a similar way to Lemma 2.2, we calculate $p_\chi(x, w, B) := P(\tau_\chi \leq T | \chi_0 = x, \chi_T = w)$ for $\chi_t$ as

$$p_\chi(x, w, B) = \exp \left( -\frac{2(B - x)(B - w)}{\epsilon^2 \tau T} \right),$$

where $\tau_\chi := \inf \{ t > 0 | \chi_t \notin D \}$, and $\tilde{\Sigma}$ is obtained in Step ii. Then, by substituting (42) to (25),

$$\hat{V}_{KI}(M) := e^{-rT} \mathbb{E} \left[ p_\chi(X_0, \hat{X}_T, B) \sum_{m=1}^{M} \epsilon^m f_{m,T} \right],$$

where $f_{m,T}$ ($m = 1, \ldots, M$) are defined in (16) and (17). Thus, the formula of adjusted approximated KI option price $\hat{V}_{KI}(3)$ is obtained by replacing $b$ and $\alpha$ in (36) by $b = \frac{1}{\epsilon^2}(B - X_0)\sqrt{\frac{\tau T}{\Sigma_T}}$ and $\alpha = \mu T \sqrt{\frac{\Sigma_T}{\Sigma_T}}$.

We show in numerical example that the results of (43) work better than those of (36) in many cases.

### 3 Numerical Examples

In this section, we examine the practical validity of our method by using parameters calibrated to the real market data. The model used in this section is CEV and SABR models expressed as follows:

$$S_t^i = S_0^i + \int_0^T \alpha S_t^i \, dt + \epsilon \int_0^T \sigma_t^i (S_t^i) \beta \, dW_t^\iota,$$

$$\sigma_t^i = \sigma_0^i + \epsilon \int_0^T \nu \sigma_t^i \, dW_t^\iota.$$  

(44)

(45)

CEV model is a special case of SABR model ($\nu \equiv 0$ in (45)). We remark that there is no matter to set $\epsilon = 1$ because the difference of $\epsilon$ is absorbed in the parameters $\sigma_0$ and $\nu$.

The parameters are calibrated to the data of EURUSD 6M maturity on September 26, 2018 downloaded from Bloomberg. The calibrated parameters by the SABR formula [21] are listed as Case (A) in Table 1.
In addition, we examine stressed parameter cases (Case (B): $\sigma_0$ is 1.5 times, Case (C): $\nu$ is 1.5 times, Case (D) the maturity is 1 year), CEV model cases (Cases (E) and (F)) and multi-asset case (Case (G)). While the number of assets in Cases (A) to (F) is one (i.e. $i = 1$), that of Case (G) is five. That is, $i = 1, \cdots, 5$, $\langle X, Y \rangle_t = \rho_{XY} t$, and the weight of each asset is set as $w_i = 0.2$ in Case (G).

We calculate the forward premium of down and in call options whose strikes are from 1.14 to 1.21, and the barrier prices are 1.11 and 1.14, and the maturity is 0.5 years (except Cases (D) and (F)).

We compare the approximated values with the results of a Monte Carlo simulation (MC) whose time partition is 1,024/year, and the number of simulations is 1,000,000. $V_{KI}(3)$ and $\tilde{V}_{KI}(3)$ are the values calculated by the formula (36) and (43), respectively. In order to adjust the barrier hitting probability introduced in Section 2.3, $t_1$ and $t_2$ in Section 2.3 are set as $t_1 = T/2$ and $t_2 = T$.

The results are in Tables 2 and 3.
<table>
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<th>Strike</th>
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<td>0.0015</td>
<td>0.0011</td>
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<td>-0.0006</td>
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<td>-0.0003</td>
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<tr>
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<td>0.0050</td>
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<td>0.0076</td>
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<td>0.0040</td>
<td>0.0034</td>
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<td>0.0083</td>
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<td>0.0064</td>
<td>0.0056</td>
<td>0.0050</td>
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<td>0.0064</td>
<td>0.0056</td>
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<td>0.0042</td>
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<td>0.0058</td>
<td>0.0050</td>
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<td>-0.0019</td>
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<td>0.0006</td>
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<td>$V_{KI}(3)$</td>
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<td>0.0006</td>
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<td>0.0003</td>
<td>0.0002</td>
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</tr>
<tr>
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<td>0.0000</td>
<td>0.0000</td>
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<tr>
<td>Case (F) Premium MC</td>
<td>0.0077</td>
<td>0.0064</td>
<td>0.0052</td>
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<td>0.0004</td>
<td>0.0003</td>
<td>0.0002</td>
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<td>Case (G) Premium MC</td>
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<td>0.0004</td>
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<tr>
<td>Diff $V_{KI}(3)$</td>
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<td>-0.0004</td>
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</table>

Table 2: Numerical Results of down and in call options with, $B = 1.11$
expresses the curvature of the volatility smile, and thus our approximation is accurate if the
difference of the distributions is large if
and
$\epsilon$
adjusted results in Table 2 are improved. Since the dstribution of LSV model is different from
stressed cases of $V$

Case (F) Premium  
MC 0.0131 0.0106 0.0085 0.0067 0.0053 0.0041 0.0032 0.0025
$V_{KI(3)}$ 0.0127 0.0101 0.0080 0.0063 0.0049 0.0038 0.0030 0.0023
$\bar{V}_{KI(3)}$ 0.0128 0.0103 0.0081 0.0064 0.0050 0.0039 0.0030 0.0023

Diff  $V_{KI(3)}$ -0.0004 -0.0005 -0.0005 -0.0004 -0.0004 -0.0003 -0.0003 -0.0002
$V_{KI(3)}$ -0.0004 -0.0004 -0.0004 -0.0003 -0.0003 -0.0002 -0.0002

Case (B) Premium  
MC 0.0256 0.0222 0.0191 0.0164 0.0140 0.0119 0.0100 0.0085
$V_{KI(3)}$ 0.0260 0.0234 0.0219 0.0192 0.0164 0.0140 0.0118 0.0100 0.0084
$\bar{V}_{KI(3)}$ 0.0253 0.0219 0.0188 0.0161 0.0137 0.0116 0.0098 0.0082

Diff  $V_{KI(3)}$ 0.0004 0.0002 0.0001 0.0001 0.0000 -0.0000 -0.0001 -0.0001
$V_{KI(3)}$ -0.0003 -0.0003 -0.0003 -0.0003 -0.0003 -0.0003 -0.0002 -0.0002

Case (C) Premium  
MC 0.0146 0.0122 0.0101 0.0083 0.0069 0.0057 0.0047 0.0039
$V_{KI(3)}$ 0.0137 0.0112 0.0092 0.0075 0.0061 0.0050 0.0041 0.0034
$\bar{V}_{KI(3)}$ 0.0139 0.0114 0.0093 0.0076 0.0062 0.0050 0.0041 0.0034

Diff  $V_{KI(3)}$ -0.0009 -0.0010 -0.0009 -0.0008 -0.0007 -0.0006 -0.0006 -0.0005
$V_{KI(3)}$ -0.0007 -0.0008 -0.0008 -0.0008 -0.0007 -0.0006 -0.0006 -0.0005

Case (D) Premium  
MC 0.0276 0.0243 0.0213 0.0186 0.0162 0.0140 0.0121 0.0105
$V_{KI(3)}$ 0.0275 0.0240 0.0205 0.0180 0.0155 0.0133 0.0114 0.0098
$\bar{V}_{KI(3)}$ 0.0271 0.0236 0.0205 0.0177 0.0153 0.0131 0.0113 0.0097

Diff  $V_{KI(3)}$ -0.0001 -0.0003 -0.0005 -0.0006 -0.0007 -0.0007 -0.0007 -0.0007
$V_{KI(3)}$ -0.0005 -0.0007 -0.0008 -0.0009 -0.0009 -0.0009 -0.0009 -0.0009

Case (E) Premium  
MC 0.0113 0.0088 0.0067 0.0050 0.0037 0.0027 0.0019 0.0013
$V_{KI(3)}$ 0.0115 0.0090 0.0069 0.0052 0.0038 0.0027 0.0019 0.0013
$\bar{V}_{KI(3)}$ 0.0114 0.0089 0.0068 0.0051 0.0037 0.0027 0.0019 0.0013

Diff  $V_{KI(3)}$ 0.0002 0.0002 0.0001 0.0001 0.0001 0.0001 0.0000 0.0000
$V_{KI(3)}$ 0.0001 0.0001 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000

Case (F) Premium  
MC 0.0270 0.0235 0.0202 0.0173 0.0147 0.0124 0.0104 0.0086
$V_{KI(3)}$ 0.0279 0.0243 0.0210 0.0180 0.0153 0.0129 0.0108 0.0090
$\bar{V}_{KI(3)}$ 0.0274 0.0239 0.0206 0.0176 0.0150 0.0127 0.0106 0.0088

Diff  $V_{KI(3)}$ 0.0009 0.0008 0.0007 0.0007 0.0006 0.0005 0.0005 0.0004
$V_{KI(3)}$ 0.0004 0.0004 0.0004 0.0004 0.0003 0.0003 0.0003 0.0002

Case (G) Premium  
MC 0.0999 0.0078 0.0060 0.0045 0.0034 0.0025 0.0018 0.0013
$V_{KI(3)}$ 0.0993 0.0072 0.0054 0.0041 0.0030 0.0022 0.0016 0.0011
$\bar{V}_{KI(3)}$ 0.0998 0.0076 0.0058 0.0043 0.0032 0.0023 0.0017 0.0012

Diff  $V_{KI(3)}$ -0.0006 -0.0006 -0.0005 -0.0004 -0.0003 -0.0003 -0.0003 -0.0002
$V_{KI(3)}$ -0.0002 -0.0002 -0.0002 -0.0002 -0.0002 -0.0002 -0.0002 -0.0001

Table 3: Numerical Results of down and in call options with, $B = 1.14$

The accuracy of approximated values is changed by the parameter sets. The larger $\sigma^j_0$, $\nu^j$
mean larger $\epsilon$, and large $\epsilon$ and $T$ make the error to be high (see Proposition 2.4). In particular, $\nu$ and $T$ have a large impact on the accuracy. The effect of $\nu$ is observed by comparing Cases (C) and (E). Since we approximate the distribution of the underlying asset price by Gaussian distribution, the difference of the distributions is large if $\nu$ is large. The parameter $\nu$ mainly expresses the curvature of the volatility smile, and thus our approximation is accurate if the curvature of the smile is small. On the other hand, the error of long maturity is large even if $\nu$ is small (c.f., Cases (E) and (F)). Since both the error terms $c_1$ and $c_2$ in Proposition 2.4 depend on $T$, our approximation works well in short maturity cases.

In general, while the results of both $V_{KI(3)}$ and $\bar{V}_{KI(3)}$ are well when the prices need
to be evaluated in a short time even in the multi-asset case (Case (G)), the errors in a few stressed cases of $V_{KI(3)}$ (e.g., Cases (C) and (D) in Table 2) are large. However, the errors of $\bar{V}_{KI(3)}$ are stable and smaller than those of $V_{KI(3)}$ in many cases. In particular, all of the adjusted results in Table 2 are improved. Since the distribution of LSV model is different from Gaussian distribution at the tail area, the adjustment works efficiently if the barrier price is far from at the money.
4 Acknowledgments

This research is supported by CARF (Center for Advanced Research in Finance).

References


