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Abstract

In this paper, we study a class of Anticipated Backward Stochastic Differential Equations (ABSDE) with jumps. The solution of the ABSDE is a triple (Y, Z, ψ) where Y is a semimartingale, and (Z, ψ) are the diffusion and jump coefficients. We allow the driver of the ABSDE to have linear growth on the uniform norm of Y's future paths, as well as quadratic and exponential growth on the spot values of (Z, ψ), respectively. The existence of the unique solution is proved for Markovian and non-Markovian settings with different structural assumptions on the driver. In the former case, some regularities on (Z, ψ) with respect to the forward process are also obtained.

Keywords : predictive mean-field type, time-advanced, quadratic growth, future path dependent driver, ABSDE

1 Introduction

As a powerful probabilistic tool to analyze general control problems, non-linear partial differential equations as well as many newly appeared financial problems, backward stochastic differential equations (BSDEs) have attracted strong research interests since the pioneering works of Bismut (1973) [5] and Pardoux & Peng (1990) [24].

Recently, Peng & Yang (2009) [26] introduced a new class, so-called anticipated (or time-advanced) BSDEs, where the drivers are dependent on the conditional expectations of the future paths of the solutions. They originally appeared as adjoint processes when dealing with optimal control problems on delayed systems. Since then various generalizations have been studied by many authors: Oksendal et al. (2011) [22] dealt with a control problem on delayed systems with jumps, Pamen (2015) [23] a stochastic differential game with delay, Xu (2011) [31], Yang & Elliott (2013) [30] studied some generalizations and conditions for the comparison principle to hold. Jeanblac et al. (2016) [14] studied anticipated BSDEs under a setting of progressive enlargement of filtration. The importance of anticipated BSDEs for financial applications is likely to grow in the coming years because of the set of new regulations (in particular, the margin rule on the independent amount). They require the financial firms to adjust the collateral (or capital) amount based on the expected future maximum loss, exposure or the variability of the mark-to-market, which naturally makes the drivers of the pricing BSDEs dependent on the expected future paths of the portfolio values.

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In this paper, we are interested in anticipated BSDEs with jumps and quadratic-exponential growth drivers. Although the properties of Lipschitz ABSDEs have been well established, the ABSDEs with quadratic growth generators have not yet appeared in the literature. We are interested in anticipated BSDEs with jumps of the following form:

$$Y_t = \xi + \int_t^T \mathbb{E}_{\mathcal{F}_r} f\Big(r, (Y_v)_{v\in[r,T]}, Y_r, Z_r, \psi_r\Big)dr - \int_t^T Z_r dW_r - \int_t^T \int_E \psi_r(e)\widetilde{\mu}(dr, de)$$

where the driver $f(t, \cdot)$ is allowed to have linear growth in $\sup_{v\in[t,T]} |Y_v|$, quadratic in Z_t and exponential growth in the jump coefficients ψ_t.

For the (non-anticipated) BSDEs with quadratic growth drivers, the first breakthrough was made by Kobylanski (2000) [18] and then followed by many authors. In the presence of jumps, in particular, they were studied by Becherer (2006) [4], Morlais (2010) [20], Ngoupeyou (2010) [21], Cohen & Elliott (2015) [6], Kazi-Tani et al. (2015) [17], Antonelli & Mancini (2016) [1], El Karoui et al. (2016) [8] and Fujii & Takahashi (2015) [12] with varying generality. An important common tool is the so called A_Γ-condition [2, 29] necessary to make the comparison principle to hold in the presence of jumps, which is then used to create a monotone sequence of regularized BSDEs.

Although A_Γ-condition is known to hold for the setting of exponential utility optimization [20], it is rather restrictive, and in fact, stronger than the local Lipschitz continuity. Furthermore, in the anticipated settings, the comparison principle does not hold generally even when the A_Γ-condition is satisfied. Although the fixed point approach [6, 17] does not rely on the comparison principle at least for small terminal values, it requires the second-order differentiability of the driver which is difficult to establish in the presence of the general path dependence.

In this paper, we firstly extend the quadratic-exponential structure condition of [3, 8] to allow the dependence on Y's future paths, and then derive the universal bounds on (Y, Z, ψ) under a general bounded terminal condition. This bounds are then used to prove a stability result under a general non-Markovian setting. Under the Markovian setting, this stability result leads to the compactness result for the deterministic map defined by $u(t, x) = Y_t^{t,x}$, which then allows us to prove the existence of the solution in the absence of the A_Γ-condition. It also provides some regularities on (Z, ψ) with respect to the forward process. As a by product, it makes the A_Γ-condition unnecessary for the existence, uniqueness and Malliavin's differentiability of quadratic-exponential growth (non-anticipated) BSDEs under the Markovian setting studied in Section 6 of [12]. For a non-Markovian setting, we reintroduce the A_Γ-condition and make use of our previous result in [12] to prove the existence of the unique solution. We also give a sufficient condition for the comparison principle to hold.

2 Preliminaries

2.1 General Setting

Let us first state the general setting to be used throughout the paper. $T > 0$ is some bounded time horizon. The space $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$ is the usual canonical space for a d-dimensional Brownian motion equipped with the Wiener measure $\mathbb{P}_W$. We also denote $(\Omega_\mu, \mathcal{F}_\mu, \mathbb{P}_\mu)$ as a product of canonical spaces $\Omega_\mu := \Omega_\mu^1 \times \cdots \times \Omega_\mu^k$, $\mathcal{F}_\mu := \mathcal{F}_\mu^1 \times \cdots \times \mathcal{F}_\mu^k$ and $\mathbb{P}_\mu^1 \times \cdots \times \mathbb{P}_\mu^k$ with some constant $k \in \mathbb{N}$, on which each μ^i is a Poisson measure with a compensator $\nu^i(de)dt$. Here, $\nu^i(de)$ is a σ-finite measure on $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$ satisfying $\int_{\mathbb{R}_0} |e|^2\nu^i(de) < \infty$. For notational simplicity, we write $(E, \mathcal{E}) := (\mathbb{R}_0^k, \mathcal{B}(\mathbb{R}_0)^k)$. Throughout the paper, we work on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, where the space $(\Omega, \mathcal{F}, \mathbb{P})$ is the product of the canonical spaces $(\Omega_W \times \Omega_\mu, \mathcal{F}_W \times \mathcal{F}_\mu, \mathbb{P}_W \times \mathbb{P}_\mu)$, and that the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is the canonical filtration completed for $\mathbb{P}$ and satisfying the usual conditions. In this construction, $(W, \mu^1, \cdots, \mu^k)$ are independent. We use a vector notation $\mu(\omega, dt, de) := (\mu^1(\omega, dt, de^1), \cdots, \mu^k(\omega, dt, de^k))$ and denote the compensated Poisson measure as $\widetilde{\mu} := \mu - \nu$. $\mathbb{F}$-predictable σ-field on $\Omega \times [0, T]$ is denoted by $\mathcal{P}$. It is well-known that the weak property of predictable representation holds in this setup (see for example [13] chapter XIII).

2.2 Notation

We denote a generic constant by C which may change line by line. We write $C = C(a, b, c, \cdots)$ when the constant depends only on the parameters $(a, b, c, \cdots)$. $\mathcal{T}_s^t$ denotes the set of $\mathbb{F}$-stopping times $\tau : \Omega \to [s, t]$. We denote the conditional expectation with respect to $\mathcal{F}_t$ by $\mathbb{E}_{\mathcal{F}_t}[\cdot]$ or $\mathbb{E}[\cdot|\mathcal{F}_t]$. Under a probability measure $\mathbb{Q}$ different from $\mathbb{P}$, we explicitly denote it, for example, by $\mathbb{E}^{\mathbb{Q}}_{\mathcal{F}_t}[\cdot]$. Sometimes we use the abbreviations $||x||_{[s,t]} := \sup_{v\in[s,t]} |x_v|$ and $\Theta_v := (Y_v, Z_v, \psi_v)$.

For $(p \geq 2)$, we introduce the following spaces.

• $\mathbb{S}^p[s, t]$ is the set of real (or vector) valued càdlàg $\mathbb{F}$-adapted processes $(X_v)_{v\in[s,t]}$ such that

$$||X||_{\mathbb{S}^p[s,t]} := \mathbb{E}\Big[\sup_{v\in[s,t]} |X_v|^p\Big]^{\frac{1}{p}} < \infty.$$

• $\mathbb{S}^\infty[s, t]$ is the set of real (or vector) valued càdlàg $\mathbb{F}$-adapted processes $(X_v)_{v\in[s,t]}$ which are essentially bounded, i.e.

$$||X||_{\mathbb{S}^\infty[s,t]} := \Big|\Big|\sup_{v\in[s,t]} |X_v|\Big|\Big|_\infty < \infty.$$

Here, $||x||_\infty := \inf\big\{c \in \mathbb{R} \ ; \ \mathbb{P}(\{|x| \leq c\}) = 1\big\}$.

• $\mathbb{H}^p[s, t]$ is the set of progressively measurable real (or vector) valued processes $(Z_v)_{v\in[s,t]}$ such that

$$||Z||_{\mathbb{H}^p[s,t]} := \mathbb{E}\Big[\Big(\int_s^t |Z_v|^2 dv\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} < \infty.$$

• $\mathbb{L}^2(E, \nu)$ (or simply $\mathbb{L}^2(\nu)$) is the set of k-dimensional vector-valued functions $\psi = (\psi^i)_{1\leq i\leq k}$ for which the each component $\psi^i : \mathbb{R}_0 \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R}^0)$-measurable and

$$||\psi||_{\mathbb{L}^2(E,\nu)} := \Big(\sum_{i=1}^k \int_{\mathbb{R}_0} |\psi^i(e)|^2 \nu^i(de)\Big)^{\frac{1}{2}} < \infty.$$

• $\mathbb{L}^\infty(E, \nu)$ (or simply $\mathbb{L}^\infty(\nu)$) is the set of functions $\psi = (\psi^i)_{1\leq i\leq k}$ for which the each component $\psi^i : \mathbb{R}_0 \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R}^0)$-measurable and bounded $\nu^i(de)$-a.e. with the standard essential supremum norm.

• $\mathbb{J}^p[s, t]$ is the set of functions $\psi = (\psi^i)_{1\leq i\leq k}$ with $\psi^i : \Omega \times [s, t] \times \mathbb{R}_0 \to \mathbb{R}$ being $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_0)$-measurable (or we simply say ψ is $\mathcal{P} \otimes \mathcal{E}$-measurable) and satisfy

$$||\psi||_{\mathbb{J}^p[s,t]} := \mathbb{E}\Big[\Big(\sum_{i=1}^k \int_s^t \int_{\mathbb{R}_0} |\psi^i_v(e)|^2 \nu^i(de)dv\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} < \infty.$$

• $\mathbb{J}^\infty[s, t]$ is the set of $\mathcal{P} \otimes \mathcal{E}$-measurable functions $\psi = (\psi^i)_{1\leq i\leq k}$ such that

$$||\psi||_{\mathbb{J}^\infty[s,t]} := \Big|\Big|\sup_{v\in[s,t]} ||\psi_v||_{\mathbb{L}^\infty(E,\nu)}\Big|\Big|_\infty < \infty,$$

i.e. essentially bounded.

For notational simplicity, hereafter we write

$$\int_s^t \int_E \psi_r(e)\widetilde{\mu}(de,dr) := \sum_{i=1}^k \int_s^t \int_{\mathbb{R}_0} \psi_r^i(e)\widetilde{\mu}^i(dr,de)$$

and use similar abbreviations for the integrations with respect to $(\mu,\nu) = (\mu^i,\nu^i)_{1\le i\le k}$.

• We denote $\mathcal{K}^p[s,t] = \mathbb{S}^p[s,t] \times \mathbb{H}^p[s,t] \times \mathbb{J}^p[s,t]$ with the norm

$$||(Y,Z,\psi)||_{\mathcal{K}^p[s,t]} := ||Y||_{\mathbb{S}^p[s,t]} + ||Z||_{\mathbb{H}^p[s,t]} + ||\psi||_{\mathbb{J}^p[s,t]}.$$

• $\mathbb{H}^2_{BMO}[s,t]$ is the set of real (or vector) valued progressively measurable processes $(Z_v)_{v\in[s,t]}$ such that

$$||Z||^2_{\mathbb{H}^2_{BMO}[s,t]} := \sup_{\tau\in\mathcal{T}_s^t} \Big|\Big|\mathbb{E}_{\mathcal{F}_\tau}\Big[\int_\tau^t |Z_r|^2 dr\Big]\Big|\Big|_\infty < \infty.$$

• $\mathbb{J}^2_B[s,t]$ is the set of $\mathcal{P}\otimes\mathcal{E}$-measurable functions such that

$$||\psi||^2_{\mathbb{J}^2_B[s,t]} := \sup_{\tau\in\mathcal{T}_s^t} \Big|\Big|\mathbb{E}_{\mathcal{F}_\tau}\Big[\int_\tau^t \int_E |\psi_r(e)|^2\nu(de)dr\Big]\Big|\Big|_\infty < \infty.$$

• $\mathbb{J}^2_{BMO}[s,t]$ is the set of $\mathcal{P}\otimes\mathcal{E}$-measurable functions such that

$$||\psi||^2_{\mathbb{J}^2_{BMO}[s,t]} := \sup_{\tau\in\mathcal{T}_s^t} \Big|\Big|\mathbb{E}_{\mathcal{F}_\tau}\Big[\int_\tau^t \int_E |\psi_r(e)|^2\nu(de)dr\Big] + (\Delta M_\tau)^2\Big|\Big|_\infty < \infty,$$

where $\Delta M_\tau := \int_E \psi_\tau(e)\mu(\{\tau\},de)$. Note that we have

$$(||\psi||^2_{\mathbb{J}^2_B[s,t]} \vee ||\psi||^2_{\mathbb{J}^\infty[s,t]}) \le ||\psi||^2_{\mathbb{J}^2_{BMO}[s,t]} \le ||\psi||^2_{\mathbb{J}^2_B[s,t]} + ||\psi||^2_{\mathbb{J}^\infty[s,t]}.$$

• $\mathbb{D}[s,t]$ is the set of real valued càdlàg functions $(q_v)_{v\in[s,t]}$.

We frequently omit $[s,t]$ if it is obvious from the context.

3 A priori estimates

3.1 Universal bounds

In this section, we consider various a priori estimates regarding anticipated quadratic-exponential growth BSDEs with jumps in a general non-Markovian setup. We are interested in the following ABSDE for $t\in[0,T]$:

$$Y_t = \xi + \int_t^T \mathbb{E}_{\mathcal{F}_r} f\Big(r,(Y_v)_{v\in[r,T]},Y_r,Z_r,\psi_r\Big)dr - \int_t^T Z_r dW_r - \int_t^T \int_E \psi_r(e)\widetilde{\mu}(dr,de), \quad (3.1)$$

where $f:\Omega\times[0,T]\times\mathbb{D}[0,T]\times\mathbb{R}\times\mathbb{R}^{1\times d}\times\mathbb{L}^2(E,\nu)\to\mathbb{R}$, and ξ is an $\mathcal{F}_T$-measurable random variable.

Assumption 3.1. *(i) The driver f is a map such that for every $(y,z,\psi)\in\mathbb{R}\times\mathbb{R}^{1\times d}\times\mathbb{L}^2(E,\nu)$ and any càdlàg $\mathbb{F}$-adapted process $(Y_v)_{v\in[0,T]}$, the process $\big(\mathbb{E}_{\mathcal{F}_t} f(t,(Y_v)_{v\in[t,T]},y,z,\psi), t\in[0,T]\big)$ is $\mathbb{F}$-progressively measurable, and the map $(y,z,\psi)\to f(\cdot,y,z,\psi)$ is continuous.*
(ii) For every $(q,y,z,\psi)\in\mathbb{D}[0,T]\times\mathbb{R}\times\mathbb{R}^{1\times d}\times\mathbb{L}^2(E,\nu)$, there exist constants $\beta,\delta\ge 0$, $\gamma>0$

and a positive progressively measurable process $(l_v, v \in [0,T])$ such that

$$\begin{aligned} -l_t - \delta\big(\sup_{v\in[t,T]}|q_v|\big) - \beta|y| - \frac{\gamma}{2}|z|^2 - \int_E j_\gamma(-\psi(e))\nu(de) \ &\le \ f\big(t,(q_v)_{v\in[t,T]},y,z,\psi\big) \\ \le l_t + \delta\big(\sup_{v\in[t,T]}|q_v|\big) + \beta|y| + \frac{\gamma}{2}|z|^2 + \int_E j_\gamma(\psi(e))\nu(de) \end{aligned}$$

$d\mathbb{P}\otimes dt$-a.e. $(\omega,t)\in\Omega\times[0,T]$, where $j_\gamma(u) := \frac{1}{\gamma}(e^{\gamma u}-1-\gamma u)$.
(iii) $||\xi||_\infty, ||l||_{\mathbb{S}^\infty} < \infty$.

Lemma 3.1. *Under Assumption 3.1, if there exists a bounded solution $(Y,Z,\psi)\in\mathbb{S}^\infty\times\mathbb{H}^2\times\mathbb{J}^2$ to the ABSDE (3.1), then $Z\in\mathbb{H}^2_{BMO}$ and $\psi\in\mathbb{J}^2_{BMO}$ (hence $\psi\in\mathbb{J}^\infty$) and they satisfy*

$$\begin{aligned} ||Z||^2_{\mathbb{H}^2_{BMO}} &\le \frac{e^{4\gamma||Y||_{\mathbb{S}^\infty}}}{\gamma^2}\Big(1+2\gamma T\big[||l||_{\mathbb{S}^\infty}+(\beta+\delta)||Y||_{\mathbb{S}^\infty}\big]\Big), \\ ||\psi||^2_{\mathbb{J}^2_{BMO}} &\le \frac{e^{4\gamma||Y||_{\mathbb{S}^\infty}}}{\gamma^2}\Big(2+4\gamma T\big[||l||_{\mathbb{S}^\infty}+(\beta+\delta)||Y||_{\mathbb{S}^\infty}\big]\Big)+4||Y||^2_{\mathbb{S}^\infty}. \end{aligned}$$

Proof. It follows from Lemma 3.1 [12] by a simple replacement $||l||_{\mathbb{S}^\infty}$ with $||l||_{\mathbb{S}^\infty}+\delta||Y||_{\mathbb{S}^\infty}$. One also needs the fact that $||\psi||^2_{\mathbb{J}^2_{BMO}} \le ||\psi||^2_{\mathbb{J}^2_B} + ||\psi||^2_{\mathbb{J}^\infty}$ and $||\psi||_{\mathbb{J}^\infty}\le 2||Y||_{\mathbb{S}^\infty}$. □

Lemma 3.2. *Under Assumption 3.1, if there exists a bounded solution $(Y,Z,\psi)\in\mathbb{S}^\infty\times\mathbb{H}^2\times\mathbb{J}^2$ to the ABSDE (3.1), then Y has the following estimate*

$$||Y||_{\mathbb{S}^\infty} \le \exp\Big(T\big(\beta+\delta e^{\beta T}\big)\Big)\big(||\xi||_\infty + T||l||_{\mathbb{S}^\infty}\big)\ .$$

Proof. Applying Mayer-Ito formula, one obtains

$$d(e^{\beta s}|Y_s|) \quad = \quad e^{\beta s}\Big(\beta|Y_s|ds + \mathrm{sign}(Y_{s-})dY_s + dL_s\Big).$$

Here, $(L_s)_{s\in[0,T]}$ is a non-decreasing process including a local time L^c as

$$dL_s = dL^c_s + \int_E\Big(|Y_{s-}+\psi_s(e)| - |Y_{s-}| - \mathrm{sign}(Y_{s-})\psi_s(e)\Big)\mu(ds,de)\ .$$

Note that $|y+\psi|-|y|-\mathrm{sign}(y)\psi = |y+\psi|-\mathrm{sign}(y)(y+\psi)\ge 0$. Let us introduce the following non-decreasing processes $(B_s)_{s\in[0,T]}$ and $(C_s)_{s\in[0,T]}$ by

$$\begin{aligned} dB_s &= -\mathrm{sign}(Y_s)\mathbb{E}_{\mathcal{F}_s}f\big(s,(Y_v)_{v\in[s,T]},\Theta_s\big)ds \\ &\quad +\Big(l_s+\delta\mathbb{E}_{\mathcal{F}_s}\big(\sup_{v\in[s,T]}|Y_v|\big)+\beta|Y_s|+\frac{\gamma}{2}|Z_s|^2+\int_E j_\gamma(\mathrm{sign}(Y_s)\psi_s(e))\nu(de)\Big)ds\ , \\ dC_s &= e^{\beta s}(dB_s+dL_s)+\frac{\gamma}{2}(e^{2\beta s}-e^{\beta s})|Z_s|^2ds \\ &\quad +\int_E\Big(j_\gamma(e^{\beta s}\mathrm{sign}(Y_s)\psi_s(e)) - e^{\beta s}j_\gamma(\mathrm{sign}(Y_s)\psi_s(e))\Big)\nu(de)ds\ . \end{aligned}$$

Notice that for $k\ge 1$, $j_\gamma(ku)-kj_\gamma(u)=\frac{1}{\gamma}(e^{k\gamma u}-ke^{\gamma u}-1+k)\ge 0$, and thus the last line is non-decreasing. One then sees

$$d\Big(e^{\beta s}|Y_s| + \int_0^s e^{\beta r}\big(l_r + \delta\mathbb{E}_{\mathcal{F}_r}(\sup_{v\in[r,T]}|Y_v|)\big)dr\Big) = e^{\beta s}\mathrm{sign}(Y_{s-})\Big(Z_s dW_s + \int_E \psi_s(e)\widetilde{\mu}(ds,de)\Big)$$
$$-\int_E j_\gamma\big(e^{\beta s}\mathrm{sign}(Y_s)\psi_s(e)\big)\nu(de)ds - \frac{\gamma}{2}|e^{\beta s}\mathrm{sign}(Y_s)Z_s|^2 ds + dC_s\ .$$

We now investigate the process $P_t, t\in[0,T]$ defined by

$$P_t := \exp\Big(\gamma e^{\beta t}|Y_t| + \gamma\int_0^t e^{\beta r}\big(l_r + \delta\mathbb{E}_{\mathcal{F}_r}(\sup_{v\in[r,T]}|Y_v|)\big)dr\Big), \quad t\in[0,T],$$

where $P\in\mathbb{S}^\infty$ is clearly seen. Applying Ito formula, one obtains

$$\begin{aligned}
dP_t &= P_{t-}\gamma d\Big(e^{\beta t}|Y_t| + \int_0^t e^{\beta r}\big(l_r + \delta\mathbb{E}_{\mathcal{F}_r}\sup_{v\in[r,T]}|Y_v|\big)dr\Big) + P_t\frac{\gamma^2}{2}|e^{\beta t}\mathrm{sign}(Y_t)Z_t|^2 dt \\
&\quad + P_{t-}\int_E\Big(e^{\gamma e^{\beta t}(|Y_{t-}+\psi_t(e)|-|Y_{t-}|)} - 1 - \gamma e^{\beta t}\mathrm{sign}(Y_{t-})\psi_t(e)\Big)\mu(dt,de) \\
&= P_{t-}\Big(\gamma e^{\beta t}\mathrm{sign}(Y_t)Z_t dW_t + \int_E\Big(\exp\big(\gamma e^{\beta t}\mathrm{sign}(Y_{t-})\psi_t(e)\big) - 1\Big)\widetilde{\mu}(dt,de) + dC'_t\Big)\ ,
\end{aligned}$$

where $(C'_s)_{s\in[0,T]}$ is another non-decreasing process defined by

$$dC'_t = \gamma dC_t + \int_E\Big(e^{\gamma e^{\beta t}(|Y_{t-}+\psi_t(e)|-|Y_{t-}|)} - e^{\gamma e^{\beta t}\mathrm{sign}(Y_{t-})\psi_t(e)}\Big)\mu(dt,de).$$

Since $(P,Y,Z,\psi)\in\mathbb{S}^\infty\times\mathbb{S}^\infty\times\mathbb{H}^2_{BMO}\times\mathbb{J}^2_{BMO}$, one sees the process P is a true submartingale. Therefore, it follows that, for any $t\in[0,T]$,

$$\begin{aligned}
\exp\Big(\gamma e^{\beta t}|Y_t|\Big) &\le \mathbb{E}_{\mathcal{F}_t}\Big[\exp\Big(\gamma e^{\beta T}|\xi| + \gamma\int_t^T e^{\beta r}\big(l_r + \delta\mathbb{E}_{\mathcal{F}_r}(\sup_{v\in[r,T]}|Y_v|)\big)dr\Big)\Big] \\
&\le \exp\Big(\gamma e^{\beta T}(||\xi||_\infty + T||l||_{\mathbb{S}^\infty}) + \gamma\delta e^{\beta T}\int_t^T ||Y||_{\mathbb{S}^\infty[r,T]}\ dr\Big) \text{ a.s.}
\end{aligned}$$

Thus, $|Y_t| \le e^{\beta T}(||\xi||_\infty + T||l||_{\mathbb{S}^\infty}) + \delta e^{\beta T}\int_t^T ||Y||_{\mathbb{S}^\infty[r,T]}\ dr$ a.s. Since the right-hand side is non-increasing in t, the same inequality holds with the left-hand side replaced by $\sup_{s\in[t,T]}|Y_s|$. Hence equivalently,

$$||Y||_{\mathbb{S}^\infty[t,T]} \le e^{\beta T}(||\xi||_\infty + T||l||_{\mathbb{S}^\infty}) + \delta e^{\beta T}\int_t^T ||Y||_{\mathbb{S}^\infty[r,T]}\ dr\ .$$

Now the backward Gronwall inequality (see, for example, Corollary 6.61 [25]), one obtains the desired result. □

As a result of Lemmas 3.1 and 3.2, one sees the norms of $||Y||_{\mathbb{S}^\infty}, ||Z||_{\mathbb{H}^2_{BMO}}, ||\psi||_{\mathbb{J}^2_{BMO}}$ are solely controlled by the set of parameters $A := (||\xi||_\infty, ||l||_{\mathbb{S}^\infty}, \delta, \beta, \gamma, T)$. In the next subsection, we introduce the local Lipschitz continuity.

3.2 Stability and Uniqueness

Assumption 3.2. *For each $M>0$, and for every (q,y,z,ψ), $(q',y',z',\psi')\in\mathbb{D}[0,T]\times\mathbb{R}\times\mathbb{R}^{1\times d}\times\mathbb{L}^2(E,\nu)$ satisfying $\sup_{v\in[0,T]}|q_v|, \sup_{v\in[0,T]}|q'_v|,\ |y|,\ |y'|,\ ||\psi||_{\mathbb{L}^\infty(\nu)},\ ||\psi'||_{\mathbb{L}^\infty(\nu)}\leq M$, there exists some positive constant K_M (depending on M) such that*

$$\begin{aligned}&|f(t,(q_v)_{v\in[t,T]},y,z,\psi)-f(t,(q'_v)_{v\in[t,T]},y',z',\psi')|\\&\quad\leq K_M\Big(\sup_{v\in[t,T]}|q_v-q'_v|+|y-y'|+||\psi-\psi'||_{\mathbb{L}^2(\nu)}\Big)\\&\quad+K_M\big(1+|z|+|z'|+||\psi||_{\mathbb{L}^2(\nu)}+||\psi'||_{\mathbb{L}^2(\nu)}\big)|z-z'|\end{aligned}\tag{3.2}$$

$d\mathbb{P}\otimes dt$-a.e. $(\omega,t)\in\Omega\times[0,T]$.

Let us introduce the two ABSDEs for $t\in[0,T]$, with $i=\{1,2\}$,

$$Y^i_t=\xi^i+\int_t^T\mathbb{E}_{\mathcal{F}_r}f^i\Big(r,(Y^i_v)_{v\in[r,T]},Y^i_r,Z^i_r,\psi^i_r\Big)dr-\int_t^T Z^i_r dW_r-\int_t^T\int_E\psi^i_r(e)\widetilde{\mu}(dr,de).\tag{3.3}$$

Let us put $\delta Y:=Y^1-Y^2,\delta Z:=Z^1-Z^2,\delta\psi:=\psi^1-\psi^2$, and

$$\delta f(r):=(f^1-f^2)\big(r,(Y^1_v)_{v\in[r,T]},Y^1_r,Z^1_r,\psi^1_r\big)\ .$$

Then, we have the following stability result.

Proposition 3.1. *Suppose that the data $(\xi^i,f^i)_{1\leq i\leq 2}$ satisfy Assumptions 3.1 and 3.2. If the two ABSDEs (3.3) have bounded solutions $(Y^i,Z^i,\psi^i)_{1\leq i\leq 2}\in\mathbb{S}^\infty\times\mathbb{H}^2\times\mathbb{J}^2$, then for any $p>2q_*$*

$$||\delta Y||_{\mathbb{S}^p[0,T]}\leq C_1\mathbb{E}\Big[|\delta\xi|^p+\Big(\int_0^T\mathbb{E}_{\mathcal{F}_r}|\delta f(r)|dr\Big)^p\Big]^{\frac{1}{p}}\tag{3.4}$$

and for any $p\geq 2,\bar{q}\geq q_*$

$$\big|\big|(\delta Y,\delta Z,\delta\psi)\big|\big|_{\mathcal{K}^p[0,T]}\leq C_2\mathbb{E}\Big[|\delta\xi|^{p\bar{q}^2}+\Big(\int_0^T\mathbb{E}_{\mathcal{F}_r}|\delta f(r)|dr\Big)^{p\bar{q}^2}\Big]^{\frac{1}{p\bar{q}^2}}\tag{3.5}$$

where $q_*\in(1,\infty)$ is a constant depending only on $(K.,A)$, $C_1=C(p,K.,A)$ and $C_2=(p,\bar{q},K.,A)$ are two positive constants.

Proof. Note that one can apply (3.2) globally with fixed K_M by choosing M larger than the bounds implied from Lemmas 3.1 and 3.2. Let fix such an M in the reminder. Define the $\mathbb{R}^d$-valued progressively measurable process $(b_r,r\in[0,T])$ by

$$b_r:=\frac{\mathbb{E}_{\mathcal{F}_r}\big[f^2(r,(Y^1_v)_{v\in[r,T]},Y^1_r,Z^1_r,\psi^1_r)-f^2(r,(Y^1_v)_{v\in[r,T]},Y^1_r,Z^2_r,\psi^1_r)\big]}{|\delta Z_r|^2}\mathbf{1}_{\delta Z_r\neq 0}\delta Z_r^\top\ .$$

Since $|b_r|\leq K_M\big(1+|Z^1_r|+|Z^2_r|+2||\psi^1_r||^2_{\mathbb{L}^2(\nu)}\big)$, there exists some constant C such that $||b||_{\mathbb{H}^2_{BMO}}\leq C$ with $C=C(K.,A)$. Thus one can define an equivalent probability measure $\mathbb{Q}$ by $\frac{d\mathbb{Q}}{d\mathbb{P}}=\mathcal{E}_T\big(\int_0^\cdot b_r^\top dW_r\big)$ where $\mathcal{E}(\cdot)$ is Doléans-Dade exponential. We have $W^{\mathbb{Q}}=W-\int_0^\cdot b_r dr$ and the Poisson measure is unchanged, $\widetilde{\mu}^{\mathbb{Q}}=\widetilde{\mu}$. We also have $\frac{d\mathbb{P}}{d\mathbb{Q}}=\mathcal{E}_T\big(-\int_0^\cdot b_r^\top dW^{\mathbb{Q}}_r\big)$. From Remark A.1, there exists some constant $r^*\in(1,\infty)$ such that the reverse Hölder inequality holds for both of the $\mathcal{E}.(\int_0^\cdot b_r^\top dW_r)$ and $\mathcal{E}.(-\int_0^\cdot b_r^\top dW^{\mathbb{Q}}_r)$ with power $\bar{r}\in(1,r^*]$. Define $q_*>1$ by $q_*:=r^*/(r^*-1)$. Note that (r^*,q_*) are solely controlled by $(K.,A)$.

Under the measure $\mathbb{Q}$, we have

$$\delta Y_t = \delta\xi + \int_t^T \mathbb{E}_{\mathcal{F}_r}\Big[\delta f(r) + f^2\big(r,(Y_v^1)_{v\in[r,T]}, Y_r^1, Z_r^2, \psi_r^1\big) - f^2\big(r,(Y_v^2)_{v\in[r,T]}, Y_r^2, Z_r^2, \psi_r^2\big)\Big]dr$$
$$- \int_t^T \delta Z_r dW_r^{\mathbb{Q}} - \int_t^T\int_E \delta\psi_r(e)\widetilde{\mu}^{\mathbb{Q}}(dr,de),\ t\in[0,T].$$

[Stability for Y] Applying Ito formula to δY^2, one obtains

$$|\delta Y_t|^2 + \int_t^T |\delta Z_r|^2 dr + \int_t^T\int_E |\delta\psi_r(e)|^2\mu(dr,de)$$
$$= |\delta\xi|^2 + \int_t^T 2\delta Y_r \mathbb{E}_{\mathcal{F}_r}\Big[\delta f(r) + f^2\big(r,(Y_v^1)_{v\in[r,T]}, Y_r^1, Z_r^2, \psi_r^1\big) - f^2\big(r,(Y_v^2)_{v\in[r,T]}, Y_r^2, Z_r^2, \psi_r^2\big)\Big]dr$$
$$- \int_t^T 2\delta Y_r \delta Z_r dW_r^{\mathbb{Q}} - \int_t^T\int_E 2\delta Y_{r-}\delta\psi_r(e)\widetilde{\mu}^{\mathbb{Q}}(dr,de)\ . \tag{3.6}$$

The last two terms are true $\mathbb{Q}$-martingales, which can be checked by reverse Hölder and energy inequalities. By taking conditional expectation $\mathbb{E}^{\mathbb{Q}}_{\mathcal{F}_t}[\cdot]$, one obtains with any $\lambda > 0$

$$|\delta Y_t|^2 + \mathbb{E}^{\mathbb{Q}}_{\mathcal{F}_t}\int_t^T |Z_r|^2 dr + \mathbb{E}^{\mathbb{Q}}_{\mathcal{F}_t}\int_t^T ||\delta\psi_r||^2_{\mathbb{L}^2(\nu)}dr \le C\mathbb{E}^{\mathbb{Q}}_{\mathcal{F}_t}\int_t^T \mathbb{E}_{\mathcal{F}_r}\big[||\delta Y||^2_{[r,T]}\big]dr$$
$$+\mathbb{E}^{\mathbb{Q}}_{\mathcal{F}_t}\Big[|\delta\xi|^2 + \frac{1}{\lambda}\Big(\int_t^T \mathbb{E}_{\mathcal{F}_r}|\delta f(r)|dr\Big)^2 + \lambda||\delta Y||^2_{[t,T]}\Big] + \frac{1}{2}\mathbb{E}^{\mathbb{Q}}_{\mathcal{F}_t}\int_t^T ||\delta\psi_r||^2_{\mathbb{L}^2(\nu)}dr$$

with some positive constant $C = C(K_\cdot, A)$. Here we have used the fact that $|\delta Y_r| \le \mathbb{E}_{\mathcal{F}_r}\big[||\delta Y||_{[r,T]}\big]$. Therefore, in particular,

$$|\delta Y_t|^2 \le \mathbb{E}^{\mathbb{Q}}_{\mathcal{F}_t}\Big[|\delta\xi|^2 + \frac{1}{\lambda}\Big(\int_t^T \mathbb{E}_{\mathcal{F}_r}|\delta f(r)|dr\Big)^2 + \lambda||\delta Y||^2_{[t,T]} + C\int_t^T \mathbb{E}_{\mathcal{F}_r}\big[||\delta Y||^2_{[r,T]}\big]dr\Big]$$
$$= \frac{1}{\mathcal{E}_t}\mathbb{E}_{\mathcal{F}_t}\Big[\mathcal{E}_T\Big(|\delta\xi|^2 + \frac{1}{\lambda}\Big(\int_t^T \mathbb{E}_{\mathcal{F}_r}|\delta f(r)|dr\Big)^2 + \lambda||\delta Y||^2_{[t,T]} + C\int_t^T \mathbb{E}_{\mathcal{F}_r}\big[||\delta Y||^2_{[r,T]}\big]dr\Big)\Big]$$

where $\mathcal{E}_s := \mathcal{E}_s(\int_0^\cdot b_r^\top dW_r)$. Choosing $\bar{q} \in [q_*, \infty)$, the reverse Hölder inequality yields

$$|\delta Y_t|^{2\bar{q}} \le C\mathbb{E}_{\mathcal{F}_t}\Big[|\delta\xi|^{2\bar{q}} + \frac{1}{\lambda^{\bar{q}}}\Big(\int_t^T \mathbb{E}_{\mathcal{F}_r}|\delta f(r)|dr\Big)^{2\bar{q}} + \Big(\int_t^T \mathbb{E}_{\mathcal{F}_r}\big[||\delta Y||^2_{[r,T]}\big]dr\Big)^{\bar{q}} + \lambda^{\bar{q}}||\delta Y||^{2\bar{q}}_{[t,T]}\Big]$$
$$\le C\mathbb{E}_{\mathcal{F}_t}\Big[|\delta\xi|^{2\bar{q}} + \frac{1}{\lambda^{\bar{q}}}\Big(\int_t^T \mathbb{E}_{\mathcal{F}_r}|\delta f(r)|dr\Big)^{2\bar{q}} + \int_t^T ||\delta Y||^{2\bar{q}}_{[r,T]}dr + \lambda^{\bar{q}}||\delta Y||^{2\bar{q}}_{[t,T]}\Big]$$

with some $C = C(\bar{q}, K_\cdot, A)$, where in the 2nd line Jensen's inequality was used. For any $p > 2\bar{q}$, applying Doob's maximal inequality, one obtains

$$\mathbb{E}\Big[||\delta Y||^p_{[s,T]}\Big] \le C\mathbb{E}\Big[|\delta\xi|^p + \frac{1}{\lambda^{\frac{p}{2}}}\Big(\int_s^T \mathbb{E}_{\mathcal{F}_r}|\delta f(r)|dr\Big)^p\Big] + C\int_s^T \mathbb{E}\Big[||\delta Y||^p_{[r,T]}\Big]dr + C\lambda^{\frac{p}{2}}\mathbb{E}\Big[||\delta Y||^p_{[s,T]}\Big]$$

with $C = C(p, \bar{q}, K_\cdot, A)$. Choosing $\lambda > 0$ small enough so that $C\lambda^{\frac{p}{2}} < 1$, the backward Gronwall inequality implies

$$\mathbb{E}\Big[\sup_{t\in[s,T]} |\delta Y_t|^p\Big] \le C\mathbb{E}\Big[|\delta\xi|^p + \Big(\int_s^T \mathbb{E}_{\mathcal{F}_r}|\delta f(r)|dr\Big)^p\Big],\quad \forall s\in[0,T]\ .$$

One sees the last inequality holds for any $p > 2q_*$. This proves (3.4). Since $1 < q_* \leq \bar{q}$, it also follows that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|\delta Y_t|^p\Big]^{\frac{1}{p}} \leq \mathbb{E}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{p\bar{q}^2}\Big]^{\frac{1}{p\bar{q}^2}} \leq C\mathbb{E}\Big[|\delta\xi|^{p\bar{q}^2} + \Big(\int_0^T \mathbb{E}_{\mathcal{F}_r}|\delta f(r)|dr\Big)^{p\bar{q}^2}\Big]^{\frac{1}{p\bar{q}^2}} \tag{3.7}$$

with $C = C(p, \bar{q}, K., A)$ for any $p \geq 2$.

[Stability for Z and ψ] From (3.6), one has with $C = C(K., A)$,

$$\begin{aligned}
&|\delta Y_t|^2 + \int_t^T |\delta Z_r|^2 dr + \int_t^T \int_E |\delta\psi_r(e)|^2 \mu(dr, de) \\
&\leq |\delta\xi|^2 + \Big(\int_t^T \mathbb{E}_{\mathcal{F}_r}|\delta f(r)|dr\Big)^2 + ||\delta Y||^2_{[t,T]} + C\int_t^T \mathbb{E}_{\mathcal{F}_r}\big[||\delta Y||^2_{[r,T]}\big]dr \\
&\quad + C\int_t^T |\delta Y_r|||\delta\psi_r||_{\mathbb{L}^2(\nu)}dr - \int_t^T 2\delta Y_r \delta Z_r dW_r^{\mathbb{Q}} - \int_t^T \int_E 2\delta Y_{r-}\delta\psi_r(e)\widetilde{\mu}^{\mathbb{Q}}(dr, de) .
\end{aligned}$$

For any $p \geq 2$, applying Burkholder-Davis-Gundy inequality[1] and Lemma A.3, one can show that there exists some constant $C = C(p, K., A)$ such that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_0^T |\delta Z_r|^2 dr\Big)^{\frac{p}{2}}\Big] + \mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_0^T \int_E |\delta\psi_r(e)|^2 \mu(dr, de)\Big)^{\frac{p}{2}}\Big] \\
&\leq C\mathbb{E}^{\mathbb{Q}}\Big[|\delta\xi|^p + \Big(\int_0^T \mathbb{E}_{\mathcal{F}_r}|\delta f(r)|dr\Big)^p + \sup_{r\in[0,T]} \mathbb{E}_{\mathcal{F}_r}\big[||\delta Y||^p_{[0,T]}\big] + ||\delta Y||^p_{[0,T]}\Big] .
\end{aligned}$$

Taking $\bar{q} \geq q_*$, the reverse Hölder and Doob's maximal inequalities give

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_0^T |\delta Z_r|^2 dr\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} + \mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_0^T \int_E |\delta\psi_r(e)|^2 \mu(dr, de)\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} \\
&\leq C\mathbb{E}\Big[|\delta\xi|^{p\bar{q}} + \Big(\int_0^T \mathbb{E}_{\mathcal{F}_r}|\delta f(r)|dr\Big)^{p\bar{q}} + \sup_{r\in[0,T]} \mathbb{E}_{\mathcal{F}_r}\big[||\delta Y||^p_{[0,T]}\big]^{\bar{q}} + ||\delta Y||^{p\bar{q}}_{[0,T]}\Big]^{\frac{1}{p\bar{q}}} \\
&\leq C\mathbb{E}\Big[|\delta\xi|^{p\bar{q}} + \Big(\int_0^T \mathbb{E}_{\mathcal{F}_r}|\delta f(r)|dr\Big)^{p\bar{q}} + ||\delta Y||^{p\bar{q}}_{[0,T]}\Big]^{\frac{1}{p\bar{q}}}.
\end{aligned}$$

The reverse Hölder inequality implies $||Z||_{\mathbb{H}^p} + ||\psi||_{\mathbb{J}^P} \leq C\big(||Z||_{\mathbb{H}^{p\bar{q}}(\mathbb{Q})} + ||\psi||_{\mathbb{J}^{p\bar{q}}(\mathbb{Q})}\big)$. Thus the estimate of (3.7) and Lemma A.3 give

$$||\delta Y||_{\mathbb{S}^p} + ||\delta Z||_{\mathbb{H}^p} + ||\delta\psi||_{\mathbb{J}^p} \leq C\mathbb{E}\Big[|\delta\xi|^{p\bar{q}^2} + \Big(\int_0^T \mathbb{E}_{\mathcal{F}_r}|\delta f(r)|dr\Big)^{p\bar{q}^2}\Big]^{\frac{1}{p\bar{q}^2}}$$

for any $p \geq 2$ and $\bar{q} \geq q_*$ with some positive constant $C = C(p, \bar{q}, K., A)$. □

We also have the following relation.

Lemma 3.3. *Under the same conditions used in Proposition 3.1, one has*

$$||\delta Z||_{\mathbb{H}^2_{BMO}} + ||\delta\psi||_{\mathbb{J}^2_{BMO}} \leq C\Big(||\delta Y||_{\mathbb{S}^\infty} + ||\delta\xi||_\infty + \sup_{t\in\mathcal{T}_0^T}\Big|\Big|\mathbb{E}_{\mathcal{F}_t}\int_t^T |\delta f(r)|dr\Big|\Big|_\infty\Big)$$

with some positive constant $C = C(K., A)$.

[1]See, for example, Theorem 48 in IV.4. of [27].

Proof. It follows from a simple modification of Lemma 3.3 (a) of [12]. □

Combining the results in this section, we obtain the uniqueness.

Corollary 3.1. *Under Assumptions 3.1 and 3.2, if the ABSDE (3.1) has a bounded solution* $(Y, Z, \psi) \in \mathbb{S}^\infty \times \mathbb{H}^2 \times \mathbb{J}^2$, *then it is unique with respect to the norm* $\mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$.

Proof. Proposition 3.1 implies the uniqueness of Y in $\mathbb{S}^p$, $\forall p \geq 2$, in particular. This also implies the uniqueness with respect to $\mathbb{S}^\infty$. If not, there exists some $c > 0$ such that $||\delta Y||_{\mathbb{S}^\infty} = c$, which implies for any $0 < b < c$, there exists a strictly positive constant $a > 0$ such that $\mathbb{P}(\sup_{t\in[0,T]} |\delta Y_t| > b) = a$. This yields $||\delta Y||^p_{\mathbb{S}^p} > b^p a > 0$, which is a contradiction. Thus the assertion follows from Proposition 3.1 and Lemma 3.3. □

4 Existence in a Markovian Setup

Let us now provide the existence result for a Markovian setting. We introduce the following forward process, for $s \in [0, T]$,

$$X^{t,x}_s = x + \int_t^{s\vee t} b(r, X^{t,x}_r)dr + \int_t^{s\vee t} \sigma(r, X^{t,x}_r)dW_r + \int_t^{s\vee t}\int_E \gamma(r, X^{t,x}_{r-}, e)\widetilde{\mu}(dr, de) \quad (4.1)$$

where $x \in \mathbb{R}^n$ and $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$, $\gamma : [0, T] \times \mathbb{R}^n \times E \to \mathbb{R}^{n\times k}$ are non-random measurable functions. Note that $X^{t,x}_s \equiv x$ for $s \leq t$.

Assumption 4.1. *There exists a positive constant K such that*
(i) $|b(t, 0)| + |\sigma(t, 0)| \leq K$ *uniformly in* $t \in [0, T]$.
(ii) $\sum_{i=1}^k |\gamma^i(t, 0, e)| \leq K(1 \wedge |e|)$ *uniformly in* $(t, e) \in [0, T] \times \mathbb{R}_0$.
(iii) uniformly in $t \in [0, T], x, x' \in \mathbb{R}^n, e \in \mathbb{R}_0$,

$$|b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq K|x - x'|,$$
$$\sum_{i=1}^k |\gamma^i(t, x, e) - \gamma^i(t, x', e)| \leq K(1 \wedge |e|)|x - x'| \ .$$

The following estimates are standard.

Lemma 4.1. *Under Assumption 4.1, there exists a unique solution to (4.1) for each (t, x) which satisfies for any $(t, x), (t, x') \in [0, T] \times \mathbb{R}^n$ and $p \geq 2$,*

$$(a) \quad \mathbb{E}\Big[\sup_{s\in[0,T]} |X^{t,x}_s|^p\Big] \leq C(1 + |x|^p)$$
$$(b) \quad \mathbb{E}\Big[\sup_{s,u\in[0,T],|s-u|\leq h} |X^{t,x}_s - X^{t,x}_u|^p\Big] \leq C(1 + |x|^p)h$$
$$(c) \quad \mathbb{E}\Big[\sup_{s\in[0,T]} |X^{t,x}_s - X^{t',x'}_s|^p\Big] \leq C\Big(|x - x'|^p + (1 + [|x| \vee |x'|]^p)|t - t'|\Big)$$

with some constant $C = C(p, K, T)$.

We are interested in the Markovian anticipated BSDE associated with $(X^{t,x}_v)_{v\in[0,T]}$:

$$Y^{t,x}_s = \xi(X^{t,x}_T) + \int_s^T \mathbf{1}_{r\geq t}\mathbb{E}_{\mathcal{F}_r} f\Big(r, X^{t,x}_r, (Y^{t,x}_v)_{v\in[r,T]}, Y^{t,x}_r, Z^{t,x}_r, \psi^{t,x}_r\Big)dr$$
$$- \int_s^T Z^{t,x}_r dW_r - \int_s^T \int_E \psi^{t,x}_r(e)\widetilde{\mu}(dr, de) \ , \quad (4.2)$$

where $f:[0,T]\times\mathbb{R}^n\times\mathbb{D}[0,T]\times\mathbb{R}\times\mathbb{R}^{1\times d}\times\mathbb{L}^2(E,\nu)\to\mathbb{R}$ and $\xi:\mathbb{R}^n\to\mathbb{R}$ are non-random measurable functions. Note that $(Y_s^{t,x},Z_s^{t,x},\psi_s^{t,x})\equiv(Y_t^{t,x},0,0)$ for $s\le t$.

Assumption 4.2. *(i)The driver f is a map such that for every $(x,y,z,\psi)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^{1\times d}\times\mathbb{L}^2(E,\nu)$ and any càdlàg $\mathbb{F}$-adapted process $(Y_v)_{v\in[0,T]}$, the process $\big(\mathbb{E}_{\mathcal{F}_t}f(t,x,(Y_v)_{v\in[t,T]},y,z,\psi),t\in[0,T]\big)$ is $\mathbb{F}$-progressively measurable.*
(ii)For every $(x,q,y,z,\psi)\in\mathbb{R}^n\times\mathbb{D}[0,T]\times\mathbb{R}\times\mathbb{R}^{1\times d}\times\mathbb{L}^2(E,\nu)$, there exist constants $\beta,\delta\ge0$, $\gamma>0$ and a positive non-random function $l:[0,T]\to\mathbb{R}$ such that

$$\begin{aligned}-l_t-\delta\big(\sup_{v\in[t,T]}|q_v|\big)-\beta|y|-\frac{\gamma}{2}|z|^2-\int_E j_\gamma(-\psi(e))\nu(de)\ &\le\ f\big(t,x,(q_v)_{v\in[t,T]},y,z,\psi\big)\\ \le l_t+\delta\big(\sup_{v\in[t,T]}|q_v|\big)+\beta|y|+\frac{\gamma}{2}|z|^2+\int_E j_\gamma(\psi(e))\nu(de)&\end{aligned}$$

dt-a.e. $t\in[0,T]$, where $j_\gamma(u)=\frac{1}{\gamma}(e^{\gamma u}-1-\gamma u)$.
(iii) $||\xi(\cdot)||_\infty,\ \sup_{t\in[0,T]}(l_t)<\infty$.

Assumption 4.3. *(i)For each $M>0$, and for every $(x,q,y,z,\psi),(x',q',y',z',\psi')\in\mathbb{R}^n\times\mathbb{D}[0,T]\times\mathbb{R}\times\mathbb{R}^{1\times d}\times\mathbb{L}^2(E,\nu)$ satisfying $|y|,|y'|,||\psi||_{\mathbb{L}^\infty(\nu)},||\psi'||_{\mathbb{L}^\infty(\nu)},\sup_{v\in[0,T]}|q_v|,\sup_{v\in[0,T]}|q'_v|\le M$, there exist some positive constants K_M (depending on M) and $K_\xi\ge0,\rho\ge0,\ \alpha\in(0,1]$ such that*

$$\begin{aligned}&|f(t,x,(q_v)_{v\in[t,T]},y,z,\psi)-f(t,x,(q'_v)_{v\in[t,T]},y',z',\psi')|\\ &\quad\le K_M\Big(\sup_{v\in[t,T]}|q_v-q'_v|+|y-y'|+||\psi-\psi'||_{\mathbb{L}^2(\nu)}\Big)\\ &\quad+K_M\big(1+|z|+|z'|+||\psi||_{\mathbb{L}^2(\nu)}+||\psi'||_{\mathbb{L}^2(\nu)}\big)|z-z'|\ ,\\ &|f(t,x,(q_v)_{v\in[t,T]},y,z,\psi)-f(t,x',(q_v)_{v\in[t,T]},y,z,\psi)|\\ &\quad\le K_M\big(1+[|x|\vee|x'|]^\rho+|z|^2+||\psi||^2_{\mathbb{L}^2(\nu)}\big)|x-x'|^\alpha\ ,\end{aligned}$$

and $|\xi(x)-\xi(x')|\le K_\xi|x-x'|^\alpha$.

Proposition 4.1. *Under Assumptions 4.1, 4.2 and 4.3, suppose that there exists a bounded solution $(Y^{t,x},Z^{t,x},\psi^{t,x})\in\mathbb{S}^\infty\times\mathbb{H}^2\times\mathbb{J}^2$ for each $(t,x)\in[0,T]\times\mathbb{R}^n$. Then the solution is unique and $(Y^{t,x},Z^{t,x},\psi^{t,x})\in\mathbb{S}^\infty\times\mathbb{H}^2_{BMO}\times\mathbb{J}^2_{BMO}$ with the norm solely controlled by $A:=(||\xi||_\infty,sup_{t\in[0,T]}l_t,\delta,\beta,\gamma,T)$, which is, in particular, independent of $(t,x)\in[0,T]\times\mathbb{R}^n$. Moreover, the deterministic map $u:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ defined by $u(t,x):=Y_t^{t,x}$ satisfies for any pair of $(t,x),(t',x')\in[0,T]\times\mathbb{R}^n$,*

$$|u(t,x)-u(t',x')|\le C\Big(1+[|x|\vee|x'|]^\rho\Big)\Big(|x-x'|^\alpha+\big(1+[|x|\vee|x'|]^\alpha\big)|t-t'|^{\frac{1}{2p\bar q^2}}\Big)$$

with some constant $C=C(\alpha,\rho,p,\bar q,K_\xi,K,K_\cdot,A)$ for any $p\ge2$ and $\bar q\in[q_*,\infty)$ such that $\alpha p\bar q^2\ge1$, where $q_*>1$ is some constant determined by $(K_\cdot,A)$.

Proof. The first part follows from Lemmas 3.1, 3.2 and Corollary 3.1.

Let us assume $t'\le t$ without loss of any generality. Put $\delta Y:=Y^{t,x}-Y^{t',x'}$,

$$\begin{aligned}\delta f(r)\ &:=\ \mathbf{1}_{r\ge t}f(r,X_r^{t,x},(Y_v^{t,x})_{v\in[r,T]},\Theta_r^{t,x})-\mathbf{1}_{r\ge t'}f(r,X_r^{t',x'},(Y_v^{t,x})_{v\in[r,T]},\Theta_r^{t,x})\\ &=\ \mathbf{1}_{r\ge t}\Big(f(r,X_r^{t,x},(Y_v^{t,x})_{v\in[r,T]},\Theta_r^{t,x})-f(r,X_r^{t',x'},(Y_v^{t,x})_{v\in[r,T]},\Theta_r^{t,x})\Big)\\ &\quad-\mathbf{1}_{t'\le r\le t}f(r,X_r^{t',x'},(Y_v^{t,x})_{v\in[r,T]},Y_t^{t,x},0,0)\ ,\end{aligned}$$

and $\delta\xi = \xi(X_T^{t,x}) - \xi(X_T^{t',x'})$. By Proposition 3.1, for any $p \geq 2, \bar{q} \in [q_*, \infty)$,

$$\begin{aligned} |u(t,x) - u(t',x')| &\leq \mathbb{E}\Big[\sup_{s\in[0,T]} |Y_s^{t,x} - Y_s^{t',x'}|^p\Big]^{\frac{1}{p}} \\ &\leq C\mathbb{E}\Big[|\delta\xi|^{p\bar{q}^2} + \Big(\int_0^T \mathbb{E}_{\mathcal{F}_r}|\delta f(r)|dr\Big)^{p\bar{q}^2}\Big]^{\frac{1}{p\bar{q}^2}} \end{aligned} \tag{4.3}$$

with $C = C(p, \bar{q}, K_\cdot, A)$. The universal bounds of Lemmas 3.1 and 3.2 imply that $||Y^{t,x}||_{\mathbb{S}^\infty}$, $||Z^{t,x}||_{\mathbb{H}^2_{BMO}}$, $||\psi^{t,x}||_{\mathbb{J}^2_{BMO}} \leq C$ with some $C = C(A)$ uniformly in (t,x). Thus one can apply fixed K_M for the whole range provided M is chosen large enough. It follows that

$$\begin{aligned} \mathbb{E}_{\mathcal{F}_r}|\delta f(r)| &\leq \mathbf{1}_{r\geq t} K_M\Big(1 + \big[|X_r^{t,x}| \vee |X_r^{t',x'}|\big]^\rho + |Z_r^{t,x}|^2 + ||\psi_r^{t,x}||^2_{\mathbb{L}^2(\nu)}\Big)|X_r^{t,x} - X_r^{t',x'}|^\alpha \\ &+ \mathbf{1}_{t'\leq r\leq t}\big(l + \delta\mathbb{E}_{\mathcal{F}_r}[||Y^{t,x}||_{[r,T]}] + \beta|Y_t^{t,x}|\big) , \end{aligned}$$

and hence, using the boundedness of $(Y^{t,x})$,

$$\begin{aligned} &\mathbb{E}\Big[\Big(\int_0^T \mathbb{E}_{\mathcal{F}_r}|\delta f(r)|dr\Big)^{p\bar{q}^2}\Big]^{\frac{1}{p\bar{q}^2}} \\ &\leq C\mathbb{E}\left[1 + \big[||X^{t,x}||_{[0,T]} \vee ||X^{t',x'}||_{[0,T]}\big]^{2\rho p\bar{q}^2} + \Big(\int_0^T |Z_r^{t,x}|^2 + ||\psi_r^{t,x}||^2_{\mathbb{L}^2(\nu)}dr\Big)^{2p\bar{q}^2}\right]^{\frac{1}{2p\bar{q}^2}} \\ &\quad \times\mathbb{E}\Big[||X^{t,x} - X^{t',x'}||^{2\alpha p\bar{q}^2}_{[0,T]}\Big]^{\frac{1}{2p\bar{q}^2}} + C|t - t'| . \end{aligned}$$

From the energy inequality[2] for $Z^{t,x} \in \mathbb{H}^2_{BMO}$, $\psi^{t,x} \in \mathbb{J}^2_{BMO}$ and the standard continuity result of Lemma 4.1, one can show the desired result straightforwardly. The contribution from $\delta\xi$ can be computed similarly. □

Remark 4.1. *Under the conditions of the above proposition, we have, for each* $s \in [0,T]$, $Y_s^{t,x} = Y_s^{s,X_s^{t,x}} = u(s, X_s^{t,x})$ *a.s. due to the uniqueness of solution* $Y^{t,x}$*. Furthermore, since the function* u *is jointly continuous,* $u(s, X_s^{t,x})_{s\in[0,T]}$ *is càdlàg* $\mathbb{F}$*-adapted. Thus, Chapter 1, Theorem 2 of [27] implies that* $Y_s^{t,x} = u(s, X_s^{t,x})\ \forall s \in [0,T]$ *a.s.*

We now introduce a sequence of regularized anticipated BSDEs with $m \in \mathbb{N}$:

$$\begin{aligned} Y_s^{m,t,x} &= \xi(X_T^{t,x}) + \int_s^T \mathbf{1}_{r\geq t}\mathbb{E}_{\mathcal{F}_r} f_m\big(r, X_r^{t,x}, (Y_v^{m,t,x})_{v\in[r,T]}, Y_r^{m,t,x}, Z_r^{m,t,x}, \psi_r^{m,t,x}\big)dr \\ &\quad - \int_s^T Z_r^{m,t,x}dW_r - \int_s^T\int_E \psi_r^{m,t,x}(e)\widetilde{\mu}(dr,de) \end{aligned} \tag{4.4}$$

where f_m is defined by, $\forall(r,x,q,y,z,\psi) \in [0,T] \times \mathbb{R}^n \times \mathbb{D}[0,T] \times \mathbb{R} \times \mathbb{R}^{1\times d} \times \mathbb{L}^2(E,\nu)$,

$$f_m\big(r,x,(q_s)_{v\in[r,T]},y,z,\psi\big) := f\big(r,x,(\varphi_m(q_s))_{v\in[r,T]},\varphi_m(y),\varphi_m(z),\varphi_m(\psi\circ\zeta_m)\big) . \tag{4.5}$$

Here, we have used a simple truncation function

$$\varphi_m(x) := \begin{cases} -m & \text{for } x \leq -m \\ x & \text{for } |x| \leq m \\ m & \text{for } x \geq m \end{cases}$$

[2]See, for example, (9.55) of Lemma 9.6.5 [7] and its proof.

and a cutoff function $\psi \circ \zeta_m(e) := \psi(e)\mathbf{1}_{|e|\geq 1/m}$, which are applied component-wise for z, ψ.

Lemma 4.2. *Suppose that the driver f satisfies Assumptions 4.2 and 4.3. Then, $(f_m)_{m\in\mathbb{N}}$ also satisfy Assumptions 4.2 and 4.3 uniformly in $m \in \mathbb{N}$. Moreover, for each $m \in \mathbb{N}$, the driver f_m is a.e. bounded and globally Lipschitz continuous with respect to (q, y, z, ψ) in the sense of Assumption B.1.*

Proof. With $|\varphi_m(x)| \leq |x|$, $|\varphi_m(x) - \varphi_m(x')| \leq |x - x'|$ and use the convexity of the function $j_\gamma(\cdot)$, the first claim is obvious. By denoting $C_m := \max_{1\leq k\leq 1} \int_{|e|\geq 1/m} \nu^i(de) < \infty$, one sees $|f_m| \leq \sup_{t\in[0,T]} l_t + (\delta + \beta)m + \frac{\gamma}{2} dm^2 + kj_\gamma(m)C_m$ a.e. by the structure condition. By noticing the fact that

$$||\varphi_m(\psi \circ \zeta_m)||^2_{\mathbb{L}^2(\nu)} \leq \sum_{i=1}^k m^2 \int_{|e|\geq 1/m} \nu^i(de) \leq km^2 C_m$$

the global Lipschitz continuity can be confirmed easily. □

We now provide our first main result.

Theorem 4.1. *Under Assumptions 4.1, 4.2 and 4.3, there exists a unique solution $(Y^{t,x}, Z^{t,x}, \psi^{t,x}) \in \mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ to the ABSDE (4.2) for each $(t, x) \in [0, T] \times \mathbb{R}^n$.*

Proof. Since the uniqueness follows from Proposition 4.1, it suffices to prove the existence. Due to the global Lipschitz continuity of f_m, Proposition B.1 implies that there exists a unique solution $(Y^{m,t,x}, Z^{m,t,x}, \psi^{m,t,x}) \in \mathcal{K}^2[0, T]$ of (4.4). Since $|\xi|$ and $|f_m|$ are bounded, we actually have $Y^{m,t,x} \in \mathbb{S}^\infty$. Therefore, Lemmas 4.2, 3.1 and 3.2 imply that there exists some constant $C = C(A)$ such that

$$||Y^{m,t,x}||_{\mathbb{S}^\infty}, ||Z^{m,t,x}||_{\mathbb{H}^2_{BMO}}, ||\psi^{m,t,x}||_{\mathbb{J}^2_{BMO}} \leq C \tag{4.6}$$

uniformly in $(m, t, x) \in \mathbb{N} \times [0, T] \times \mathbb{R}^n$. Furthermore, Lemma 4.2 and Proposition 4.1 imply that the deterministic map $u_m : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ defined by $u_m(t, x) := Y_t^{m,t,x}$ satisfies the local Hölder continuity uniformly in m with $C = C(\alpha, \rho, p, \bar{q}, K_\xi, K, K_\cdot, A)$ such that

$$|u_m(t, x) - u_m(t', x')| \leq C\Big(1 + [|x| \vee |x'|]^\rho\Big)\Big(|x - x'|^\alpha + \big(1 + [|x| \vee |x'|]^\alpha\big)|t - t'|^{\frac{1}{2p\bar{q}^2}}\Big) .$$

From (4.6), it is also clear that $\sup_{m\geq 1} \sup_{(t,x)\in[0,T]\times\mathbb{R}^n} |u_m(t, x)| \leq C$.

Let us now confirm the compactness result for $(u_m)_{m\in\mathbb{N}}$. By defining the compact set $\mathbb{K}_j$ with $j \in \mathbb{N}$ by $\mathbb{K}_j := [0, T] \times \overline{B}_j(\mathbb{R}^n) \subset \mathbb{R}^{n+1}$, we have $\bigcup_{j=1}^\infty \mathbb{K}_j = [0, T] \times \mathbb{R}^n$. Here, $\overline{B}_j(\mathbb{R}^n)$ is a closed ball in $\mathbb{R}^n$ of radius j centered at the origin. Arzelà-Ascoli theorem (see, Section 10.1 [28]) tells that there exists a subsequence $(m^{(1)}) \subset (m)$ such that, $\exists u^{(1)} \in C(\mathbb{K}_1)$, $(u_{m^{(1)}})$ converges uniformly to $u^{(1)}$ on $\mathbb{K}_1$. Since the sequence $(u_{m^{(1)}})$ is also bounded and equicontinuous, there exists a further subsequence $(m^{(2)}) \subset (m^{(1)})$ such that, $\exists u^{(2)} \in C(\mathbb{K}_2)$, $(u_{m^{(2)}})$ converges uniformly to $u^{(2)}$ on $\mathbb{K}_2$. By construction, it is clear that $u^{(2)}|_{\mathbb{K}_1} = u^{(1)}$. Continue the above procedures and construct a diagonal sequence as

$$(m^{(m)})_{m\geq 1} := \{1^{(1)}, 2^{(2)}, \cdots, j^{(j)}, \cdots\} .$$

From Lemma 2 in Section 10.1 [28] implies that there exists a subsequence $(m') \subset (m^{(m)})$ and some function $u : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ such that $(u_{m'})$ converges to u pointwise on the whole $[0, T] \times \mathbb{R}^n$ space. Moreover, by the above construction of the sequence $(m^{(m)})$, it follows that the function is actually continuous $u \in C([0,T]\times\mathbb{R}^n)$ to which $(u_{m'})$ converges uniformly to any compact subset $\mathbb{K}_R$.

In the reminder, we work on the sequence (m') (and possibly its further subsequences). Define the càdlàg $\mathbb{F}$-adapted process $(Y_s^{t,x})_{s\in[0,T]}$ by $Y_s^{t,x} := u(s, X_s^{t,x}), \quad \forall(\omega,s)\in\Omega\times[0,T]$. Using the uniform boundedness of $(u_{m'}, u)$ and Chebyshev's inequality, one obtains

$$||Y^{m',t,x} - Y^{t,x}||^p_{\mathbb{S}^p} \le \mathbb{E}\Big[\sup_{s\in[0,T]}\big|u_{m'}(s,X_s^{t,x}) - u(s,X_s^{t,x})\big|^p \mathbf{1}_{\{\sup_{s\in[0,T]}|X_s^{t,x}|\le R\}}\Big] + C\Big(\frac{1+|x|}{R}\Big)^j$$

for any $p, R, j > 0$ with some m-independent constant C. Since $(u_{m'})$ converges uniformly to u on any compact set, one concludes $Y^{m',t,x} \to Y^{t,x}$ in $\mathbb{S}^p$ with $\forall p>0$. Therefore, by extracting further subsequence (still denoted by (m')), we have $\lim_{m'\to\infty}\sup_{s\in[0,T]}|Y_s^{m',t,x} - Y_s^{t,x}| = 0$ $\mathbb{P}$-a.s. and hence, in particular, $||Y^{m',t,x} - Y^{t,x}||_{\mathbb{S}^\infty} \to 0$.

With $m_1, m_2 \in (m')$, $\delta Y^{m_1,m_2} := Y^{m_1,t,x} - Y^{m_2,t,x}$, $\delta Z^{m_1,m_2} := Z^{m_1,t,x} - Z^{m_2,t,x}$ and $\delta\psi^{m_1,m_2} := \psi^{m_1,t,x} - \psi^{m_2,t,x}$, Ito formula yields for any $\tau \in \mathcal{T}_0^T$,

$$\begin{aligned}
&\mathbb{E}_{\mathcal{F}_\tau}\int_\tau^T |\delta Z_r^{m_1,m_2}|^2 dr + \mathbb{E}_{\mathcal{F}_\tau}\int_\tau^T ||\delta\psi_r^{m_1,m_2}||^2_{\mathbb{L}^2(\nu)} dr \\
&\le 2||\delta Y^{m_1,m_2}||_{\mathbb{S}^\infty}\mathbb{E}_{\mathcal{F}_\tau}\int_\tau^T \sum_{i=1}^2 |f_m(r, X_r^{t,x}, (Y_v^{m_i,t,x})_{v\in[r,T]}, \Theta_r^{m_i,t,x})| dr\ .
\end{aligned}$$

Since the conditional expectation of the 2nd line is bounded by $C\sum_{i=1}^2\big(1 + ||Y^{m_i,t,x}||_{\mathbb{S}^\infty} + ||Z^{m_i,t,x}||^2_{\mathbb{H}^2_{BMO}} + ||\psi^{m_i,t,x}||^2_{\mathbb{J}^2_{BMO}}\big) \le C$, with $C = C(K., A)$, the right-hand side converges to zero as $m_1, m_2 \to \infty$ uniformly in $\tau \in \mathcal{T}_0^T$. Therefore $\exists(Z^{t,x}, \psi^{t,x}) \in \mathbb{H}^2_{BMO}\times\mathbb{J}^2_{BMO}$ such that $Z^{m',t,x} \to Z^{t,x}$ in $\mathbb{H}^2_{BMO}$ and $\psi^{m',t,x} \to \psi^{t,x}$ in $\mathbb{J}^2_{BMO}$.

Proving that $(Y^{t,x}, Z^{t,x}, \psi^{t,x})$ provides a solution of (4.2) can be done in the standard manner. In particular, one can extract a subsequence (still denoted by (m')) such that $\sup_{s\in[0,T]}|Y_s^{m',t,x} - Y_s^{t,x}| \to 0$ a.s., $Z^{m',t,x} \to Z^{t,x}$ $d\mathbb{P}\otimes ds$-a.e., $\psi^{m,t,x} \to \psi^{t,x}$ $d\mathbb{P}\otimes\nu(de)ds$-a.e., $\sup_{s\in[0,T]}\Big|\int_s^T (Z_r^{m',t,x} - Z_r^{t,x})dW_r\Big| \to 0$ a.s. and $\sup_{s\in[0,T]}\Big|\int_s^T\int_E(\psi_r^{m',t,x}(e) - \psi_r^{t,x}(e))\widetilde{\mu}(dr,de)\Big| \to 0$ a.s. Since $f_m \to f$ locally uniformly, it follows that

$$f_{m'}(s, X_s^{t,x}, (Y_v^{m',t,x})_{v\in[s,T]}, \Theta_s^{m',t,x}) \to f(s, X_s^{t,x}, (Y_v^{t,x})_{v\in[s,T]}, \Theta_s^{t,x})$$

$d\mathbb{P}\otimes ds$-a.e. By the same arguments given in Lemma 2.5 in [18], one can choose (m') in such a way that $G := \sup_{m'}|Z^{m',t,x}|^2$ and $H := \sup_{m'}||\psi^{m',t,x}||^2_{\mathbb{L}^2(\nu)}$ are in $\mathbb{L}^1(\Omega\times[0,T])$. Since $|f_{m'}| \le C(1 + G + H)$ a.s.with some $C = C(K., A)$, we have

$$\int_0^T |f_{m'}(r, X_r^{t,x}, (Y_v^{m',t,x})_{v\in[r,T]}, \Theta_r^{m',t,x}) - f(r, X_r^{t,x}, (Y_v^{t,x})_{v\in[r,T]}, \Theta_r^{t,x})| dr \to 0 \text{ a.s.}$$

by the Lebesgue's dominated convergence theorem. This finishes the proof. □

Remark 4.2. *In the above proof of Theorem 4.1, the convergence actually occurs in the entire sequence of (m) not only the subsequence (m'). If this is not the case, there must be a subsequence $(\hat{m}) \subset (m)$ such that $||Y^{m_j,t,x} - Y^{t,x}||_{\mathbb{S}^\infty} > c$ with some $c > 0$ for every $m_j \in (\hat{m})$. However, by repeating the same procedures done in the proof, we can extract a further subsequence $(\hat{m}') \subset (\hat{m})$ such that, $\exists(\widetilde{Y}^{t,x}, \widetilde{Z}^{t,x}, \widetilde{\psi}^{t,x})$, $(Y^{m_j,t,x}, Z^{m_j,t,x}, \psi^{m_j,t,x}) \to (\widetilde{Y}^{t,x}, \widetilde{Z}^{t,x}, \widetilde{\psi}^{t,x})$ in $\mathbb{S}^\infty\times\mathbb{H}^2_{BMO}\times\mathbb{J}^2_{BMO}$ as $(\hat{m}') \ni m_j \to \infty$. One can show that it also provides the solution to (4.2). By the uniqueness of solution, $\widetilde{Y}^{t,x} = Y^{t,x}$ in $\mathbb{S}^\infty$, which* *contradicts the assumption.*

5 Some regularity results

Due to the general path-dependence of (Y) in the driver, it is difficult to establish Malliavin's differentiability. Interestingly, we can apply the method similar to Lemma 15 in Fromm & Imkeller (2013) [10] or Lemma 2.5.14 in Fromm (2014) [11] to derive some useful regularity results on the control variables. The method only needs the fundamental Lebesgue's differentiation theorem.[3]

Lemma 5.1. *Under Assumptions 4.1, 4.2 and 4.3 with $\alpha = 1$, the control variables of the solution to the ABSDE (4.2) satisfy the estimate for every (t,x)*

$$|Z_s^{t,x}| \le C\big(1+|X_s^{t,x}|^{1+\rho}\big), \quad ||\psi_s^{t,x}||_{\mathbb{L}^2(\nu)} \le C\big(1+|X_{s-}^{t,x}|^{1+\rho}\big)$$

for $d\mathbb{P}\otimes ds$-a.e. $(\omega,s)\in\Omega\times[0,T]$ with some constant $C=C(\rho,K_\xi,K,K.,A)$.

Proof. For notational simplicity, let us fix the initial data (t,x) and omit the associated superscripts in the reminder of the proof. We start from the regularized ABSDE (4.4). Choose any $s'\in[0,T)$ and define $\delta W_s := W_s - W_{s'}$ for $s\in[s',T]$. An application of Ito formula to $(Y^m\delta W^\top)$ yields

$$\begin{aligned} Y_s^m \delta W_s^\top &= \int_{s'}^s Z_r^m dr - \int_{s'}^s \mathbf{1}_{r\ge t}\delta W_r^\top \mathbb{E}_{\mathcal{F}_r} f_m\big(r, X_r, (Y_v^m)_{v\in[r,T]}, \Theta_r^m\big)dr \\ &+ \int_{s'}^s \delta W_r^\top Z_r^m dW_r + \int_{s'}^s\int_E \delta W_r^\top \psi_r^m(e)\widetilde{\mu}(dr,de) + \int_{s'}^s Y_r^m dW_r^\top \ . \end{aligned} \tag{5.1}$$

Since $(Y^m, Z^m, \psi^m)\in\mathbb{S}^\infty\times\mathbb{H}^2_{BMO}\times\mathbb{J}^2_{BMO}$, one can show easily that the last three terms are true martingales. Notice that

$$\begin{aligned} &\mathbb{E}\Big[\int_0^T \big|\mathbf{1}_{r\ge t} W_r^\top \mathbb{E}_{\mathcal{F}_r} f_m\big(r, X_r, (Y_v^m)_{v\in[r,T]}, \Theta_r^m\big)\big| dr\Big] \\ &\le C\mathbb{E}\Big[||W||^2_{[0,T]}\Big]^{\frac{1}{2}}\mathbb{E}\Big[\Big(\int_0^T \big(1+|Z_r^m|^2+||\psi_r^m||^2_{\mathbb{L}^2(\nu)}\big)dr\Big)^2\Big]^{\frac{1}{2}} \le C \end{aligned}$$

with $C=C(K.,A)$. Thus Lebesgue's differentiation theorem implies that,

$$\begin{aligned} &\lim_{s\downarrow s'}\frac{1}{s-s'}\int_{s'}^s \mathbf{1}_{r\ge t} W_r^\top \mathbb{E}_{\mathcal{F}_r} f_m\big(r, X_r, (Y_v^m)_{v\in[r,T]}, \Theta_r^m\big)dr \\ &\qquad = \mathbf{1}_{s'\ge t} W_{s'}^\top \mathbb{E}_{\mathcal{F}_{s'}} f_m\big(s', X_{s'}, (Y_v^m)_{v\in[s',T]}, \Theta_{s'}^m\big) \ \text{ a.s.} \end{aligned}$$

for dt-a.e. $s'\in[0,T)$. Similarly one obtains for dt-a.e. $s'\in[0,T)$,

$$\begin{aligned} &\lim_{s\downarrow s'}\frac{1}{s-s'}\int_{s'}^s Z_r^m dr = Z_{s'}^m \ \text{ a.s.} \\ &\lim_{s\downarrow s'}\frac{1}{s-s'}\int_{s'}^s \mathbf{1}_{r\ge t}\mathbb{E}_{\mathcal{F}_r} f_m\big(r, X_r, (Y_v^m)_{v\in[r,T]}, \Theta_r^m\big)dr \\ &\qquad = \mathbf{1}_{s'\ge t}\mathbb{E}_{\mathcal{F}_{s'}} f_m\big(s', X_{s'}, (Y_v^m)_{v\in[s',T]}, \Theta_{s'}^m\big)dr \ \text{ a.s.} \end{aligned}$$

Since $Z^m\in\mathbb{H}^2$, we can also take s' such that $\mathbb{E}[|Z_{s'}^m|]<\infty$ a.e. in $[0,T)$.

[3]See, for example, Section E.4, Theorem 6 [9].

As in Lemma 2.5.14 of [11], we introduce the stopping time $\tau:\Omega\to(s',T]$ such that the following inequalities hold for all $s\in(s',T]$:

- $\displaystyle \Big|\frac{1}{s-s'}\int_{s'}^{\tau\wedge s} Z_r^m dr\Big| \le |Z_{s'}^m|+1 \quad \text{a.s.}$
- $\displaystyle \Big|\frac{1}{s-s'}\int_{s'}^{\tau\wedge s} \mathbf{1}_{r\ge t}\mathbb{E}_{\mathcal{F}_r} f_m\big(r,X_r,(Y_v^m)_{v\in[r,T]},\Theta_r^m\big)dr\Big|$
  $\displaystyle \le \mathbf{1}_{s'\ge t}\Big|\mathbb{E}_{\mathcal{F}_{s'}} f_m\big(s',X_{s'},(Y_v^m)_{v\in[s',T]},\Theta_{s'}^m\big)\Big|+1 \quad \text{a.s.}$
- $\displaystyle \Big|\frac{1}{s-s'}\int_{s'}^{\tau\wedge s} \mathbf{1}_{r\ge t}W_r^\top\mathbb{E}_{\mathcal{F}_r} f_m\big(r,X_r,(Y_v^m)_{v\in[r,T]},\Theta_r^m\big)dr\Big|$
  $\displaystyle \le \mathbf{1}_{s'\ge t}\Big|W_{s'}^\top\mathbb{E}_{\mathcal{F}_{s'}} f_m\big(s',X_{s'},(Y_v^m)_{v\in[s',T]},\Theta_{s'}^m\big)\Big|+1 \quad \text{a.s.}$

Then one can show from (5.1) and the fact that $\tau(\omega)\wedge s=s$ for sufficiently small $s\in(s',T]$,

$$Z_{s'}^m=\lim_{s\downarrow s'}\mathbb{E}_{\mathcal{F}_{s'}}\Big[\frac{1}{s-s'}Y_{\tau\wedge s}^m(W_{\tau\wedge s}-W_{s'})^\top\Big]$$

$d\mathbb{P}\otimes dt$-a.e. $(\omega,s')\in\Omega\times[0,T)$ by the dominated convergence theorem. One sees

$$\begin{aligned}&\Big|\mathbb{E}_{\mathcal{F}_{s'}}\Big[\frac{1}{s-s'}Y_{\tau\wedge s}^m(W_{\tau\wedge s}-W_{s'})^\top\Big]\Big|\\ &\le\Big|\mathbb{E}_{\mathcal{F}_{s'}}\Big[\frac{1}{s-s'}Y_{s}^m(W_{\tau\wedge s}-W_{s'})^\top\Big]\Big|+\Big|\mathbb{E}_{\mathcal{F}_{s'}}\Big[\frac{1}{s-s'}(Y_s^m-Y_{\tau\wedge s}^m)(W_{\tau\wedge s}-W_{s'})^\top\Big]\Big|,\end{aligned}$$

where the second term yields

$$\begin{aligned}&\Big|\mathbb{E}_{\mathcal{F}_{s'}}\Big[\frac{1}{s-s'}(Y_s^m-Y_{\tau\wedge s}^m)(W_{\tau\wedge s}-W_{s'})^\top\Big]\Big|\\ &=\mathbb{E}_{\mathcal{F}_{s'}}\Big[\frac{1}{s-s'}\mathbb{E}_{\mathcal{F}_{\tau\wedge s}}\big[Y_s^m-Y_{\tau\wedge s}^m\big](W_{\tau\wedge s}-W_{s'})^\top\Big]\Big|\\ &\le\mathbb{E}_{\mathcal{F}_{s'}}\Big[\frac{1}{s-s'}\int_{\tau\wedge s}^{s}\mathbb{E}_{\mathcal{F}_{\tau\wedge s}}|f_m(r,X_r,(Y_v^m)_{v\in[r,T]},\Theta_r^m)|dr(W_{\tau\wedge s}-W_{s'})^\top\Big]\\ &\le C_m\mathbb{E}_{\mathcal{F}_{s'}}\Big[|W_{\tau\wedge s}-W_{s'}|^2\Big]^{\frac12}\le C_m\sqrt{s-s'}\to0\quad s\downarrow s'.\end{aligned}$$

Here, we have used the fact that $|f_m|$ is essentially bounded for each m (see Lemma 4.2). The first term gives the estimate

$$\begin{aligned}&\Big|\mathbb{E}_{\mathcal{F}_{s'}}\Big[\frac{1}{s-s'}u_m(s,X_s)(W_{\tau\wedge s}-W_{s'})^\top\Big]\Big|=\Big|\mathbb{E}_{\mathcal{F}_{s'}}\Big[\frac{1}{s-s'}\big(u_m(s,X_s)-u_m(s,X_{s'})\big)(W_{\tau\wedge s}-W_{s'})^\top\Big]\Big|\\ &\le\frac{1}{\sqrt{s-s'}}\mathbb{E}_{\mathcal{F}_{s'}}\Big[|u_m(s,X_s)-u_m(s,X_{s'})|^2\Big]^{\frac12}\le\frac{C}{\sqrt{s-s'}}\mathbb{E}_{\mathcal{F}_{s'}}\Big[\big(1+[|X_s|\vee|X_{s'}|]^{2\rho}\big)|X_s-X_{s'}|^2\Big]^{\frac12}\\ &\le\frac{C}{\sqrt{s-s'}}\mathbb{E}_{\mathcal{F}_{s'}}\Big[\big(1+|X_{s'}|^{2\rho}+|X_s-X_{s'}|^{2\rho}\big)|X_s-X_{s'}|^2\Big]^{\frac12}\\ &\le C(1+|X_{s'}|^{1+\rho})\quad\text{a.s.}\end{aligned}$$

where Proposition 4.1 and Lemma 4.1 were used. Thus we have $d\mathbb{P}\otimes dt$-a.e.

$$|Z_{s'}^m|\le C(1+|X_{s'}|^{1+\rho})$$

with $C = C(\rho, K_\xi, K, K_\cdot, A)$. It is known from the proof of Theorem 4.1 that $Z^m \to Z$ $d\mathbb{P} \otimes dt$-a.e. under an appropriate subsequence, and hence the first claim follows.

The joint continuity of u implies $Y_{s-} = \lim_{r\uparrow s} u(r, X_r) = u(s, X_{s-})$ and hence

$$\begin{aligned}\int_E |\psi_s(e)|^2 \nu(de) &= \int_E |u(s, X_{s-} + \gamma(s, X_{s-}, e)) - u(s, X_{s-})|^2 \nu(de) \\ &\le C \int_E \Big(1 + |X_{s-}|^{2\rho} + |\gamma(s, X_{s-}, e)|^{2\rho}\Big) |\gamma(s, X_{s-}, e)|^2 \nu(de) \\ &\le C(1 + |X_{s-}|^{2(1+\rho)}) \int_E |e|^2 \nu(de) \le C(1 + |X_{s-}|^{2(1+\rho)}) ,\end{aligned}$$

which proves the second claim. □

6 A non-Markovian setting

6.1 Existence

In order to obtain the existence result in a non-Markovian setting, we need an additional so-called A_Γ-condition on the driver, which is rather restrictive but plays a crucial role in almost every existing work on quadratic growth BSDEs with jumps.

Assumption 6.1. *For each $M > 0$, for every $q \in \mathbb{D}[0, T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^{1\times d}$, $\psi, \psi' \in \mathbb{L}^2(E, \nu)$ with $\sup_{v\in[0,T]} |q_v|, |y|, ||\psi||_{\mathbb{L}^\infty(\nu)}, ||\psi'||_{\mathbb{L}^\infty(\nu)} \le M$ there exists a $\mathcal{P} \otimes \mathcal{E}$-measurable process $\Gamma^{q,y,z,\psi,\psi'}$ such that, $d\mathbb{P} \otimes dt$-a.e.,*

$$f\big(t, (q_v)_{v\in[t,T]}, y, z, \psi\big) - f\big(t, (q_v)_{v\in[t,T]}, y, z, \psi'\big) \le \int_E \Gamma_t^{q,y,z,\psi,\psi'}(e)\big(\psi(e) - \psi'(e)\big)\nu(de)$$

with $C_M^1(1 \wedge |e|) \le \Gamma_t^{q,y,z,\psi,\psi'}(e) \le C_M^2(1 \wedge |e|)$ with two M dependent constants satisfying $C_M^1 > -1$ and $C_M^2 \ge 0$.

We introduce a regularized ABSDE with some positive constant $m > 0$:

$$\begin{aligned}Y_t^m &= \xi + \int_t^T \mathbb{E}_{\mathcal{F}_r} f_m\big(r, (Y_v^m)_{v\in[r,T]}, Y_r^m, Z_r^m, \psi_r^m\big) dr \\ &\quad - \int_t^T Z_r^m dW_r - \int_t^T \int_E \psi_r^m(e) \widetilde{\mu}(dr, de), \ t \in [0, T] \end{aligned} \tag{6.1}$$

with the definition $f_m\big(t, (q_v)_{v\in[t,T]}, y, z, \psi\big) := f\big(t, (\varphi_m(q_v))_{v\in[t,T]}, y, z, \psi\big)$ for every $(\omega, t, q, y, z, \psi) \in \Omega \times [0, T] \times \mathbb{D}[0, T] \times \mathbb{R} \times \mathbb{R}^{1\times d} \times \mathbb{L}^2(E, \nu)$. φ_m is the truncation function used previously.

Lemma 6.1. *If the driver f satisfies Assumptions 3.1, 3.2 and 6.1, then the driver (f_m) defined above also satisfies the same conditions uniformly in m. Moreover, if there exists a bounded solution $(Y^m, Z^m, \psi^m) \in \mathbb{S}^\infty \times \mathbb{H}^2 \times \mathbb{J}^2$ to the ABSDE (6.1), then it is unique and belongs to $\mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ with the norms $||Y^m||_{\mathbb{S}^\infty}, ||Z^m||_{\mathbb{H}^2_{BMO}}, ||\psi^m||_{\mathbb{J}^2_{BMO}} \le C$ with some constant C depending only on $A = (||\xi||_\infty, ||l||_{\mathbb{S}^\infty}, \delta, \beta, \gamma, T)$.*

Proof. The first claim is obvious. The second claim follows from Lemmas 3.1, 3.2 and Corollary 3.1. □

Theorem 6.1. *Under Assumptions 3.1, 3.2 and 6.1, there exists a unique solution $(Y, Z, \psi) \in \mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ to the ABSDE (3.1).*

Proof. Uniqueness follows from Corollary 3.1. Notice that it suffices to prove the existence of solution $(Y^m, Z^m, \psi^m) \in \mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ of (6.1) for each m. In fact, by choosing m bigger than the bound given in Lemma 3.2, one sees (Y^m, Z^m, ψ^m) actually provides the solution for (3.1). Let fix such an m in the reminder.

Let us put $Y^{m,0} \equiv 0$ and define a sequence of BSDEs with $n \in \mathbb{N}$ such that

$$Y^{m,n}_t = \xi + \int_t^T \mathbb{E}_{\mathcal{F}_r} f_m\big(r, (Y^{m,n-1}_v)_{v\in[r,T]}, Y^{m,n}_r, Z^{m,n}_r, \psi^{m,n}_r\big)dr \\ - \int_t^T Z^{m,n}_r dW_r - \int_t^T \int_E \psi^{m,n}_r(e)\widetilde{\mu}(dr, de),\ t \in [0,T]\ . \tag{6.2}$$

The driver for the BSDE (6.2) can be seen as $\widetilde{f}_m(r, y, z, \psi) := \mathbb{E}_{\mathcal{F}_r} f\big(r, (Y^{m,n-1}_v)_{v\in[r,T]}, y, z, \psi\big)$. By replacing l_r by $l_r + \delta m$, one sees the data $(\xi, \widetilde{f}_m)$ satisfy Assumptions 3.1, 3.2 and 4.1 in [12] for non-anticipated quadratic-exponential growth BSDEs. Therefore, Theorem 4.1 [12] implies that there exists a (unique) solution $(Y^{m,n}, Z^{m,n}, \psi^{m,n}) \in \mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ for each $n \geq 1$. Furthermore, as a special case of the universal bounds, one sees $||Y^{m,n}||_{\mathbb{S}^\infty}, ||Z^{m,n}||_{\mathbb{H}^2_{BMO}}, ||\psi^{m,n}||_{\mathbb{J}^2_{BMO}} \leq C$ with $C = C(||\xi||_\infty, ||l||_{\mathbb{S}^\infty} + \delta m, \beta, \gamma, T)$.

Let denote $\delta Y^{m,n} := Y^{m,n} - Y^{m,n-1}$. Replacing l_r by $l_r + \delta m$, then putting $\delta = 0$, and considering the drivers $f^1(r, y, z, \psi) := f_m(r, (Y^{m,n}_v)_{v\in[r,T]}, y, z, \psi)$, $f^2(r, y, z, \psi) := f_m(r, (Y^{m,n-1}_v)_{v\in[r,T]}, y, z, \psi)$, one sees that $(f^i)^2_{i=1}$ satisfy Assumptions 3.1 and 3.2. Thus one can apply the stability results in Proposition 3.1 to the BSDE (6.2). In particular, by (3.4), one has for any $p \geq 2q_*$ and $0 < h \leq T$,

$$\mathbb{E}\Big[\sup_{t\in[T-h,T]} |\delta Y^{m,n+1}_t|^p\Big] \leq C\mathbb{E}\Big[\Big(\int_{T-h}^T \mathbb{E}_{\mathcal{F}_r}\big|f_m(r, (Y^{m,n}_v)_{v\in[r,T]}, \Theta^{m,n+1}_r) \\ - f_m(r, (Y^{m,n-1}_v)_{v\in[r,T]}, \Theta^{m,n+1}_r)\big|dr\Big)^p\Big] \leq Ch^p\mathbb{E}\Big[\sup_{t\in[T-h,T]} |\delta Y^{m,n}_t|^p\Big]$$

with some constant $C = C(p, K., ||\xi||_\infty, ||l||_{\mathbb{S}^\infty} + \delta m, \beta, \gamma, T)$. By choosing h small enough so that $Ch^p < 1$, it becomes a strict contraction and thus $(Y^{m,n}_v, v \in [T-h, T])_{n\geq 1}$ forms a Cauchy sequence in $\mathbb{S}^p[T-h, T]$.

By extracting an appropriate subsequence $(n') \subset (n)$, one has $||\delta Y^{m,n'}||_{\mathbb{S}^\infty[T-h,T]} \to 0$ as $n' \to \infty$. Applying Ito formula to $(\delta Y^{m,n'})^2$ and repeating the same procedures used in last part of the proof in Theorem 4.1, one can show that $\exists (Y^m, Z^m, \psi^m) \in (\mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO})_{[T-h,T]}$, $(Y^{m,n'}, Z^{m,n'}, \psi^{m,n'}) \to (Y^m, Z^m, S^m)$ in the corresponding norm, and that $(Y^m_v, Z^m_v, \psi^m_v)_{v\in[T-h,T]}$ solves the ABSDE (6.1) for the period $[T-h, T]$.[4]

Now, let us replace $(Y^{m,n}, Z^{m,n}, \psi^{m,n})_{n\in\mathbb{N}}$ by (Y^m, Z^m, ψ^m) for $(\omega, s) \in \Omega \times [T-h, T]$ in (6.2). Then for $t \leq T-h$, we have

$$Y^{m,n}_t = Y^m_{T-h} + \int_t^{T-h} \mathbb{E}_{\mathcal{F}_r} f_m\big(r, (Y^{m,n-1}_v)_{v\in[r,T]}, Y^{m,n}_r, Z^{m,n}_r, \psi^{m,n}_r\big)dr \\ - \int_t^{T-h} Z^{m,n}_r dW_r - \int_t^{T-h} \int_E \psi^{m,n}_r(e)\widetilde{\mu}(dr, de).$$

[4]Thanks to the uniqueness of the solution of (6.1), the same arguments used in Remark 4.2 guarantee that the above convergence actually occurs in the entire sequence (n).

An application of Proposition 3.1 with the data $(Y^m_{T-h}, f^1), (Y^m_{T-h}, f^2)$ yields,

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[T-2h,T-h]}|\delta Y^{m,n+1}|^p\Big] &\leq C\mathbb{E}\Big[\Big(\int_{T-2h}^{T-h}\mathbb{E}_{\mathcal{F}_r}|\delta f(r)|dr\Big)^p\Big] \\ &\leq Ch^p\mathbb{E}\Big[\sup_{t\in[T-2h,T]}|\delta Y^{m,n}_t|^p\Big] = Ch^p\mathbb{E}\Big[\sup_{t\in[T-2h,T-h]}|\delta Y^{m,n}_t|^p\Big]\end{aligned}$$

where the fact $Y^{m,n}_s = Y^m_s$, $s \in [T-h, T]$ is used in the 2nd line. Thus one can extend the solution to the period $[T-2h, T-h]$ by the same procedures used in the previous step. Since coefficient C can be taken independently of the specific period, the whole period $[0, T]$ can be covered by a finite number of partitions. Notice here that, as one can see from the proof of Proposition 3.1, the coefficient C depends on the essential supremum of the terminal value $||\xi||_\infty$ only through the local Lipschitz constant K_M and the universal bounds controlling M as well as the coefficients of the reverse Hölder inequality. Hence the appearance of the new terminal value Y^m_{T-h} does not change the size of the coefficient C. This finishes the proof for the existence of a bounded solution to (6.1) for each m. □

6.2 Comparison principle

For completeness, we give a sufficient condition for the comparison principle to hold for our ABSDE in the rest of this section. In non-anticipated settings, i.e. when there is no future path-dependence of $(Y_v)_{v\in[0,T]}$ in the driver f, it is known that the comparison principle holds for quadratic-exponential growth BSDEs in the presence of A_Γ-condition (See, Lemma C.1.). For the current anticipated setting, we need an additional assumption same as the one used in Theorem 5.1 of [26]. Consider the two ABSDEs with $i \in \{1, 2\}$,

$$Y^i_t = \xi_i + \int_t^T \mathbb{E}_{\mathcal{F}_r} f_i\big(r, (Y^i_v)_{v\in[r,T]}, Y^i_r, Z^i_r, \psi^i_r\big)dr - \int_t^T Z^i_r dW_r - \int_t^T\int_E \psi^i_r(e)\widetilde{\mu}(dr, de)$$

for $t \in [0, T]$.

Theorem 6.2. *Suppose the data $(\xi_i, f_i)_{1\leq i\leq 2}$ satisfy Assumptions 3.1, 3.2 and 6.1. Moreover, f_2 is increasing in $(q_v)_{v\in[0,T]}$, i.e. $f_2(r, (q_v)_{v\in[r,T]}, y, z, \psi) \leq f_2(r, (q'_v)_{v\in[r,T]}, y, z, \psi)$ for every $(r, y, z, \psi) \in [0, T] \times \mathbb{R} \times \mathbb{R}^{1\times d} \times \mathbb{L}^2(E, \nu)$ and $q, q' \in \mathbb{D}[0, T]$, if $q_v \leq q'_v$ $\forall v \in [r, T]$. If $\xi_1 \leq \xi_2$ a.s. and $f_1(r, (q_v)_{v\in[r,T]}, y, z, \psi) \leq f_2(r, (q_v)_{v\in[r,T]}, y, z, \psi)$ $d\mathbb{P} \otimes dr$-a.e. for every $(q, y, z, \psi) \in \mathbb{D}[0, T] \times \mathbb{R} \times \mathbb{R}^{1\times d} \times \mathbb{L}^2(E, \nu)$, then $Y^1_t \leq Y^2_t$ $\forall t \in [0, T]$ a.s.*

Proof. Firstly, let us regularize the driver f_2 by f'_2 defined as, for every (r, q, y, z, ψ),

$$f'_2(r, (q_v)_{r\in[t,T]}, y, z, \psi) := f_2\big(r, (\varphi_m(q_v))_{v\in[r,T]}, y, z, \psi\big) \tag{6.3}$$

with some truncation level m satisfying $m > (||Y^1||_{\mathbb{S}^\infty} \vee ||Y^2||_{\mathbb{S}^\infty})$. Consider a sequence of non-anticipated BSDEs with $n \in \mathbb{N}$ by

$$\begin{aligned}Y^{2,n}_t &= \xi_2 + \int_t^T \mathbb{E}_{\mathcal{F}_r} f'_2\big(r, (Y^{2,n-1}_v)_{v\in[r,T]}, Y^{2,n}_r, Z^{2,n}_r, \psi^{2,n}_r\big)dr \\ &\quad - \int_t^T Z^{2,n}_r dW_r - \int_t^T\int_E \psi^{2,n}_r(e)\widetilde{\mu}(dr, de), \quad t \in [0, T]\end{aligned} \tag{6.4}$$

under the condition $Y^{2,0} = Y^1$. By the proof of Theorem 6.1, there exists $h > 0$ such that $(Y^{2,n}, Z^{2,n}, \psi^{2,n}) \to (Y^2, Z^2, \psi^2)$ in $\mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ as $n \to \infty$ for the period $[T-h, T]$. Note that the constraint $\varphi_m(\cdot)$ becomes passive at least for large enough n.

Firstly, let us focus on the period $[T-h,T]$. Set $\widetilde{f}_1(r,y,z,\psi) = \mathbb{E}_{\mathcal{F}_r} f_1(r,(Y^1_v)_{v\in[r,T]},y,z,\psi)$ and $\widetilde{f}_2(r,y,z,\psi) = \mathbb{E}_{\mathcal{F}_r} f'_2(r,(Y^1_v)_{v\in[r,T]},y,z,\psi) = \mathbb{E}_{\mathcal{F}_r} f_2(r,(Y^1_v)_{v\in[r,T]},y,z,\psi)$. Applying Lemma C.1, one obtains $Y^1_t = Y^{2,0}_t \le Y^{2,1}_t$ $\forall t\in[T-h,T]$ a.s. Then using the new definition

$$\widetilde{f}_1(r,y,z,\psi) = \mathbb{E}_{\mathcal{F}_r} f'_2(r,(Y^{2,0}_v)_{v\in[r,T]},y,z,\psi),$$
$$\widetilde{f}_2(r,y,z,\psi) = \mathbb{E}_{\mathcal{F}_r} f'_2(r,(Y^{2,1}_v)_{v\in[r,T]},y,z,\psi),$$

and the hypothesis that the driver is increasing in $q\in\mathbb{D}[0,T]$, Lemma C.1 yields $Y^{2,1}_t \le Y^{2,2}$ $\forall t\in[T-h,T]$ a.s. By repeating the same arguments, one sees $Y^1_t \le Y^{2,n-1}_t \le Y^{2,n}_t$ $\forall t\in[T-h,T]$ a.s. for every $n\in\mathbb{N}$. Since $Y^{2,n}$ converges to Y^2 in $\mathbb{S}^\infty[T-h,T]$, one concludes $Y^1_t \le Y^2_t$ $\forall t\in[T-h,T]$ a.s.

Let us now replace $Y^{2,n}_t$ by Y^2_t for all $t\in[T-h,T]$ in (6.4), and consider a sequence of non-anticipated BSDEs $n\in\mathbb{N}$

$$\begin{aligned} Y^{2,n}_t &= Y^2_{T-h} + \int_t^{T-h} \mathbb{E}_{\mathcal{F}_r} f'_2\big(r,(Y^{2,n-1}_v)_{v\in[r,T]},Y^{2,n}_r,Z^{2,n}_r,\psi^{2,n}_r\big)dr \\ &\quad - \int_t^{T-h} Z^{2,n}_r dW_r - \int_t^{T-h}\int_E \psi^{2,n}_r(e)\widetilde{\mu}(dr,de) \end{aligned}$$

with the initial condition $Y^{2,0}_t = \begin{cases} Y^1_t, & t\in[0,T-h) \\ Y^2_t, & t\in[T-h,T]\end{cases}$ for the next short period $t\in[T-2h,T-h]$. By the result of the previous step, one has $Y^1_t \le Y^{2,0}_t$ $\forall t\in[T-2h,T]$ a.s. Now, let us set $\widetilde{f}_1(r,y,z,\psi) = \mathbb{E}_{\mathcal{F}_r} f_1(r,(Y^1_v)_{v\in[r,T]},y,z,\psi)$, $\widetilde{f}_2(r,y,z,\psi) = \mathbb{E}_{\mathcal{F}_r} f'_2(r,(Y^{2,0}_v)_{v\in[r,T]},y,z,\psi)$, where the latter is equal to $\mathbb{E}_{\mathcal{F}_r} f_2(r,(Y^{2,0}_v)_{v\in[r,T]},y,z,\psi)$. By applying Lemma C.1 to the data $(Y^1_{T-h},\widetilde{f}_1)$, $(Y^2_{T-h},\widetilde{f}_2)$, one obtains $Y^1_t \le Y^{2,1}_t$ $\forall t\in[T-2h,T-h]$ a.s. Since $Y^{2,1}_t = Y^2_t$ for $t\in[T-h,T]$, one concludes $Y^1_t \le Y^{2,0}_t \le Y^{2,1}_t$ $\forall t\in[T-2h,T]$ a.s. Similarly, applying Lemma C.1 with $\widetilde{f}_1(r,y,z,\psi) = \mathbb{E}_{\mathcal{F}_r} f'_2(r,(Y^{2,n-2}_v)_{v\in[r,T]},y,z,\psi)$, $\widetilde{f}_2(r,y,z,\psi) = \mathbb{E}_{\mathcal{F}_r} f'_2(r,(Y^{2,n-1}_v)_{v\in[r,T]},y,z,\psi)$ yields $Y^{2,n-1}_t \le Y^{2,n}_t$ $\forall t\in[T-2h,T]$ a.s. for every $n\ge 2$. As in the previous step, the proof of Theorem 6.1 implies $Y^{2,n}\to Y^2$ in $\mathbb{S}^\infty[T-2h,T-h]$. Since $Y^{2,n}_t = Y^2_t$ for $t\in[T-h,T]$ by construction, one actually has $Y^{2,n}\to Y^2$ in $\mathbb{S}^\infty[T-2h,T]$. It follows that $Y^1_t \le Y^2_t$ $\forall t\in[T-2h,T]$ a.s. Repeating the same procedures finite number of times, one obtains the desired result. □

A Some preliminary results

Let us remind some important properties of BMO-martingales. For our purpose, it is enough to focus on continuous ones. When $Z\in\mathbb{H}^2_{BMO}$, $M_\cdot := \int_0^\cdot Z_r dW_r$ is a continuous BMO-martingale with $||M||_{BMO} = ||Z||_{\mathbb{H}^2_{BMO}}$.

Lemma A.1 (reverse Hölder inequality)**.** *Let M be a continuous BMO-martingale. Then, Doléans-Dade exponential $\big(\mathcal{E}_t(M), t\in[0,T]\big)$ is a uniformly integrable martingale, and for every stopping time $\tau\in\mathcal{T}^T_0$, there exists some $r>1$ such that $\mathbb{E}\left[\mathcal{E}_T(M)^r|\mathcal{F}_\tau\right] \le C\mathcal{E}_\tau(M)^r$ with some positive constant $C = C(r,||M||_{BMO})$.*

Proof. See Kazamaki (1979) [15], and also Remark 3.1 of Kazamaki (1994) [16]. □

Lemma A.2. *Let M be a square integrable continuous martingale and $\hat{M} := \langle M\rangle - M$. Then, $M \in BMO(\mathbb{P})$ if and only if $\hat{M} \in BMO(\mathbb{Q})$ with $d\mathbb{Q}/d\mathbb{P} = \mathcal{E}_T(M)$. Furthermore, $||\hat{M}||_{BMO(\mathbb{Q})}$ is determined by some function of $||M||_{BMO(\mathbb{P})}$ and vice versa.*

Proof. See Theorem 3.3 and Theorem 2.4 in [16]. □

Remark A.1. *For continuous martingales, Theorem 3.1 [16] also tells that there exists some decreasing function $\Phi(r)$ with $\Phi(1+) = \infty$ and $\Phi(\infty) = 0$ such that if $||M||_{BMO(\mathbb{P})}$ satisfies $||M||_{BMO(\mathbb{P})} < \Phi(r)$ then $\mathcal{E}(M)$ satisfies the reverse Hölder inequality with power r. This implies together with Lemma A.2, one can take a common positive constant $\bar{r}$ satisfying $1 < \bar{r} \le r^*$ such that both of the $\mathcal{E}(M)$ and $\mathcal{E}(\hat{M})$ satisfy the reverse Hölder inequality with power $\bar{r}$ under the respective probability measure $\mathbb{P}$ and $\mathbb{Q}$. Furthermore, the upper bound r^* is determined only by $||M||_{BMO(\mathbb{P})}$ (or equivalently by $||M||_{BMO(\mathbb{Q})}$).*

Let us also remind the following result.

Lemma A.3. *(Chapter 1, Section 9, Lemma 6 [19]) For any $\Psi \in \mathbb{J}^p$ with $p \ge 2$, there exists some constant $C = C(p)$ such that*

$$\mathbb{E}\Big[\Big(\int_0^T\int_E |\Psi_r(e)|^2\nu(de)dr\Big)^{\frac{p}{2}}\Big] \le C\mathbb{E}\Big[\Big(\int_0^T\int_E |\Psi_r(e)|^2\mu(dr,de)\Big)^{\frac{p}{2}}\Big]\ .$$

B Existence and uniqueness results for Lipschitz case

Anticipated BSDEs under the global Lipschitz condition have been studied by many authors. Our setup is a bit different from the standard one, in particular at the terminal condition and also at the point where the continuity of the driver is defined with respect to the uniform norm of the path rather than $\mathbb{L}^2[0,T]$-norm. For readers' convenience, we provide a proof under our particular setup. It is restricted to the simplest form relevant for our purpose. One can readily generalize it to multi-dimensional setups with the future (Z,ψ)-dependence (See [22] among others.).

Let us consider the ABSDE for $t \in [0,T]$

$$Y_t = \xi + \int_t^T \mathbb{E}_{\mathcal{F}_r} f\big(r, (Y_v)_{v\in[r,T]}, Y_r, Z_r, \psi_r\big)dr - \int_t^T Z_r dW_r - \int_t^T\int_E \psi_r(e)\widetilde{\mu}(dr,de) \quad \text{(B.1)}$$

where $f : \Omega\times[0,T]\times\mathbb{D}[0,T]\times\mathbb{R}\times\mathbb{R}^{1\times d}\times\mathbb{L}^2(E,\nu)\to\mathbb{R}$ and ξ is an $\mathcal{F}_T$-measurable random variable.

Assumption B.1. *(i) The driver f is a map such that for every $(y,z,\psi) \in \mathbb{R}\times\mathbb{R}^{1\times d}\times\mathbb{L}^2(E,\nu)$ and any càdlàg $\mathbb{F}$-adapted process $(Y_v)_{v\in[0,T]}$, the process $\big(\mathbb{E}_{\mathcal{F}_t} f(t,(Y_v)_{v\in[t,T]},y,z,\psi), t \in [0,T]\big)$ is progressively measurable.*
(ii) For every $(q,y,z,\psi),(q',y',z',\psi') \in \mathbb{D}[0,T]\times\mathbb{R}\times\mathbb{R}^{1\times d}\times\mathbb{L}^2(E,\nu)$, there exists some positive constant K such that

$$\begin{aligned}&\big|f\big(t,(q_v)_{v\in[t,T]},y,z,\psi\big) - f\big(t,(q'_v)_{v\in[t,T]},y',z',\psi'\big)\big|\\ &\quad\le K\Big(\sup_{v\in[t,T]}|q_v-q'_v| + |y-y'| + |z-z'| + ||\psi-\psi'||_{\mathbb{L}^2(\nu)}\Big)\end{aligned}$$

$d\mathbb{P}\otimes dt$-a.e. $(\omega,t)\in\Omega\times[0,T]$.
(iii) $\mathbb{E}\Big[|\xi|^2 + \Big(\int_0^T |f(r,0,0,0,0)|dr\Big)^2\Big] < \infty$.

Proposition B.1. *Under Assumption B.1, there exists a unique solution* $(Y, Z, \psi) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{J}^2$ *to the ABSDE (B.1).*

Proof. We prove the claim by constructing a strictly contracting map $\Phi : \mathcal{K}^2[0,T] \ni (Y^k, Z^k, \psi^k) \mapsto \Phi(Y^k, Z^k, \psi^k) =: (Y^{k+1}, Z^{k+1}, \psi^{k+1}) \in \mathcal{K}^2[0,T]$ defined by

$$Y_t^{k+1} = \xi + \int_t^T \mathbb{E}_{\mathcal{F}_r} f\big(r, (Y_v^k)_{v\in[r,T]}, Y_r^k, Z_r^k, \psi_r^k\big) dr - \int_t^T Z_r^{k+1} dW_r - \int_t^T \int_E \psi_r^{k+1}(e) \widetilde{\mu}(dr, de)$$

with $k \in \mathbb{N}_0$ and $(Y^0, Z^0, \psi^0) \equiv (0,0,0)$. It is easy to see that the map is well-defined. Let

$$\delta Y^{k+1} := Y^{k+1} - Y^k, \quad \delta Z^{k+1} := Z^{k+1} - Z^k, \quad \delta\psi^{k+1} := \psi^{k+1} - \psi^k, \quad \Theta^k := (Y^k, Z^k, \psi^k) \ .$$

We consider the norm $||\cdot||_{\mathcal{K}^2_\beta}$ equivalent to $||\cdot||_{\mathcal{K}^2}$ defined with some $\beta > 0$

$$||(Y, Z, \psi)||^2_{\mathcal{K}^2_\beta} := \mathbb{E}\Big[\sup_{r\in[0,T]} |e^{\beta r} Y_r|^2\Big] + \mathbb{E}\int_0^T |e^{\beta r} Z_r|^2 dr + \mathbb{E}\int_0^T ||e^{\beta r}\psi_r||^2_{\mathbb{L}^2(\nu)} dr \ .$$

Applying Ito formula to $e^{2\beta t}|\delta Y_t^{k+1}|^2$, one obtains for any $t \in [0,T]$

$$\begin{aligned}
&e^{2\beta t}|\delta Y_t^{k+1}|^2 + \int_t^T e^{2\beta r}|\delta Z_r^{k+1}|^2 dr + \int_t^T \int_E e^{2\beta r}|\delta\psi_r^{k+1}(e)|^2 \mu(dr, de) \\
&= \int_t^T e^{2\beta r}\Big(2\delta Y_r^{k+1} \mathbb{E}_{\mathcal{F}_r}\big[f(r, (Y_v^k)_{v\in[r,T]}, \Theta_r^k) - f(r, (Y_v^{k-1})_{v\in[r,T]}, \Theta_r^{k-1})\big] - 2\beta|\delta Y_r^{k+1}|^2\Big) dr \\
&- \int_t^T e^{2\beta r} 2\delta Y_r^{k+1} \delta Z_r^{k+1} dW_r - \int_t^T \int_E e^{2\beta r} 2\delta Y_{r-}^{k+1} \delta\psi_r^{k+1}(e) \widetilde{\mu}(dr, de) \ . \qquad \text{(B.2)}
\end{aligned}$$

For any $\epsilon > 0$, one has

$$\begin{aligned}
&2\delta Y_r^{k+1} \mathbb{E}_{\mathcal{F}_r}\big[f(r, (Y_v^k)_{v\in[r,T]}, \Theta_r^k) - f(r, (Y_v^{k-1})_{v\in[r,T]}, \Theta_r^{k-1})\big] - 2\beta|\delta Y_r^{k+1}|^2 \\
&\leq 2K|\delta Y_r^{k+1}|\Big(2\mathbb{E}_{\mathcal{F}_r}\big[||\delta Y^k||_{[r,T]}\big] + |\delta Z_r^k| + ||\delta\psi_r^k||_{\mathbb{L}^2(\nu)}\Big) - 2\beta|\delta Y_r^{k+1}|^2 \\
&\leq \Big(\frac{6K^2}{\epsilon} - 2\beta\Big)|\delta Y_r^{k+1}|^2 + \epsilon\Big(\mathbb{E}_{\mathcal{F}_r}\big[||\delta Y^k||^2_{[r,T]}\big] + |\delta Z_r^k|^2 + ||\delta\psi_r^k||^2_{\mathbb{L}^2(\nu)}\Big) \ .
\end{aligned}$$

Thus, choosing $\beta = \beta(\epsilon) = 3K^2/\epsilon$ and taking expectation with $t = 0$ yields

$$||e^{\beta\cdot}\delta Z^{k+1}||^2_{\mathbb{H}^2} + ||e^{\beta\cdot}\delta\psi^{k+1}||^2_{\mathbb{J}^2} \leq \epsilon\Big(T||e^{\beta\cdot}\delta Y^k||^2_{\mathbb{S}^2} + ||e^{\beta\cdot\delta}\delta Z^k||^2_{\mathbb{H}^2} + ||e^{\beta\cdot}\delta\psi^k||^2_{\mathbb{J}^2}\Big) \ . \qquad \text{(B.3)}$$

Next, let us apply the BDG inequality (Theorem 48 in IV.4. of [27]) to (B.2). Then there exists some constant C such that

$$\begin{aligned}
&\mathbb{E}\Big[||e^{\beta\cdot}\delta Y^{k+1}||^2_{[0,T]}\Big] \leq \epsilon\Big(T||e^{\beta\cdot}\delta Y^k||^2_{\mathbb{S}^2} + ||e^{\beta\cdot\delta}\delta Z^k||^2_{\mathbb{H}^2} + ||e^{\beta\cdot}\delta\psi^k||^2_{\mathbb{J}^2}\Big) \\
&+C\mathbb{E}\Big[\Big(\int_0^T |e^{\beta r}\delta Y_r^{k+1}|^2 |e^{\beta r}\delta Z_r^{k+1}|^2 dr\Big)^{\frac{1}{2}}\Big] + C\mathbb{E}\Big[\Big(\int_0^T \int_E |e^{\beta r}\delta Y_{r-}^{k+1}|^2 |e^{\beta r}\delta\psi_r^{k+1}(e)|^2 \mu(dr, de)\Big)^{\frac{1}{2}}\Big] \\
&\leq \epsilon\Big(T||e^{\beta\cdot}\delta Y^k||^2_{\mathbb{S}^2} + ||e^{\beta\cdot\delta}\delta Z^k||^2_{\mathbb{H}^2} + ||e^{\beta\cdot}\delta\psi^k||^2_{\mathbb{J}^2}\Big) + \frac{1}{2}\mathbb{E}\Big[||e^{\beta\cdot}\delta Y^{k+1}||^2_{[0,T]}\Big] \\
&+C\Big(||e^{\beta\cdot}\delta Z^{k+1}||^2_{\mathbb{H}^2} + ||e^{\beta\cdot}\delta\psi^{k+1}||^2_{\mathbb{J}^2}\Big) \ .
\end{aligned}$$

Thus, with some constant C (which is independent of ϵ, β),

$$||e^{\beta\cdot}\delta Y^{k+1}||^2_{\mathbb{S}^2} \le 2\epsilon\Big(T||e^{\beta\cdot}\delta Y^k||^2_{\mathbb{S}^2}+||e^{\beta\cdot\delta}\delta Z^k||^2_{\mathbb{H}^2}+||e^{\beta\cdot}\delta\psi^k||^2_{\mathbb{J}^2}\Big)+C\Big(||e^{\beta\cdot}\delta Z^{k+1}||^2_{\mathbb{H}^2}+||e^{\beta\cdot}\delta\psi^{k+1}||^2_{\mathbb{J}^2}\Big)\,.$$

Combining with (B.3), one obtains

$$\big|\big|(\delta Y^{k+1}, \delta Z^{k+1}, \delta\psi^{k+1})\big|\big|^2_{\mathcal{K}^2_{\beta(\epsilon)}} \le \epsilon(C+3)(T\vee 1)\big|\big|(\delta Y^k, \delta Z^k, \delta\psi^k)\big|\big|^2_{\mathcal{K}^2_{\beta(\epsilon)}}$$

and hence by choosing ϵ so that $\epsilon(C+3)(T\vee 1) < 1$ (and $\beta(\epsilon)$ accordingly) makes the map Φ strict contraction with respect to the norm $\mathcal{K}^2_{\beta(\epsilon)}$. This proves the existence as well as the uniqueness. □

C Comparison principle for non-anticipated settings

Consider the two BSDEs with $i = \{1, 2\}$,

$$Y^i_t = \xi_i + \int_t^T \widetilde{f}_i(r, Y^i_r, Z^i_r, \psi^i_r)dr - \int_t^T Z^i_r dW_r - \int_t^T \int_E \psi^i_r(e)\widetilde{\mu}(dr, de) \tag{C.1}$$

for $t \in [0, T]$.

Lemma C.1. *Suppose $(\xi, \widetilde{f}_i)_{1\le i\le 2}$ satisfy Assumptions 3.1, 3.2 and 4.1 of [12], which correspond to Assumptions 3.1, 3.2 and 6.1 of the current paper without the Y's future path dependence, respectively. If $\xi_1 \le \xi_2$ a.s. and $\widetilde{f}_1(r, y, z, \psi) \le \widetilde{f}_2(r, y, z, \psi)$ $d\mathbb{P}\otimes dr$-a.e. for every $(y, z, \psi) \in \mathbb{R}\times\mathbb{R}^{1\times d}\times\mathbb{L}^2(E, \nu)$, then $Y^1_t \le Y^2_t$ $\forall t \in [0, T]$ a.s.*

Proof. One can prove it in the same way as Theorem 2.5 of [29]. By Theorem 4.1 [12], there exists a unique solution $(Y^i, Z^i, \psi^i)_{1\le i\le 2} \in \mathbb{S}^\infty\times\mathbb{H}^2_{BMO}\times\mathbb{J}^2_{BMO}$ to the BSDEs (C.1) satisfying the universal bounds. Let us put $\delta Y := Y^1 - Y^2$, $\delta Z := Z^1 - Z^2$, $\delta\psi := \psi^1 - \psi^2$, $\delta\widetilde{f}(r) := (\widetilde{f}_1 - \widetilde{f}_2)(r, Y^1_r, Z^1_r, \psi^1_r)$. We also introduce the two progressively measurable processes $(a_r)_{r\in[0,T]}$, $(b_r)_{r\in[0,T]}$ given by

$$a_r := \frac{\widetilde{f}_2(r, Y^1_r, Z^1_r, \psi^1_r) - \widetilde{f}_2(r, Y^2_r, Z^1_r, \psi^1_r)}{\delta Y_r}\mathbf{1}_{\delta Y_r\ne 0},\ b_r := \frac{\widetilde{f}_2(r, Y^2_r, Z^1_r, \psi^1_r) - \widetilde{f}_2(r, Y^2_r, Z^2_r, \psi^1_r)}{|\delta Z_r|^2}\mathbf{1}_{\delta Z_r\ne 0}\delta Z_r^\top.$$

Note that $a \in \mathbb{S}^\infty$ and $b \in \mathbb{H}^2_{BMO}$ due to the universal bounds and the local Lipschitz continuity. By Assumption 4.1 of [12], which is the A_Γ-condition, there exists a $\mathbb{P}\otimes\mathcal{E}$-measurable process Γ such that

$$\begin{aligned}\delta Y_t \;\le\; & \delta\xi + \int_t^T \Big(\delta\widetilde{f}(r) + a_r\delta Y_r + b_r Z_r + \int_{\mathbb{E}}\Gamma_r(e)\delta\psi_r(e)\nu(de)\Big)dr \\ & -\int_t^T \delta Z_r dW_r - \int_t^T\int_E \delta\psi_r(e)\widetilde{\mu}(dr, de)\end{aligned} \tag{C.2}$$

satisfying $C_1(1\wedge|e|) \le |\Gamma(e)| \le C_2(1\wedge|e|)$ with some constant $C_1 > -1$ and $C_2 \ge 0$. Here the fact that $Y^i \in \mathbb{S}^\infty, \psi^i \in \mathbb{J}^\infty$ was used. Since $\mathcal{M} := \int_0^\cdot b_r^\top dW_r + \int_0^\cdot\int_E \Gamma_r(e)\widetilde{\mu}(dr, de)$ is a BMO-martingale with jump size strictly bigger than -1, one can define an equivalent measure $\mathbb{Q}$ by $d\mathbb{Q}/d\mathbb{P} = \mathcal{E}_T(\mathcal{M})$. Thus one obtains from (C.2)

$$\delta Y_t \le \mathbb{E}^{\mathbb{Q}}_{\mathcal{F}_t}\Big[e^{R_{t,T}}\delta\xi + \int_t^T e^{R_{t,r}}\delta\widetilde{f}(r)dr\Big]$$

with $R_{t,s} := \int_t^s a_r dr$. This proves the claim. $\square$

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