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Hedging and Pricing Illiquid Options with Market Impacts

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Abstract

In this paper, we consider hedging and pricing of illiquid options on an untradable underlying asset, where an alternative asset is used as a hedging instrument. Particularly, we consider the situation where the trade price of the hedging instrument is subject to market impacts caused by the hedger and the liquidity costs paid as a spread from the mid price. Pricing illiquid options, which often appears in trading of structured products, is a critical issue in practice because of its difficulties in hedging mainly due to untradablity of the underlying asset as well as the liquidity costs and market impacts of the hedging instrument. Firstly, by setting the problem under a discrete time model, where the optimal hedging strategy is defined by the local risk-minimization, we present algorithms to obtain the option price along with the hedging strategy by an asymptotic expansion. Moreover, we provide numerical examples. This model enables the estimation of the effect of both the market impacts and the liquidity costs on option prices, which is important in practice.

Keywords: Incomplete market, Derivatives pricing, Market impact

1 Introduction

Pricing illiquid options, such as options on illiquid commodities, foreign exchange rates, and unlisted stocks, is an important but difficult problem in practice. These options usually appear as a part of structured derivative products, where an alternative asset with similar price movements is used as a hedging instrument instead of the untradable underlying asset. For example, options on Indonesian oil are hedged with WTI future which is subject to market impacts and liquidity costs. Moreover, options on BRLJPY (Brazilian real against JPY) are hedged by BRLUSD (Brazilian real against U.S. dollar) which also accompanies market impacts and liquidity costs. However, in most cases, the liquidity costs paid as a spread from the mid price and the market impacts caused by the hedging activities of the hedger are not negligible in trading, and estimation of these effects is vital in the risk management of banks.

Taking these into account, we consider a hedging and pricing problem of illiquid options, where the underlying asset is untradable and the trade price of the alternative asset is subject to the liquidity costs and the market impacts caused by the hedger. There are only a few literatures that deal with derivatives pricing with market impacts and liquidity costs because of its difficulties in solving the associated non-linear PDE. (For instance, Li and Almgren [13]

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studies an intraday hedging problem of an option, where it considers minimization of the variance of the portfolio value at the end of the day together with the liquidity costs. Saito and Takahashi [19] extends Li and Almgren [13] to the pricing problem of derivatives of the entire period, which is from the trade date to maturity. Guéant and Pu [11] considers maximization of an expected utility at maturity. All the works reduce the problems to solving a HJB equation.) They deal with options on the underlying asset with some liquidity, that is, the underlying asset is still tradable. Unlike these works, our study deals with illiquid options whose underlying asset is untradable and hedged with a correlated underlying asset, which is a more difficult issue in practice. Moreover, we consider the case where the market impact parameter, a proportionality factor determining the size of the market impact, deterministically changes over time, while the related literatures assume the parameter to be a constant. This enables us to estimate the effect of the market impact on the derivatives prices depending on the economic scenarios, which is particularly meaningful after the financial crisis.

Firstly, we set the problem under a discrete time model, where the optimal hedging strategy is defined by the local risk-minimization. Then, we provide algorithms to obtain the hedging strategy as well as the option price, showing that the strategy is uniquely determined by backward induction when the market impact parameter is a constant and the first order asymptotic expansion of the strategy is obtained when the parameter is time-dependent.

Following the fundamental works on modeling of limit order books and its application to the optimal execution problems, (see Alfonsi et al. [2],[3],[4], Bertsimas and Lo [9], Predoiu et al. [15], Almgren and Chriss [5],[6], Bank and Baum [7], Chen et al. [10], Obizhaeva and Wang [14], Roch and Soner [17], for example), we assume a block shape for the limit order book of the hedging instrument, where the spread from the mid price is proportional to the trade volume. We also assume that the hedger is a sole large trader whose hedging activity causes market impacts on the price of the hedging instrument.

Moreover, we apply the local risk-minimization to the derivatives pricing with market impacts and liquidity costs because of its mathematical tractability to work on intricate optimality problems. (For other applications of the local risk-minimization to different topics in finance, see Schweizer [20], Lamberton et al. [12] for option pricing with transaction costs and Barbarin [8] for an application to an insurance problem.) This optimality criterion aims to find a locally optimal strategy that minimizes a quadratic error in the hedging at each time point.

The paper is organized as follows: after the next section introduces the local risk-minimization in discrete time, Section 3 shows a recursive procedure to obtain the local risk-minimizing strategy in the case of a constant market impact parameter. Section 4 presents backward induction to derive the hedging strategy in the case of a time-dependent market impact parameter. In particular, the exact strategy and the first order expansion are obtained for cash settlement and physical settlement, respectively. Section 5 provides numerical examples in the case of a time-dependent market impact parameter for both cash and physical settlements. Finally, Section 6 concludes. Appendices provide a supplement of Section 5.2 and proofs of a lemma, proposition, and theorem, as well as calculation processes of Section 4.

2 The model

In this section, we introduce the local risk-minimization in discrete time, where liquidity costs and market impacts in the hedging instrument are considered. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $\{\mathcal{F}_k\}_{k=0,1,\dots,T}$ be a filtration. Let $L^p(\Omega)$ be the set of p-th integrable real-valued random variables for $p \geq 1$. We assume an economy that consists of a money market account and a hedging instrument with time points $k = 0, 1, \dots, T$. Let $\eta_k \in L^2(\Omega)$ and $\theta_{k+1} \in L^4(\Omega)$ $(k = 0, 1, \dots, T)$ be $\mathcal{F}_k$-measurable random variables, which represent the position of the money market account and the hedging instrument at time k, respectively. Here, the positive and the negative sign of η and θ indicate a long position and a short position, respectively. We call the set of the pairs $\phi := \{(\eta_k, \theta_{k+1})\}_{k=0,1,\dots,T}$ *a trading strategy*. We assume the interest rate to be zero for simplicity, which means that the price of the money market account is always 1. Hereafter, we denote $X_k - X_{k-1}$ by ΔX_k $(k = 1, \dots, T)$ for any process X, and set $\Delta\theta_1 = \theta_1$.

2.1 Trade price and market impact

Firstly, we model the liquidity cost and the market impact in the hedging instrument. Let $S(k, \Delta\theta_{k+1})$ be the trade price of the hedging instrument when the hedger buys an amount $\Delta\theta_{k+1}$ at k $(k = 0, \dots, T)$. Let $\{\tilde{S}_k\}_{k=0,\dots,T} \subset L^2(\Omega)$ be a $\mathcal{F}_k$-adapted process, which we call *the unaffected price process*. This price process is a hypothetical mid-price process in the case where there is no market impact when the hedger does not trade any hedging instruments. We assume that the hedger is the only large trader who causes market impacts on the hedging instrument. Let M and $\lambda_0, \lambda_1, \dots, \lambda_{T-1}$ be positive constants, and we denote the sequence $\{\lambda\}_{k=0,\dots,T-1}$ by λ. We define $S(k, \Delta\theta_{k+1})$ as follows.

$$S(k, \Delta\theta_{k+1}) = S(k, 0) + M\Delta\theta_{k+1}, \; k = 0, \dots, T, \tag{1}$$

$$S(0, 0) = \tilde{S}_0, \tag{2}$$

$$S(k, 0) = \tilde{S}_k + \sum_{j=1}^{k} 2\lambda_{j-1} M\Delta\theta_j, \; k = 1, \dots, T. \tag{3}$$

Equation (1) explains that when the hedger buys $\Delta\theta_{k+1}$ of the hedging instrument at time k, the average price $S(k, \Delta\theta_{k+1})$ is the sum of the mid price $S(k, 0)$ and $M\Delta\theta_{k+1}$, which is proportional to the the trade volume. We observe that this corresponds to the case of the limit order book with uniform order density $\frac{1}{2M}$. In fact, if the hedger buys $\Delta\theta_{k+1}$ of the hedging instrument in the order book, the hedger takes the offer orders with the price from $S(k, 0)$ to $S(k, 0) + 2M\Delta\theta_{k+1}$. In this case, the hedger results in buying θ_{k+1} of the hedging instrument with the average price $S(k, 0) + M\Delta\theta_{k+1}$, since

$$\frac{1}{\Delta\theta_{k+1}} \int_{S(k,0)}^{S(k,0)+2M\Delta\theta_{k+1}} x \frac{1}{2M} dx = S(k, 0) + M\Delta\theta_{k+1}. \tag{4}$$

Equation (3) shows the market impacts by the hedger on the mid price of the hedging instrument. By (3), we have

$$\Delta S(k, 0) = \Delta\tilde{S}_k + 2\lambda_{k-1} M\Delta\theta_k, \; k = 1, \dots, T. \tag{5}$$

This implies that the change in the mid price $\Delta S(k, 0)$ consists of the change in the unaffected price $\Delta\tilde{S}_k$ and the market impact $2\lambda_{k-1}M\Delta\theta_k$ caused by the hedger, where $2M\Delta\theta_k$ is the spread between the mid price and the maximum offer price at $k - 1$. (For the general relation between the shape of the limit order book and the trade price, see Saito [18], for instance.)

2.2 Local risk-minimization in discrete time

Secondly, we define the local risk-minimization in discrete time. We assume the following for the option payoff of the untradable underlying asset. Let $\bar{\eta}_T, \bar{\theta}_{T+1}$ be $\mathcal{F}_T$-measurable random variables, which are independent of $\{\theta_{k+1}\}_{k=0,\dots,T-1}$. We call $(\bar{\eta}_T, \bar{\theta}_{T+1})$ *an option payoff* at T.

Let $\{A_k\}_{k=0,\ldots,T}$ be a $\mathcal{F}_k$-adapted process, which is the price process of the untradable underlying asset. For example, we can consider a call option payoff in cash settlement $(\bar{\eta}_T, \bar{\theta}_{T+1})$,

$$\begin{aligned}\bar{\eta}_T &= \max(A_T - K_A, 0),\\ \bar{\theta}_{T+1} &= 0,\end{aligned} \tag{6}$$

where $K_A > 0$. This option pays $A_T - K_A$ in cash at T only when $A_T \geq K_A$.

Another example is an option payoff in physical settlement $(\bar{\eta}_T, \bar{\theta}_{T+1})$, where

$$\begin{aligned}\bar{\eta}_T &= -K_S 1_{\{A_T \geq K_A\}},\\ \bar{\theta}_{T+1} &= 1_{\{A_T \geq K_A\}},\end{aligned} \tag{7}$$

and $K_A, K_S > 0$. This option delivers one unit of the hedging instrument at a price K_S at T only when $A_T \geq K_A$. Here, the price of the untradable underlying asset is used as a reference price to determine the option payoff. Hereafter, we consider trading strategies $\phi = \{(\eta_k, \theta_{k+1})\}_{k=0,1,\ldots,T}$ that satisfy $\eta_T = \bar{\eta}_T$ and $\theta_{T+1} = \bar{\theta}_{T+1}$, which means that the hedger matches the portfolio (η_T, θ_{T+1}) with the option payoff at maturity.

Next, we define the value process, cost process, and perturbation of a trading strategy.

Definition 1. *The value process of a trading strategy* $\phi = \{(\eta_k, \theta_{k+1})\}_{k=0,1,\ldots,T}$ *is a* $\mathcal{F}_k$*-adapted process satisfying*

$$V_k(\phi) = \eta_k + \theta_{k+1} S(k, 0), \ k = 0, \ldots, T. \tag{8}$$

Definition 2. *The cost process of a trading strategy* ϕ *is a* $\mathcal{F}_k$*-adapted process satisfying*

$$\begin{aligned}C_k(\phi) &= V_k(\phi) - \sum_{j=1}^{k} \theta_j \Delta\tilde{S}_j - \sum_{j=1}^{k} 2\lambda_{j-1} M \theta_j \Delta\theta_j + \sum_{j=1}^{k} M(\Delta\theta_{j+1})^2\\ &= V_k(\phi) - \left(\sum_{j=1}^{k} \theta_j \Delta S(j, 0) - \sum_{j=1}^{k} M(\Delta\theta_{j+1})^2\right), \ k = 1, \ldots, T,\end{aligned} \tag{9}$$

$$C_0(\phi) = V_0(\phi). \tag{10}$$

Definition 3. *A perturbation of a trading strategy* $\phi = \{(\eta_j, \theta_{j+1})\}_{j=0,1,\ldots,T}$ *at time* k $(k = 0, \ldots, T)$ *is a trading strategy* $\tilde{\phi} = \{(\tilde{\eta}_j, \tilde{\theta}_{j+1})\}_{j=0,1,\ldots,T}$ *satisfying*

$$\eta_j = \tilde{\eta}_j \ , j = 0, \ldots, k-1, k+1, \ldots, T, \tag{11}$$

$$\theta_{j+1} = \tilde{\theta}_{j+1} \ , j = 0, \ldots, k-1, k+1, \ldots, T. \tag{12}$$

Equation (8) shows that V_k is a mid mark-to-market of the portfolio at k. We also interpret

$$\sum_{j=1}^{k} \theta_j \Delta S(j, 0) - \sum_{j=1}^{k} M(\Delta\theta_{j+1})^2 \tag{13}$$

in (9) as the trading gain of the hedging instrument from time 0 to k together with the liquidity cost paid. The perturbation of a trading strategy ϕ at k in (11) and (12) is a trading strategy that has the same portfolio profile as the strategy ϕ except for at k.

Then, we define the risk process of a trading strategy and the local risk-minimizing strategy.

Definition 4. *The risk process of a trading strategy ϕ is a $\mathcal{F}_k$-adapted process $\{R_k(\phi)\}_{k=0,1,\dots,T}$ satisfying*

$$R_k(\phi) = \mathbf{E}\Big[(C_T(\phi) - C_k(\phi))^2 \Big| \mathcal{F}_k\Big], \quad k = 0, 1, \dots, T. \tag{14}$$

Definition 5. *A trading strategy ϕ is called the local risk-minimizing strategy if for all $k = 0, \dots, T$, for any perturbation $\tilde{\phi}$ of ϕ at time k,*

$$R_k(\phi) \le R_k(\tilde{\phi}), \ \mathbf{P} - a.s. \tag{15}$$

These definitions have the following interpretation. Suppose that the hedger trades by self-financing starting from the position (η_k, θ_{k+1}) at k ($k = 0, \dots, T$). At $k+1$, the hedger buys $\Delta\theta_{k+2}$ of the hedging instrument spending cash of $\Delta\theta_{k+2} S(k+1, \Delta\theta_{k+2})$. Continuing the trading until $k = T - 1$ and buying $\Delta\theta_{T+1} = \bar{\theta}_{T+1} - \theta_T$ of the hedging instrument at maturity T, the hedger ends up with the portfolio

$$\left(\eta_k - \sum_{j=k+1}^{T} \Delta\theta_{j+1} S(j, \Delta\theta_{j+1}), \ \bar{\theta}_{T+1}\right). \tag{16}$$

The difference between the portfolio and the option payoff is

$$\bar{\eta}_T - \left(\eta_k - \sum_{j=k+1}^{T} \Delta\theta_{j+1} S(j, \Delta\theta_{j+1})\right) \tag{17}$$

in the money market account, which can be rewritten as

$$\begin{aligned}
&\bar{\eta}_T - \left(\eta_k - \sum_{j=k+1}^{T} \Delta\theta_{j+1} S(j, \Delta\theta_{j+1})\right) \\
&= \bar{\eta}_T - \eta_k + \sum_{j=k+1}^{T} \Delta\theta_{j+1} S(j, 0) + \sum_{j=k+1}^{T} M(\Delta\theta_{j+1})^2 \\
&= \left(\bar{\eta}_T + \bar{\theta}_{T+1} S(T, 0)\right) - (\eta_k + \theta_{k+1} S(k, 0)) - \sum_{j=k+1}^{T} \theta_j \Delta S(j, 0) + \sum_{j=k+1}^{T} M(\Delta\theta_{j+1})^2 \\
&= \left(\bar{\eta}_T + \bar{\theta}_{T+1} S(T, 0)\right) - (\eta_k + \theta_{k+1} S(k, 0)) - \sum_{j=k+1}^{T} \theta_j \Delta \tilde{S}_j \\
&\quad - \sum_{j=k+1}^{T} 2\lambda_{j-1} M \theta_j \Delta\theta_j + \sum_{j=k+1}^{T} M(\Delta\theta_{j+1})^2 \\
&= C_T(\phi) - C_k(\phi).
\end{aligned} \tag{18}$$

Then, the risk process in (14) is equivalent to the conditional expectation of the quadratic hedging error of the self-financing strategy. Moreover, the local risk-minimizing strategy defined by (15) is a trading strategy that minimizes the hedging error of the corresponding self-financing strategy among all the perturbations at any time point k.

Finally, we introduce the following Lemma that characterizes the property of the local risk-minimizing strategy. As in the case without market impact in Lamberton et al. [12], the martingale property of the cost process of the local risk-minimizing strategy holds. We provide the proof in Appendix B.

Lemma 1. *If ϕ is a local risk-minimizing strategy, then $\{C_k(\phi)\}_{k=0,1,\dots,T}$ is a martingale.*

3 The case where λ is a constant

In this section, we explicitly solve for a local risk-minimizing strategy in the case where the market impact parameter λ is a constant. Firstly, we introduce a proposition that presents an equivalent definition of the local risk-minimizing strategy when λ is a constant. We give the proof in Appendix C.

Proposition 1. *Assume that $\lambda_0 = \lambda_1 = \cdots = \lambda_{T-1}$ in (1). Then ϕ is a local risk-minimizing strategy if and only if*

(i) $\{C_k(\phi)\}_{k=0,1,\dots,T}$ is a martingale,

(ii) for all $k = 0, 1, \dots, T-2$,

$$\mathrm{Var}\Big[C_{k+2}(\phi) - C_k(\phi)\Big|\mathcal{F}_k\Big] = \min_{\tilde{\phi}\in\Gamma_k} \mathrm{Var}\Big[C_{k+2}(\tilde{\phi}) - C_k(\tilde{\phi})\Big|\mathcal{F}_k\Big], \tag{19}$$

$$\mathrm{Var}\Big[C_T(\phi) - C_{T-1}(\phi)\Big|\mathcal{F}_k\Big] = \min_{\tilde{\phi}\in\Gamma_{T-1}} \mathrm{Var}\Big[C_T(\tilde{\phi}) - C_{T-1}(\tilde{\phi})\Big|\mathcal{F}_{T-1}\Big], \tag{20}$$

where Γ_k is a set of perturbations of ϕ at time k.

This proposition shows that the local risk-minimizing strategy defined by (14) and (15) also minimizes the conditional variance of the two-period difference of the cost process. From Proposition 1, the local risk-minimizing strategy is determined uniquely by the following backward induction. (See Appendix D for details.)

$$\begin{aligned}\theta_T =&\mathrm{Cov}\Big[\bar{\eta}_T + \bar{\theta}_{T+1}\tilde{S}_T + M\bar{\theta}^2_{T+1}, \Delta\tilde{S}_T + 2M\bar{\theta}_{T+1} - 2\lambda M\bar{\theta}_{T+1}\Big|\mathcal{F}_{T-1}\Big]\\ &/\mathrm{Var}\Big[\Delta\tilde{S}_T + 2M\bar{\theta}_{T+1} - 2\lambda M\bar{\theta}_{T+1}\Big|\mathcal{F}_{T-1}\Big].\end{aligned} \tag{21}$$

For $k = 0, \dots, T-2$,

$$\begin{aligned}&\theta_{k+1}\\ &= \mathrm{Cov}\Big[\eta_{k+2} + \theta_{k+3}(\tilde{S}_{k+2} + 2\lambda M\theta_{k+2}) - \theta_{k+2}\Delta\tilde{S}_{k+2} - 2\lambda M\theta^2_{k+2} + M(\Delta\theta_{k+3})^2 + M\theta^2_{k+2},\\ &\quad \Delta\tilde{S}_{k+1} - 2\lambda M\theta_{k+2} + 2M\theta_{k+2}\Big|\mathcal{F}_k\Big]\\ &\quad /\mathrm{Var}\Big[\Delta\tilde{S}_{k+1} - 2\lambda M\theta_{k+2} + 2M\theta_{k+2}\Big|\mathcal{F}_k\Big].\end{aligned} \tag{22}$$

$$\begin{aligned}\eta_k = \mathbf{E}\Big[&\eta_{k+1} + \theta_{k+2}(\tilde{S}_{k+1} + 2\lambda M\theta_{k+1}) - \theta_{k+1}\Delta\tilde{S}_{k+1} - 2\lambda M\theta_{k+1}\Delta\theta_{k+1} + M(\Delta\theta_{k+2})^2\Big|\mathcal{F}_k\Big]\\ &- \theta_{k+1}S(k,0).\end{aligned} \tag{23}$$

1. Calculate θ_T by (21).

2. For given $\eta_{k+2}, \theta_{k+2}, \theta_{k+3}$ $(k = 0, \dots, T-2)$, calculate θ_{k+1} by (22).

3. For given $\eta_{k+1}, \theta_{k+2}, \theta_{k+1}, \theta_k$, calculate η_k $(k = 0, \dots, T-1)$ by (23).

The next theorem indicates that the strategy obtained by the backward induction is a unique local risk-minimizing strategy. We also give the proof in Appendix D.

Theorem 1. *Let $\{\theta_j\}_{j=1,\dots,T}$ be a sequence of random variables recursively defined by (21),(22) satisfying*

$$\mathrm{Var}\Big[\Delta\tilde{S}_{k+1} - 2\lambda M\theta_{k+2} + 2M\theta_{k+2}\Big|\mathcal{F}_k\Big] \neq 0, \tag{24}$$

$$\theta_{k+1} \in L^4(\Omega), (k = 0, 1, \dots, T), \tag{25}$$

$$\tilde{S}_k\theta_j \in L^2(\Omega) \text{ for all } k = 0, 1, \dots, T \text{ and } j = 1, \dots, T+1 \tag{26}$$

at each step. Then $\{(\eta_k, \theta_{k+1})\}_{k=0,\dots,T}$ is a unique local risk-minimizing strategy.

4 The case where λ is time-dependent

Finally, we solve for a local risk-minimization in the case where the market impact parameter λ is time-dependent, which corresponds to a situation where the degree of the market impact varies over time due to economic environment changes.

We first show the equations that $\{\theta_{k+1}\}_{k=0,\dots,T-1}$, the position of the hedging instrument in the local risk-minimizing strategy, satisfies. Then, we shall observe that a unique local risk-minimizing strategy is obtained in the case of cash settlement. In physical settlement, although the strategy is not determined uniquely, considering the first order expansion with respect to λ around $\lambda = 0$, which corresponds to the no market impact case, the approximate strategy is obtained by backward induction.

From Proposition 1, the equations that $\{\theta_{k+1}\}_{k=0,\dots,T-1}$ satisfies are obtained as follows. (See Appendices E & F for details.)

$$\begin{aligned}\theta_T =&\mathrm{Cov}\Big[\bar{\eta}_T + \bar{\theta}_{T+1}(\tilde{S}_T - 2\sum_{j=1}^{T-1}\Delta\lambda_j M\theta_j) + M\bar{\theta}^2_{T+1}, \Delta\tilde{S}_T + 2M\bar{\theta}_{T+1} - 2\lambda_{T-1}M\bar{\theta}_{T+1}\Big|\mathcal{F}_{T-1}\Big]\\ &/\mathrm{Var}\Big[\Delta\tilde{S}_T + 2M\bar{\theta}_{T+1} - 2\lambda_{T-1}M\bar{\theta}_{T+1}\Big|\mathcal{F}_{T-1}\Big].\end{aligned} \tag{27}$$

For $k = 0, \dots, T-2$,

$$\begin{aligned}\theta_{k+1} = \mathrm{Cov}\Big[&\bar{\eta}_T + \bar{\theta}_{T+1}(\tilde{S}_T + 2\lambda_{T-1}M\theta_T - 2\sum_{j=1,j\neq k+1}^{T-1}\Delta\lambda_j M\theta_j) - \sum_{j=k+2}^{T}\theta_j\Delta\tilde{S}_j\\ &- \sum_{j=k+3}^{T}2\lambda_{j-1}M\theta_j\Delta\theta_j - 2\lambda_{k+1}M\theta^2_{k+2} + \sum_{j=k+2}^{T}M(\Delta\theta_{j+1})^2 + M\theta^2_{k+2},\\ &\Delta\tilde{S}_{k+1} + 2\Delta\lambda_{k+1}M\bar{\theta}_{T+1} - 2\lambda_{k+1}M\theta_{k+2} + 2M\theta_{k+2}\Big|\mathcal{F}_k\Big]\\ &/\mathrm{Var}\Big[\Delta\tilde{S}_{k+1} + 2\Delta\lambda_{k+1}M\bar{\theta}_{T+1} - 2\lambda_{k+1}M\theta_{k+2} + 2M\theta_{k+2}\Big|\mathcal{F}_k\Big].\end{aligned} \tag{28}$$

Notice that in (28), since θ_{k+1} includes $\theta_1, \dots, \theta_k$ in its expression, $\{\theta_{k+1}\}_{k=0,\dots,T-1}$ cannot be obtained by backward induction. This dependence on the historical positions comes from to the expression of the accumulated market impact $\sum_{j=1}^{k} 2\lambda_{j-1}M\Delta\theta_j$ of $S(k, 0)$ in (3). In the case where λ is a constant, the accumulated market impact becomes $2\lambda M\theta_k$ by cancellation and $\theta_1, \dots, \theta_k$ do not appear in the expression of θ_{k+1} as in (22).

4.1 Cash settlement

In cash settlement, since the position of the hedging instrument is cleared at maturity, $\bar{\theta}_{T+1} = \theta_{T+1} = 0$. Substituting $\theta_{T+1} = 0,\ \eta_T = \bar{\eta}_T$ in (27) and (28), we have

$$\theta_T = \frac{\mathrm{Cov}\Big[\bar{\eta}_T, \Delta\tilde{S}_T \Big| \mathcal{F}_{T-1}\Big]}{\mathrm{Var}\Big[\Delta\tilde{S}_T \Big| \mathcal{F}_{T-1}\Big]}, \tag{29}$$

$$\begin{aligned}\theta_{k+1} = \mathrm{Cov}\Big[&\bar{\eta}_T - \sum_{j=k+2}^{T} \theta_j \Delta\tilde{S}_j - \sum_{j=k+3}^{T} 2\lambda_{j-1} M\theta_j \Delta\theta_j - 2\lambda_{k+1} M\theta_{k+2}^2 + \sum_{j=k+2}^{T} M(\Delta\theta_{j+1})^2 + M\theta_{k+2}^2,\\ &\Delta\tilde{S}_{k+1} - 2\lambda_{k+1} M\theta_{k+2} + 2M\theta_{k+2} \Big| \mathcal{F}_k\Big]\\ &/\mathrm{Var}\Big[\Delta\tilde{S}_{k+1} - 2\lambda_{k+1} M\theta_{k+2} + 2M\theta_{k+2} \Big| \mathcal{F}_k\Big],\ (k = 0, \ldots, T-2).\end{aligned} \tag{30}$$

Since θ_{k+1} does not depend on $\theta_1, \ldots, \theta_k$, $\{\theta_{k+1}\}_{k=0,\ldots,T-1}$ is obtained in the following manner.

1. Calculate θ_T by (29).
2. For given $\{\theta_j\}_{j=k+2,\ldots,T+1}$, calculate θ_{k+1} by (30). $(k = 0, \ldots, T-2)$
3. Calculate $\{\eta_k\}_{k=0,1,\ldots,T-1}$ by

$$\eta_k = \mathbf{E}\Big[\bar{\eta}_T - \sum_{j=k+1}^{T} \theta_j \Delta\tilde{S}_j - \sum_{j=k+1}^{T} 2\lambda_{j-1} M\theta_j \Delta\theta_j + \sum_{j=k+1}^{T} M(\Delta\theta_{j+1})^2 \Big| \mathcal{F}_k\Big] - \theta_{k+1} S(k, 0), \tag{31}$$

which follows from the martingale property of $\{C_k(\phi)\}_{k=0,1,\ldots,T}$ in Lemma 1 and (9).

4.2 Physical settlement

In physical settlement, since $\bar{\theta}_{T+1} = \theta_{T+1} \neq 0$, the simultaneous equations (27), (28) cannot be reduced to a solvable form. However, we can obtain the first order expansion of the local risk-minimizing strategy in a recursive manner if we expand the strategy with respect to $\{\lambda_i\}_{i=0,1,\ldots,T-1}$ around $\lambda_0 = \cdots = \lambda_{T-1} = 0$. We assume differentiability of $\theta_j\ (j = 1, \ldots, T)$ with respect to $\lambda_i\ (i = 0, \ldots, T-1)$.

Expanding θ_j with respect to $\{\lambda_i\}_{i=0,1,\ldots,T-1}$ around $\lambda_0 = \cdots = \lambda_{T-1} = 0$, we have

$$\theta_j = \sum_{m_0,\ldots,m_{T-1}=0}^{\infty} c^j_{m_0,\ldots,m_{T-1}} \lambda_0^{m_0} \ldots \lambda_{T-1}^{m_{T-1}}, \tag{32}$$

where

$$c^j_{m_0,\ldots,m_{T-1}} = \frac{1}{m_0! \ldots m_{T-1}!} \frac{\partial^{m_0+\cdots+m_{T-1}} \theta_j}{\partial\lambda_0^{m_0} \ldots \partial\lambda_{T-1}^{m_{T-1}}}\Big|_{\lambda_0=0,\ldots,\lambda_{T-1}=0}. \tag{33}$$

Let $\{\tilde{\theta}_{k+1}\}_{k=0,\ldots,T-1}$ be the first order expansion of $\{\theta_{k+1}\}_{k=0,\ldots,T-1}$ in (32) defined by

$$\tilde{\theta}_{k+1} = \theta_{k+1}^{(0)} + \sum_{i=1}^{T-1} \partial_{\lambda_i} \theta_{k+1}^{(0)} \lambda_i, \tag{34}$$

where

$$\theta_j^{(0)} = c^j_{0,\dots,0} = \theta_j\big|_{\lambda_0=0,\dots,\lambda_{T-1}=0}\ (j=1,\dots,T), \tag{35}$$

$$\partial_{\lambda_i}\theta_j^{(0)} := \frac{\partial\theta_j}{\partial\lambda_i}\big|_{\lambda_0=0,\dots,\lambda_{T-1}=0},\ i=0,\dots,T-1,\ j=1,\dots,T. \tag{36}$$

Then $\{\tilde{\theta}_{k+1}\}_{k=0,\dots,T-1}$ is calculated as follows. (See Appendices G & H for details.)

$$\begin{aligned}
\tilde{\theta}_T = & \frac{\mathrm{Cov}\Big[\bar{\eta}_T + \bar{\theta}_{T+1}\tilde{S}_T + M\bar{\theta}^2_{T+1}, \Delta\tilde{S}_T + 2M\bar{\theta}_{T+1}\Big|\mathcal{F}_{T-1}\Big]}{\mathrm{Var}\Big[\Delta\tilde{S}_T + 2M\bar{\theta}_{T+1}\Big|\mathcal{F}_{T-1}\Big]} \\
& + \Big\{\mathrm{Cov}[\bar{\theta}_{T+1}(-2\sum_{j=1}^{T-1}\Delta\lambda_j M\theta_j^{(0)}), \Delta\tilde{S}_T + 2M\bar{\theta}_{T+1}\Big|\mathcal{F}_{T-1}\Big] \\
& \quad + \mathrm{Cov}\Big[\bar{\eta}_T + \bar{\theta}_{T+1}\tilde{S}_T + M\bar{\theta}^2_{T+1}, -2\lambda_{T-1}M\bar{\theta}_{T+1}\Big|\mathcal{F}_{T-1}\Big]\Big\} \\
& \quad \Big/ \mathrm{Var}[\Delta\tilde{S}_T + 2M\bar{\theta}_{T+1}\Big|\mathcal{F}_{T-1}\Big] \\
& - 2\mathrm{Cov}\Big[-2\lambda_{T-1}M\bar{\theta}_{T+1}, \Delta\tilde{S}_T + 2M\bar{\theta}_{T+1}\Big|\mathcal{F}_{T-1}\Big] \\
& \quad \times \mathrm{Cov}\Big[\bar{\eta}_T + \bar{\theta}_{T+1}\tilde{S}_T + M\bar{\theta}^2_{T+1}, \Delta\tilde{S}_T + 2M\bar{\theta}_{T+1}\Big|\mathcal{F}_{T-1}\Big] \\
& \quad \Big/ \mathrm{Var}\Big[\Delta\tilde{S}_T + 2M\bar{\theta}_{T+1}\Big|\mathcal{F}_{T-1}\Big]^2.
\end{aligned} \tag{37}$$

For $k=0,\dots,T-2$,

$$
\begin{aligned}
&\tilde{\theta}_{k+1} \\
&= \frac{\mathrm{Cov}\Big[\bar{\eta}_T + \bar{\theta}_{T+1}\tilde{S}_T - \sum_{j=k+2}^{T}\theta_j^{(0)}\Delta\tilde{S}_j + \sum_{j=k+2}^{T} M(\Delta\theta_{j+1}^{(0)})^2 + M\theta_{k+2}^{(0)2}, \Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}\Big|\mathcal{F}_k\Big]}{\mathrm{Var}\Big[\Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}\Big|\mathcal{F}_k\Big]} \\
&+ \Big\{\mathrm{Cov}\Big[\bar{\theta}_{T+1}(2\lambda_{T-1}M\theta_T^{(0)} - 2\sum_{j=1, j\neq k+1}^{T-1}\Delta\lambda_j M\theta_j^{(0)}) - \sum_{j=k+2}^{T}(\sum_{i=0}^{T-1}\partial_{\lambda_i}\theta_j^{(0)}\lambda_i)\Delta\tilde{S}_j - \sum_{j=k+3}^{T} 2\lambda_{j-1}M\theta_j^{(0)}\Delta\theta_j^{(0)} \\
&\quad - 2\lambda_{k+1}M\theta_{k+2}^{(0)2} + 2\sum_{j=k+2}^{T} M\Delta\theta_{j+1}^{(0)}(\sum_{i=0}^{T-1}(\partial_{\lambda_i}\theta_{j+1}^{(0)} - \partial_{\lambda_i}\theta_j^{(0)})\lambda_i) + 2M\theta_{k+2}^{(0)}(\sum_{i=0}^{T-1}\partial_{\lambda_i}\theta_{k+2}^{(0)}\lambda_i), \\
&\quad \Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}\Big|\mathcal{F}_k\Big] \\
&+ \mathrm{Cov}\Big[\bar{\eta}_T + \bar{\theta}_{T+1}\tilde{S}_T - \sum_{j=k+2}^{T}\theta_j^{(0)}\Delta\tilde{S}_j + \sum_{j=k+2}^{T} M(\Delta\theta_{j+1}^{(0)})^2 + M\theta_{k+2}^{(0)2}, \\
&\quad 2\Delta\lambda_{k+1}M\bar{\theta}_{T+1} - 2\lambda_{k+1}M\theta_{k+2}^{(0)} + 2M(\sum_{i=0}^{T-1}\partial_{\lambda_i}\theta_{k+2}^{(0)}\lambda_i)\Big|\mathcal{F}_k\Big]\Big\} \\
&\Big/\mathrm{Var}\Big[\Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}\Big|\mathcal{F}_k\Big] \\
&- 2\mathrm{Cov}\Big[\Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}, 2\Delta\lambda_{k+1}M\bar{\theta}_{T+1} - 2\lambda_{k+1}M\theta_{k+2}^{(0)} + 2M(\sum_{i=0}^{T-1}\partial_{\lambda_i}\theta_{k+2}^{(0)}\lambda_i)\Big|\mathcal{F}_k\Big] \\
&\times \mathrm{Cov}\Big[\bar{\eta}_T + \bar{\theta}_{T+1}\tilde{S}_T - \sum_{j=k+2}^{T}\theta_j^{(0)}\Delta\tilde{S}_j + \sum_{j=k+2}^{T} M(\Delta\theta_{j+1}^{(0)})^2 + M\theta_{k+2}^{(0)2}, \Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}\Big|\mathcal{F}_k\Big] \\
&\Big/\mathrm{Var}\Big[\Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}\Big|\mathcal{F}_k\Big]^2. \qquad (38)
\end{aligned}
$$

We define the first order expansion of the money market account position in the local risk-minimizing strategy $\{\tilde{\eta}_k\}_{k=0,\dots,T}$ by

$$
\tilde{\eta}_k = \mathbf{E}\Big[\bar{\eta}_T + \bar{\theta}_{T+1}S(T,0) - \sum_{j=k+1}^{T}\tilde{\theta}_j\Delta\tilde{S}_j - \sum_{j=k+1}^{T} 2\lambda_{j-1}M\tilde{\theta}_j\Delta\tilde{\theta}_j + \sum_{j=k+1}^{T} M(\Delta\tilde{\theta}_{j+1})^2\Big|\mathcal{F}_k\Big] - \tilde{\theta}_{k+1}S(k,0). \tag{39}
$$

We also give the expression of $\tilde{\theta}_{k+1}$ $(k=0,\dots,T-1)$ in the form of (34) in Appendix I.

Equation (37) indicates that $\tilde{\theta}_T$ is obtained explicitly when $\{\theta_j^{(0)}\}_{j=1,\dots,T}$, the zeroth order expansion of $\{\theta_j\}_{j=1,\dots,T}$, is given. As in Section 3, this zeroth order expansion, the constant market impact parameter case where $\lambda=0$, is determined recursively.

In addition, since $\tilde{\theta}_{k+1}$ includes $\{\theta_j^{(0)}\}_{j=1,\dots,k,k+2,\dots,T}$ and $\{\partial_{\lambda_i}\theta_j^{(0)}\}_{j=k+2,\dots,T,\ i=0,\dots,T-1}$ as in (38), $\{\tilde{\theta}_{k+1}\}_{k=0,\dots,T-1}$ is obtained in the following procedure.

1. Calculate $\{\theta_j^{(0)}\}_{j=1,\dots,T}$ by (21),(22) and (23) as the constant market impact case where $\lambda=0$.
2. For given $\{\theta_j^{(0)}\}_{j=1,\dots,T}$, calculate $\tilde{\theta}_T$ along with $\partial_{\lambda_i}\theta_T^{(0)}$ $(i=0,\dots,T-1)$ by (37).
3. For given $\{\theta_j^{(0)}\}_{j=1,\dots,T}$ and $\{\partial_{\lambda_i}\theta_j^{(0)}\}_{i=0,\dots,T-1,j=k+2,\dots,T}$, calculate $\tilde{\theta}_{k+1}$ along with $\partial_{\lambda_i}\theta_{k+1}^{(0)}$ $(i=0,\dots,T-1)$ by (38).

5 Numerical examples

In this section, we provide numerical examples of the local risk-minimizing strategies and the option prices in a two-period case because of its clarity and in order to compare the result of the asymptotic expansion with the exact values obtained in Appendix A.

Let $\Omega = \{\omega_1, \omega_2, \dots, \omega_{16}\}$, $\mathbf{P}(\{\omega_i\}) = \frac{1}{16}$ for $i = 1, 2, \dots, 16$, $\mathcal{F} = \mathcal{F}_2 = 2^\Omega$, $\mathcal{F}_1 = \sigma(\{\omega_1, \omega_2, \omega_3, \omega_4\}, \dots, \{\omega_{13}, \omega_{14}, \omega_{15}, \omega_{16}\})$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$. We assume the following two-dimensional binomial distribution for $\{A_k\}_{k=0,1,2}$, the price process of the untradable underlying asset, and $\{\tilde{S}_k\}_{k=0,1,2}$, the unaffected price of the hedging instrument, in Section 2.

For $k = 0, 1$,

$$\tilde{S}_{k+1} = \tilde{S}_k \exp(-\frac{1}{2}\sigma^{\tilde{S}}\Delta t + \sigma^{\tilde{S}}\sqrt{\Delta t}\xi_{1,k+1}), \tag{40}$$

$$A_{k+1} = A_k \exp(-\frac{1}{2}\sigma^A \Delta t + \sigma^A \sqrt{\Delta t}(\rho \xi_{1,k+1} + \sqrt{1-\rho^2}\xi_{2,k+1})), \tag{41}$$

where the random variables $\xi_{1,k}$ and $\xi_{2,k}$ take the following values at each ω_i $(i = 1, \dots, 16)$,

$$\xi_{1,1}(\omega_i) = \begin{cases} +1, & i = 1,2,3,4,5,6,7,8 \\ -1, & i = 9,10,11,12,13,14,15,16, \end{cases} \tag{42}$$

$$\xi_{2,1}(\omega_i) = \begin{cases} +1, & i = 1,2,3,4,9,10,11,12 \\ -1, & i = 5,6,7,8,13,14,15,16, \end{cases} \tag{43}$$

$$\xi_{1,2}(\omega_i) = \begin{cases} +1, & i = 1,2,5,6,9,10,13,14 \\ -1, & i = 3,4,7,8,11,12,15,16, \end{cases} \tag{44}$$

$$\xi_{2,2}(\omega_i) = \begin{cases} +1, & i = 1,3,5,7,9,11,13,15 \\ -1, & i = 2,4,6,8,10,12,14,16. \end{cases} \tag{45}$$

For instance, $\xi_{1,1} = +1$, $\xi_{2,1} = +1$ for $\omega = \omega_1, \omega_2, \omega_3, \omega_4$, and $\xi_{1,1} = +1$, $\xi_{2,1} = -1$ for $\omega = \omega_5, \omega_6, \omega_7, \omega_8$.

The trade price of the hedging instrument $S(k, \Delta\theta_{k+1})$ when buying $\Delta\theta_{k+1}$ at k is as in (1)-(3) in Section 4.1, namely,

$$S(k, \Delta\theta_{k+1}) = S(k, 0) + M\Delta\theta_{k+1},\ k = 0, \dots, T, \tag{46}$$

$$S(0, 0) = \tilde{S}_0, \tag{47}$$

$$S(k, 0) = \tilde{S}_k + \sum_{j=1}^{k} 2\lambda_{j-1} M \Delta\theta_j,\ k = 1, \dots, T. \tag{48}$$

In the following examples, we consider the case where the hedger holds a short position of a call option on the underlying asset and replicates the payoff in order to deliver it to the buyer at the maturity.

5.1 Cash settlement

Firstly, we show numerical examples of the hedging strategy and the option price for cash settlement. By (29) and (30) in Section 4.1, we calculate them for different market impact parameters λ, correlations ρ, and thinness of the limit orders M. Let the option maturity $T = 2$. We consider the the call option payoff at T in cash settlement in (6) in Section 4.2, that is, $(\bar{\eta}_2, \bar{\theta}_3)$ where $\bar{\eta}_2 = (A_2 - K_A)^+$ and $\bar{\theta}_3 = 0$. We assume the strike price and the parameters in (40) and (41) to be $K_A = 100$, $\tilde{S}_0 = 100$, $\sigma^{\tilde{S}} = 10\%$, $A_0 = 100$, $\sigma^A = 10\%$, $\Delta t = 1$.

Table 1: $\rho = 1$: Local risk-minimizing strategy in cash settlement, $\lambda_0 = 0.1, \lambda_1 = 0.9$

M	5	4	3	2	1	0
$\theta_2(\omega_i),\ i = 1,2,3,4$	0.9546	0.9546	0.9546	0.9546	0.9546	0.9546
$\theta_2(\omega_i),\ i = 5,6,7,8$	0.9546	0.9546	0.9546	0.9546	0.9546	0.9546
$\theta_2(\omega_i),\ i = 9,10,11,12$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\theta_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
θ_1	0.5446	0.5408	0.5369	0.5330	0.5290	0.5249
R_0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
P_0	8.0967	7.4957	6.9085	6.3353	5.7764	5.2321

Table 2: $\rho = 0.7$: Local risk-minimizing strategy in cash settlement, $\lambda_0 = 0.1, \lambda_1 = 0.9$

M	5	4	3	2	1	0
$\theta_2(\omega_i),\ i = 1,2,3,4$	0.7081	0.7081	0.7081	0.7081	0.7081	0.7081
$\theta_2(\omega_i),\ i = 5,6,7,8$	0.3167	0.3167	0.3167	0.3167	0.3167	0.3167
$\theta_2(\omega_i),\ i = 9,10,11,12$	0.3958	0.3958	0.3958	0.3958	0.3958	0.3958
$\theta_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
θ_1	0.3833	0.3809	0.3784	0.3759	0.3734	0.3708
R_0	46.7251	46.5793	46.4331	46.2865	46.1395	45.9921
P_0	6.8481	6.5600	6.2784	6.0032	5.7344	5.4721

Table 3: $\rho = 0.5$: Local risk-minimizing strategy in cash settlement, $\lambda_0 = 0.1, \lambda_1 = 0.9$

M	5	4	3	2	1	0
$\theta_2(\omega_i),\ i = 1,2,3,4$	0.4972	0.4972	0.4972	0.4972	0.4972	0.4972
$\theta_2(\omega_i),\ i = 5,6,7,8$	0.2148	0.2148	0.2148	0.2148	0.2148	0.2148
$\theta_2(\omega_i),\ i = 9,10,11,12$	0.3121	0.3121	0.3121	0.3121	0.3121	0.3121
$\theta_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
θ_1	0.2748	0.2731	0.2713	0.2696	0.2678	0.2660
R_0	59.9914	59.9136	59.8354	59.7567	59.6777	59.5982
P_0	6.3902	6.2423	6.0976	5.9563	5.8181	5.6832

Table 4: $\rho = 1$: Local risk-minimizing strategy in cash settlement, $\lambda_0 = 0.2, \lambda_1 = 0.4$

M	5	4	3	2	1	0
$\theta_2(\omega_i),\ i = 1,2,3,4$	0.9546	0.9546	0.9546	0.9546	0.9546	0.9546
$\theta_2(\omega_i),\ i = 5,6,7,8$	0.9546	0.9546	0.9546	0.9546	0.9546	0.9546
$\theta_2(\omega_i),\ i = 9,10,11,12$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\theta_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
θ_1	0.6208	0.6052	0.5881	0.5692	0.5483	0.5249
R_0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
P_0	9.2715	8.3769	7.5221	6.7104	5.9458	5.2321

Table 5: $\rho = 0.7$: Local risk-minimizing strategy in cash settlement, $\lambda_0 = 0.2, \lambda_1 = 0.4$

M	5	4	3	2	1	0
$\theta_2(\omega_i),\ i = 1,2,3,4$	0.7081	0.7081	0.7081	0.7081	0.7081	0.7081
$\theta_2(\omega_i),\ i = 5,6,7,8$	0.3167	0.3167	0.3167	0.3167	0.3167	0.3167
$\theta_2(\omega_i),\ i = 9,10,11,12$	0.3958	0.3958	0.3958	0.3958	0.3958	0.3958
$\theta_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
θ_1	0.4400	0.4273	0.4141	0.4002	0.3858	0.3708
R_0	50.2396	49.4184	48.5834	47.7342	46.8704	45.9921
P_0	7.2212	6.8225	6.4482	6.0986	5.7734	5.4721

Table 6: $\rho = 0.5$: Local risk-minimizing strategy in cash settlement, $\lambda_0 = 0.2, \lambda_1 = 0.4$

M	5	4	3	2	1	0
$\theta_2(\omega_i),\ i = 1,2,3,4$	0.4972	0.4972	0.4972	0.4972	0.4972	0.4972
$\theta_2(\omega_i),\ i = 5,6,7,8$	0.2148	0.2148	0.2148	0.2148	0.2148	0.2148
$\theta_2(\omega_i),\ i = 9,10,11,12$	0.3121	0.3121	0.3121	0.3121	0.3121	0.3121
$\theta_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
θ_1	0.3154	0.3062	0.2966	0.2867	0.2765	0.2660
R_0	61.8051	61.3920	60.9652	60.5243	60.0687	59.5982
P_0	6.5808	6.3758	6.1838	6.0046	5.8378	5.6832

Tables 1-3 show the position of the hedging instrument θ_1, θ_2 in the local risk-minimizing strategy and the option price P_0 with the time-dependent market impact parameter $\lambda_0 = 0.1, \lambda_1 = 0.9$. We define the option price P_0 as the cost to construct the initial portfolio, namely,

$$P_0 = \eta_0 + \theta_1 S(0, \theta_1). \tag{49}$$

R_0 is the risk process at the initial time in (14) in Section 2.2, which also measures the quadratic hedging error from the payoff at the maturity. The correlation ρ is set to be $100\%, 70\%, 50\%$ in Tables 1-3, respectively, while M is set as $5, 4, 3, 2, 1, 0$ for all the correlations. Similarly, Tables 4-6 illustrate those with $\lambda_0 = 0.2, \lambda_1 = 0.4$.

Then, as expected, in all the cases, we observe that the option price P_0 increases as M increases, which implies that when the order book is thin, the replication cost of the option payoff for the seller is high. We also observe that R_0 increases as ρ decreases or M increases. This indicates that the replication error is large when the correlation between the untradable asset and the hedging instrument is low or the limit order book is thin.

5.2 Physical settlement

Next, we present numerical examples of the local risk-minimizing strategy and the option price in physical settlement. Similarly to the cash settlement case, we calculate the first order expansion of these by (37) and (38) in Section 4.2 for different λ, ρ, and M by comparing these approximations with the exact values, which are obtained as in Appendix A. We assume the following payoff $(\bar{\eta}_2, \bar{\theta}_3)$ in physical settlement at maturity $T = 2$ as in Section 2.2, namely,

$$\begin{aligned} \bar{\theta}_3 &= 1_{\{A_2 > K_A\}}, \\ \bar{\eta}_2 &= -K_S 1_{\{A_2 > K_A\}}, \end{aligned} \tag{50}$$

where $K_A, K_S > 0$. In this settlement, one unit of the hedging instrument is delivered to the buyer at the price K_S at the maturity only when A_2, the reference price of the untradable underlying asset, satisfies $A_2 > K_A$. We assume the strike prices to be $K_A = K_S = 100$, and set the same parameters as the previous subsection for $\tilde{S}_0,\ \sigma^{\tilde{S}},\ A_0,\ \sigma^A,\ \Delta t$.

Table 7: $\rho = 1$: First order expansion of local risk-minimizing strategy in physical settlement, $\lambda_0 = 0.1, \lambda_1 = 0.9$

M	5	4	3	2	1	0
$\tilde{\theta}_2(\omega_i),\ i = 1,2,3,4$	0.9027	0.9154	0.9280	0.9398	0.9495	0.9546
$\tilde{\theta}_2(\omega_i),\ i = 5,6,7,8$	0.9027	0.9154	0.9280	0.9398	0.9495	0.9546
$\tilde{\theta}_2(\omega_i),\ i = 9,10,11,12$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\tilde{\theta}_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\theta_2(\omega_i),\ i = 1,2,3,4$	0.9434	0.9461	0.9487	0.9510	0.9530	0.9546
$\theta_2(\omega_i),\ i = 5,6,7,8$	0.9434	0.9461	0.9487	0.9510	0.9530	0.9546
$\theta_2(\omega_i),\ i = 9,10,11,12$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\theta_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\tilde{\theta}_1$	0.5449	0.5419	0.5380	0.5332	0.5282	0.5249
θ_1	0.5379	0.5358	0.5335	0.5309	0.5281	0.5249
R_0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\tilde{P}_0$	8.0003	7.4332	6.8689	6.3116	5.7655	5.2321
P_0	7.9659	7.4080	6.8548	6.3070	5.7657	5.2321

Table 8: $\rho = 0.7$: First order expansion of local risk-minimizing strategy in physical settlement, $\lambda_0 = 0.1, \lambda_1 = 0.9$

M	5	4	3	2	1	0
$\tilde{\theta}_2(\omega_i),\ i = 1,2,3,4$	0.9796	0.9836	0.9851	0.9841	0.9812	0.9773
$\tilde{\theta}_2(\omega_i),\ i = 5,6,7,8$	0.5739	0.5516	0.5295	0.5086	0.4907	0.4773
$\tilde{\theta}_2(\omega_i),\ i = 9,10,11,12$	0.0270	0.0128	0.0005	-0.0098	-0.0186	-0.0277
$\tilde{\theta}_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\theta_2(\omega_i),\ i = 1,2,3,4$	0.9812	0.9823	0.9827	0.9821	0.9804	0.9773
$\theta_2(\omega_i),\ i = 5,6,7,8$	0.5146	0.5092	0.5030	0.4957	0.4872	0.4773
$\theta_2(\omega_i),\ i = 9,10,11,12$	0.0055	0.0011	-0.0042	-0.0106	-0.0184	-0.0277
$\theta_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\tilde{\theta}_1$	0.4632	0.4541	0.4430	0.4295	0.4132	0.3937
θ_1	0.4767	0.4635	0.4486	0.4320	0.4137	0.3937
R_0	18.7942	18.5105	18.1795	17.7960	17.3556	16.8563
$\tilde{P}_0$	6.0535	5.5643	5.0888	4.6321	4.2004	3.7997
P_0	6.1271	5.6032	5.1050	4.6365	4.2008	3.7997

Table 9: $\rho = 0.5$: First order expansion of local risk-minimizing strategy in physical settlement, $\lambda_0 = 0.1, \lambda_1 = 0.9$

M	5	4	3	2	1	0
$\tilde{\theta}_2(\omega_i),\ i = 1,2,3,4$	0.9645	0.9699	0.9738	0.9761	0.9770	0.9773
$\tilde{\theta}_2(\omega_i),\ i = 5,6,7,8$	0.5588	0.5380	0.5182	0.5006	0.4865	0.4773
$\tilde{\theta}_2(\omega_i),\ i = 9,10,11,12$	0.3848	0.4237	0.4566	0.4798	0.4930	0.5000
$\tilde{\theta}_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\theta_2(\omega_i),\ i = 1,2,3,4$	0.9728	0.9740	0.9750	0.9759	0.9767	0.9773
$\theta_2(\omega_i),\ i = 5,6,7,8$	0.5062	0.5009	0.4953	0.4895	0.4835	0.4773
$\theta_2(\omega_i),\ i = 9,10,11,12$	0.4702	0.4760	0.4819	0.4878	0.4939	0.5000
$\theta_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\tilde{\theta}_1$	0.5249	0.5214	0.5173	0.5141	0.5126	0.5125
θ_1	0.5217	0.5197	0.5178	0.5158	0.5141	0.5125
R_0	31.3333	31.3059	31.2751	31.2411	31.2044	31.1653
$\tilde{P}_0$	5.2295	4.6877	4.1544	3.6337	3.1230	2.6161
P_0	5.2042	4.6772	4.1549	3.6374	3.1246	2.6161

Table 10: $\rho = 1$: First order expansion of local risk-minimizing strategy in physical settlement, $\lambda_0 = 0.2, \lambda_1 = 0.4$

M	5	4	3	2	1	0
$\tilde{\theta}_2(\omega_i),\ i = 1,2,3,4$	0.8794	0.8926	0.9069	0.9222	0.9382	0.9546
$\tilde{\theta}_2(\omega_i),\ i = 5,6,7,8$	0.8794	0.8926	0.9069	0.9222	0.9382	0.9546
$\tilde{\theta}_2(\omega_i),\ i = 9,10,11,12$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\tilde{\theta}_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\theta_2(\omega_i),\ i = 1,2,3,4$	0.8876	0.8989	0.9112	0.9245	0.9390	0.9546
$\theta_2(\omega_i),\ i = 5,6,7,8$	0.8876	0.8989	0.9112	0.9245	0.9390	0.9546
$\theta_2(\omega_i),\ i = 9,10,11,12$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\theta_2(\omega_i),\ i = 13,14,15,16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\tilde{\theta}_1$	0.5546	0.5518	0.5478	0.5421	0.5346	0.5249
θ_1	0.5723	0.5663	0.5590	0.5501	0.5389	0.5249
R_0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\tilde{P}_0$	8.2030	7.6190	7.0280	6.4315	5.8316	5.2321
P_0	8.3241	7.6969	7.0720	6.4511	5.8366	5.2321

Table 11: $\rho = 0.7$: First order expansion of local risk-minimizing strategy in physical settlement, $\lambda_0 = 0.2, \lambda_1 = 0.4$

M	5	4	3	2	1	0
$\tilde{\theta}_2(\omega_i),\ i = 1, 2, 3, 4$	0.9277	0.9387	0.9492	0.9591	0.9684	0.9773
$\tilde{\theta}_2(\omega_i),\ i = 5, 6, 7, 8$	0.6144	0.5917	0.5664	0.5387	0.5089	0.4773
$\tilde{\theta}_2(\omega_i),\ i = 9, 10, 11, 12$	0.0949	0.0699	0.0447	0.0197	-0.0046	-0.0277
$\tilde{\theta}_2(\omega_i),\ i = 13, 14, 15, 16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\theta_2(\omega_i),\ i = 1, 2, 3, 4$	0.9296	0.9394	0.9492	0.9589	0.9683	0.9773
$\theta_2(\omega_i),\ i = 5, 6, 7, 8$	0.6051	0.5846	0.5617	0.5363	0.5082	0.4773
$\theta_2(\omega_i),\ i = 9, 10, 11, 12$	0.0911	0.0677	0.0437	0.0195	-0.0045	-0.0277
$\theta_2(\omega_i),\ i = 13, 14, 15, 16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\tilde{\theta}_1$	0.4524	0.4435	0.4334	0.4219	0.4087	0.3937
θ_1	0.4578	0.4469	0.4351	0.4223	0.4085	0.3937
R_0	19.4753	19.1222	18.6798	18.1484	17.5353	16.8563
$\tilde{P}_0$	6.1669	5.6753	5.1909	4.7156	4.2512	3.7997
P_0	6.1876	5.6844	5.1936	4.7157	4.2509	3.7997

Table 12: $\rho = 0.5$: First order expansion of local risk-minimizing strategy in physical settlement, $\lambda_0 = 0.2, \lambda_1 = 0.4$

M	5	4	3	2	1	0
$\tilde{\theta}_2(\omega_i),\ i = 1, 2, 3, 4$	0.9239	0.9353	0.9464	0.9571	0.9674	0.9773
$\tilde{\theta}_2(\omega_i),\ i = 5, 6, 7, 8$	0.6106	0.5883	0.5636	0.5367	0.5078	0.4773
$\tilde{\theta}_2(\omega_i),\ i = 9, 10, 11, 12$	0.3365	0.3647	0.3960	0.4293	0.4639	0.5000
$\tilde{\theta}_2(\omega_i),\ i = 13, 14, 15, 16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\theta_2(\omega_i),\ i = 1, 2, 3, 4$	0.9259	0.9361	0.9465	0.9570	0.9673	0.9773
$\theta_2(\omega_i),\ i = 5, 6, 7, 8$	0.6014	0.5813	0.5591	0.5344	0.5072	0.4773
$\theta_2(\omega_i),\ i = 9, 10, 11, 12$	0.3486	0.3718	0.3990	0.4298	0.4637	0.5000
$\theta_2(\omega_i),\ i = 13, 14, 15, 16$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\tilde{\theta}_1$	0.5234	0.5217	0.5196	0.5172	0.5149	0.5125
θ_1	0.5342	0.5309	0.5270	0.5225	0.5176	0.5125
R_0	30.7850	31.0966	31.3037	31.3892	31.3431	31.1653
$\tilde{P}_0$	5.3100	4.7505	4.2010	3.6632	3.1358	2.6161
P_0	5.3525	4.7807	4.2202	3.6726	3.1383	2.6161

Tables 7-12 show the first order expansion of θ_1, θ_2, the position of the hedging instrument in the local risk-minimizing strategy, in physical settlement, which we denote by $\tilde{\theta}_1$, $\tilde{\theta}_2$. The time-dependent market impact parameter $\{\lambda_k\}_{k=0,1}$ is set to be $\lambda_0 = 0.1, \lambda_1 = 0.9$ for Tables 7-9, and $\lambda_0 = 0.2, \lambda_1 = 0.4$ for Tables 10-12. These tables also provide the exact values of θ_1, θ_2, which can be computed by solving the cubic equation (58) in Appendix A. The thinness of the order book M is set to be $5, 4, 3, 2, 1, 0$ for all the correlations $\rho = 100\%, 70\%, 50\%$. $\tilde{P}_0$ is the mid value and the replication cost of the approximate initial portfolio defined by

$$\tilde{P}_0 = \tilde{\eta}_0 + \tilde{\theta}_1 S(0, \tilde{\theta}_1). \tag{51}$$

Then we observe that in all the cases, $\tilde{P}_0$, the approximation of the option price P_0, is close to the exact value in those cases. For example, the largest relative error of $\tilde{P}_0$ is 1.44% when $M = 5$ in Table 10. Furthermore, for the option price and the quadratic hedging error, we notice the same features as the cash settlement case, that is, the option price increases as the order book is thinner, and the quadratic hedging error increases as the correlation between the hedging instrument and the untradable underlying asset lowers. The increase in the quadratic hedging error by a thinner limit order book density holds for all the cases except for Tables 12 where the error is almost unchanged throughout different values in M.

6 Conclusion

In this paper, we have considered the pricing and hedging problem of illiquid options, where the underlying asset is untradable and an alternative asset is used as a hedging instrument. Particularly, we have considered the situation where the trade price of the hedging instrument is subject to market impacts caused by the hedger and the liquidity costs paid as a spread from the mid price. Pricing of illiquid options, which often appears in trading of structured products, is an important issue in practice because of its difficulty in hedging mainly due to untradablity of the underlying asset as well as market impacts and liquidity costs of the hedging instrument. Firstly, we have set the problem under a discrete time model, where the optimal hedging strategy is defined by the local risk-minimization. Then, after showing that the local risk-minimizing strategy is uniquely determined by backward induction in the case of a constant market impact parameter, we have provided the algorithm to obtain the strategy in the case of a time-dependent market impact parameter, which is especially important when estimating hedging cost and effect of market impact in the scenario of economic environment changes. In particular, the exact strategy is obtained by backward induction in cash settlement, and the first order expansion of the strategy expanded with respect to the parameter is obtained recursively in physical settlement. Finally, we have provided the numerical examples of both physical and cash settlements in the case of a time-dependent market impact parameter. This model is useful in estimation of the effect of the market impacts as well as the liquidity costs on derivatives prices.

A Physical settlement in two-period case with time-dependent λ

In this appendix, we show that the exact local risk-minimizing strategy satisfying (27) and (28) in Section 4 is obtained in two-period cases. By (27) and (28), in the two-period case,

$$\theta_2 = \frac{\mathrm{Cov}[\eta_2 + \theta_3(\tilde{S}_2 - 2(\lambda_1 - \lambda_0)M\theta_1) + M\theta_3^2, -2\lambda_1 M\theta_3 + \Delta\tilde{S}_2 + 2M\theta_3|\mathcal{F}_1]}{\mathrm{Var}[-2\lambda_1 M\theta_3 + \Delta\tilde{S}_2 + 2M\theta_3|\mathcal{F}_1]}, \tag{52}$$

$$\begin{aligned}\theta_1 =& \mathrm{Cov}[\eta_2 + \theta_3(\tilde{S}_2 + 2\lambda_1 M\theta_2) - \theta_2\Delta\tilde{S}_2 + M(\Delta\theta_3)^2 + M\theta_2^2 - 2\lambda_1 M\theta_2^2,\\ &\Delta\tilde{S}_1 + 2M\theta_2 - 2\lambda_1 M\theta_2 + 2(\lambda_1 - \lambda_0)M\theta_3)]\\ &/\mathrm{Var}[\Delta\tilde{S}_1 + 2M\theta_2 - 2\lambda_1 M\theta_2 + 2(\lambda_1 - \lambda_0)M\theta_3].\end{aligned} \tag{53}$$

Rewriting θ_2 as

$$\begin{aligned}\theta_2 =& -2(\lambda_1 - \lambda_0)M\frac{\mathrm{Cov}[\theta_3, -2\lambda_1 M\theta_3 + \Delta\tilde{S}_2 + 2M\theta_3|\mathcal{F}_1]}{\mathrm{Var}[-2\lambda_1 M\theta_3 + \Delta\tilde{S}_2 + 2M\theta_3|\mathcal{F}_1]}\theta_1\\ &+ \frac{\mathrm{Cov}[\eta_2 + \theta_3\tilde{S}_2 + M\theta_3^2, -2\lambda_1 M\theta_3 + \Delta\tilde{S}_2 + 2M\theta_3|\mathcal{F}_1]}{\mathrm{Var}[-2\lambda_1 M\theta_3 + \Delta\tilde{S}_2 + 2M\theta_3|\mathcal{F}_1]}\\ :=& \alpha_1\theta_1 + \beta_1,\end{aligned} \tag{54}$$

and substituting this into the equation (53), we have

$$\begin{aligned}
&\theta_1 \mathrm{Var}[\Delta\tilde{S}_1 + 2(1-\lambda_1)M(\alpha_1\theta_1+\beta_1) + 2(\lambda_1-\lambda_0)M\theta_3] \\
&= \mathrm{Cov}[\eta_2 + \theta_3(\tilde{S}_2 + 2\lambda_1 M(\alpha_1\theta_1+\beta_1)) - (\alpha_1\theta_1+\beta_1)\Delta\tilde{S}_2 \\
&\quad + M(\theta_3 - (\alpha_1\theta_1+\beta_1))^2 + (1-2\lambda_1)M(\alpha_1\theta_1+\beta_1)^2, \\
&\quad \Delta\tilde{S}_1 + 2M\theta_2 - 2\lambda_1 M\theta_2 + 2(\lambda_1-\lambda_0)M\theta_3)].
\end{aligned} \tag{55}$$

Both sides of (55) are calculated as follows.

$$\begin{aligned}
\mathrm{LHS} = {}& \theta_1^3(\mathrm{Var}[2(1-\lambda_1)M\alpha_1]) \\
&+ \theta_1^2(2\mathrm{Cov}[2(1-\lambda_1)M\alpha_1\theta_1, \Delta\tilde{S}_1 + 2(1-\lambda_1)M\beta_1 + 2(\lambda_1-\lambda_0)M\theta_3]) \\
&+ \theta_1 \mathrm{Var}[\Delta\tilde{S}_1 + 2(1-\lambda_1)M\beta_1 + 2(\lambda_1-\lambda_0)M\theta_3],
\end{aligned} \tag{56}$$

$$\begin{aligned}
\mathrm{RHS} = {}& \theta_1^3 \mathrm{Cov}[2(1-\lambda_1)\alpha_1^2 M, 2(1-\lambda_1)M\alpha_1] \\
&+ \theta_1^2(\mathrm{Cov}[2(1-\lambda_1)\alpha_1^2 M, \Delta\tilde{S}_1 + 2(1-\lambda_1)M\beta_1 + 2(\lambda_1-\lambda_0)M\theta_3)] \\
&+ \mathrm{Cov}[2\lambda_1 M\alpha_1\theta_3 - \alpha_1\Delta\tilde{S}_2 - 2\alpha_1 M(\theta_3-\beta_1) + 2M(1-2\lambda_1)\alpha_1\beta_1, 2(1-\lambda_1)M\alpha_1]) \\
&+ \theta_1(\mathrm{Cov}[2\lambda_1 M\alpha_1\theta_3 - \alpha_1\Delta\tilde{S}_2 - 2\alpha_1 M(\theta_3-\beta_1) + 2M(1-2\lambda_1)\alpha_1\beta_1, \\
&\Delta\tilde{S}_1 + 2(1-\lambda_1)M\beta_1 + 2(\lambda_1-\lambda_0)M\theta_3)] \\
&+ \mathrm{Cov}[\eta_2 + \theta_3(\tilde{S}_2 + 2\lambda_1 M\beta_1) - \beta_1\Delta\tilde{S}_2 + M(\theta_3-\beta_1)^2 + M(1-2\lambda_1)\beta_1^2, 2(1-\lambda_1)M\alpha_1]) \\
&+ \mathrm{Cov}[\eta_2 + \theta_3(\tilde{S}_2 + 2\lambda_1 M\beta_1) - \beta_1\Delta\tilde{S}_2 + M(\theta_3-\beta_1)^2 + M(1-2\lambda_1)\beta_1^2, \\
&\Delta\tilde{S}_1 + 2(1-\lambda_1)M\beta_1 + 2(\lambda_1-\lambda_0)M\theta_3].
\end{aligned} \tag{57}$$

Rearranging the equation (55) with respect to θ_1 as

$$f(\theta_1) := a_3\theta_1^3 + a_2\theta_1^2 + a_1\theta_1^1 + a_0 = 0, \tag{58}$$

we reduce the original problem to solving this cubic equation on θ_1. After solving (58) for θ_1, by substituting θ_1 into (54), we obtain θ_2.

Remark 1. *Note that the conditions for the equation (58) to have three solutions are*

$$a_3 \neq 0, \tag{59}$$

$$a_2^2 - 3a_3a_1 > 0, \tag{60}$$

and

$$f\Big(\frac{-a_2 + \sqrt{a_2^2 - 3a_3a_1}}{3a_3}\Big) f\Big(\frac{-a_2 - \sqrt{a_2^2 - 3a_3a_1}}{3a_3}\Big) < 0. \tag{61}$$

By these conditions, we observe that in Tables 8,9,11 and 12, there is a unique local risk-minimizing strategy, and in Tables 7 & 10, there are three local risk-minimizing strategies. In all the cases in Tables 7 & 10, the cubic equation has one solution in between 0 *and* 1*. The other two solutions take a significantly large value, which corresponds to an excessively high initial replication cost. For example, Tables 13&14 show coefficients of the cubic equation and its solutions when* $M = 5$ *in Table 7. In this case,* $(\theta_1, P_0) = (0.5379, 7.97)$, $(54.00, 26782)$, $(71.81, 47424)$. *Similarly in Tables 15&16, we observe that, when* $M = 1$ *in Table 10,* $(\theta_1, P_0) = (0.5389, 5.84)$, $(1007.98, 1635691)$, $(1024.57, 1689988)$. *In Tables 7&10, we assume the solution* $\theta_1 \in [0, 1]$ *to be the true value that* $\tilde{\theta}_1$ *approximates.*

Table 13: $M = 5, \lambda_0 = 0.1, \lambda_1 = 0.9$: Coefficients of Cubic Equation

M	5
λ_0	0.1
λ_1	0.9
a_3	-0.04
a_2	5.19
a_1	-162.11
a_0	85.70

Table 14: $M = 5, \lambda_0 = 0.1, \lambda_1 = 0.9$: Solutions of Cubic Equation

θ_1	0.5379	54.00	71.81
$\theta_2(\omega_i),\ i = 1, 2, 3, 4$	0.9434	-17.72	-23.93
$\theta_2(\omega_i),\ i = 5, 6, 7, 8$	0.9434	-17.72	-23.93
$\theta_2(\omega_i),\ i = 9, 10, 11, 12$	0.00	0.00	0.00
$\theta_2(\omega_i),\ i = 13, 14, 15, 16$	0.00	0.00	0.00
$\eta_1(\omega_i),\ i = 1, 2, 3, 4$	-92.90	-7,155	-13,671
$\eta_1(\omega_i),\ i = 5, 6, 7, 8$	-92.90	-7,155	-13,671
$\eta_1(\omega_i),\ i = 9, 10, 11, 12$	0.00	0.00	0.00
$\eta_1(\omega_i),\ i = 13, 14, 15, 16$	0.00	0.00	0.00
P_0	7.97	26,782	47,424
R_0	0.00	0	89,316

Table 15: $M = 1, \lambda_0 = 0.2, \lambda_1 = 0.4$: Coefficients of Cubic Equation

M	1
λ_0	0.2
λ_1	0.4
a_3	-0.00011
a_2	0.22
a_1	-113.33
a_0	61.01

Table 16: $M = 1, \lambda_0 = 0.2, \lambda_1 = 0.4$: Solutions of Cubic Equation

θ_1	0.5389	1,007.98	1,024.57
$\theta_2(\omega_i),\ i = 1, 2, 3, 4$	0.9390	-16.49	-16.78
$\theta_2(\omega_i),\ i = 5, 6, 7, 8$	0.9390	-16.49	-16.78
$\theta_2(\omega_i),\ i = 9, 10, 11, 12$	0.00	0.00	0.00
$\theta_2(\omega_i),\ i = 13, 14, 15, 16$	0.00	0.00	0.00
$\eta_1(\omega_i),\ i = 1, 2, 3, 4$	-92.59	-4,962	-5,159
$\eta_1(\omega_i),\ i = 5, 6, 7, 8$	-92.59	-4,962	-5,159
$\eta_1(\omega_i),\ i = 9, 10, 11, 12$	0.00	0.00	0.00
$\eta_1(\omega_i),\ i = 13, 14, 15, 16$	0.00	0.00	0.00
P_0	5.84	1,635,691	1,689,988
R_0	0.00	0	32,186

B Proof of Lemma 1

In this appendix, we give the proof of Lemma 1 in Section 2.2. It is enough to show that $\mathbf{E}[(C_T(\phi) - C_k(\phi))|\mathcal{F}_k] = 0,$ for $k = 0, 1, \ldots, T$. Consider $\tilde{\phi}$, a perturbation of ϕ, such that

$$\tilde{\eta}_k = \eta_k + \mathbf{E}[(C_T(\phi) - C_k(\phi))|\mathcal{F}_k], \tag{62}$$

$$\tilde{\eta}_j = \eta_j \ (j \neq k), \tag{63}$$

$$\tilde{\theta}_{j+1} = \theta_{j+1} \ (j = 0, 1, \ldots, T). \tag{64}$$

Then, by the definitions of $\tilde{\phi}$ and the cost process,

$$\begin{aligned}
R_k(\tilde{\phi}) &= \mathbf{E}[(C_T(\tilde{\phi}) - C_k(\tilde{\phi}))^2|\mathcal{F}_k] \\
&= \mathbf{E}[(C_T(\phi) - C_k(\phi) + C_k(\phi) - C_k(\tilde{\phi}))^2|\mathcal{F}_k] \\
&= \mathbf{E}[((C_T(\phi) - C_k(\phi)) - \mathbf{E}[(C_T(\phi) - C_k(\phi))|\mathcal{F}_k]))^2|\mathcal{F}_k] \\
&= \mathbf{E}[(C_T(\phi) - C_k(\phi))^2|\mathcal{F}_k] - \mathbf{E}[(C_T(\phi) - C_k(\phi)|\mathcal{F}_k]^2 \\
&\leq \mathbf{E}[(C_T(\phi) - C_k(\phi))^2|\mathcal{F}_k] = R_k(\phi).
\end{aligned} \tag{65}$$

As $R_k(\tilde{\phi}) \geq R_k(\phi)$, $\mathbf{E}[(C_T(\phi) - C_k(\phi)|\mathcal{F}_k] = 0,\ \mathbf{P} - a.s.$ □

C Proof of Proposition 1

Let us first show the if part. As $\{C_k(\phi)\}_{k=0,1,\ldots,T}$ is a martingale, for any $\tilde{\phi} \in \Gamma_{T-1}$,

$$\begin{aligned}
R_{T-1}(\phi) &= \mathbf{E}[(C_T(\phi) - C_{T-1}(\phi))^2|\mathcal{F}_{T-1}] \\
&= \mathrm{Var}[C_T(\phi) - C_{T-1}(\phi)|\mathcal{F}_{T-1}] \\
&\leq \mathrm{Var}[C_T(\tilde{\phi}) - C_{T-1}(\tilde{\phi})|\mathcal{F}_{T-1}] \\
&\leq \mathbf{E}[(C_T(\tilde{\phi}) - C_{T-1}(\tilde{\phi}))^2|\mathcal{F}_{T-1}] \\
&= R_{T-1}(\tilde{\phi}).
\end{aligned} \tag{66}$$

For $k = 0, 1, \ldots, T-2$, note that by the definition of $C_k(\phi)$, $C_T(\phi) - C_{k+2}(\phi) = C_T(\tilde{\phi}) - C_{k+2}(\tilde{\phi})$, for any $\tilde{\phi} \in \Gamma_k$.

In fact,

$$\begin{aligned}
&C_T(\phi) - C_{k+2}(\phi) \\
&= V_T(\phi) - V_{k+2}(\phi) - \sum_{j=k+3}^{T} \theta_j \Delta \tilde{S}_j - \sum_{j=k+3}^{T} 2\lambda M \theta_j \Delta\theta_j + \sum_{j=k+3}^{T} M(\Delta\theta_{j+1})^2 \\
&= (\bar{\eta}_T + \bar{\theta}_{T+1}(\tilde{S}_T + 2\lambda M\theta_T)) - (\eta_{k+2} + \theta_{k+3}(\tilde{S}_{k+2} + 2\lambda M\theta_{k+2})) \\
&\quad - \sum_{j=k+3}^{T} \theta_j \Delta \tilde{S}_j - \sum_{j=k+3}^{T} 2\lambda M \theta_j \Delta\theta_j + \sum_{j=k+3}^{T} M(\Delta\theta_{j+1})^2 \\
&= C_T(\tilde{\phi}) - C_{k+2}(\tilde{\phi}).
\end{aligned} \tag{67}$$

Therefore,

$$\begin{aligned}
R_k(\phi) &= \mathbf{E}[(C_T(\phi) - C_k(\phi))^2|\mathcal{F}_k] \\
&= \mathbf{E}[(C_T(\phi) - C_{k+2}(\phi))^2|\mathcal{F}_k] + \mathbf{E}[(C_{k+2}(\phi) - C_k(\phi))^2|\mathcal{F}_k] \\
&= \mathbf{E}[(C_T(\tilde{\phi}) - C_{k+2}(\tilde{\phi}))^2|\mathcal{F}_k] + \mathrm{Var}[C_{k+2}(\phi) - C_k(\phi)|\mathcal{F}_k] \\
&\leq \mathbf{E}[(C_T(\tilde{\phi}) - C_{k+2}(\tilde{\phi}))^2|\mathcal{F}_k] + \mathrm{Var}[C_{k+2}(\tilde{\phi}) - C_k(\tilde{\phi})|\mathcal{F}_k] \\
&\leq \mathbf{E}[(C_T(\tilde{\phi}) - C_{k+2}(\tilde{\phi}))^2|\mathcal{F}_k] + \mathbf{E}[(C_{k+2}(\tilde{\phi}) - C_k(\tilde{\phi}))^2|\mathcal{F}_k] \\
&= \mathbf{E}[(C_T(\tilde{\phi}) - C_k(\tilde{\phi}))^2|\mathcal{F}_k] \\
&= R_k(\tilde{\phi}).
\end{aligned} \tag{68}$$

In the second line from the last, we used the fact that

$$\mathbf{E}[(C_T(\tilde{\phi}) - C_{k+2}(\tilde{\phi}))(C_{k+2}(\tilde{\phi}) - C_k(\tilde{\phi}))|\mathcal{F}_k] = 0, \tag{69}$$

which follows from the tower property and $C_T(\phi) - C_{k+2}(\phi) = C_T(\tilde{\phi}) - C_{k+2}(\tilde{\phi})$.

Next, we show the only if part. (i) is shown in Lemma 1.

For (ii), for $k = 0, 1, \ldots, T$,

$$\begin{aligned}
&\mathbf{E}[(C_T(\phi) - C_k(\phi))^2|\mathcal{F}_k] \\
&= \mathrm{Var}[C_T(\phi) - C_{k+2}(\phi)|\mathcal{F}_k] + \mathrm{Var}[C_{k+2}(\phi) - C_k(\phi)|\mathcal{F}_k].
\end{aligned} \tag{70}$$

For any $\tilde{\phi}$, a perturbation of ϕ at time k,

$$\begin{aligned}
\mathbf{E}[(C_T(\tilde{\phi}) - C_k(\tilde{\phi}))^2|\mathcal{F}_k] &= \mathbf{E}[(C_T(\phi) - C_{k+2}(\phi) + C_{k+2}(\tilde{\phi}) - C_k(\tilde{\phi}))^2|\mathcal{F}_k] \\
&= \mathbf{E}[(C_T(\phi) - C_{k+2}(\phi))^2|\mathcal{F}_k] + \mathbf{E}[(C_{k+2}(\tilde{\phi}) - C_k(\tilde{\phi}))^2|\mathcal{F}_k] \\
&\geq \mathrm{Var}[C_T(\phi) - C_{k+2}(\phi)|\mathcal{F}_k] + \mathrm{Var}[C_{k+2}(\tilde{\phi}) - C_k(\tilde{\phi})|\mathcal{F}_k] \\
&= \mathrm{Var}[C_T(\phi) - C_{k+2}(\phi)|\mathcal{F}_k] + \mathrm{Var}[C_{k+2}(\phi) - C_k(\phi)|\mathcal{F}_k] \\
&= \mathbf{E}[(C_T(\phi) - C_k(\phi))^2|\mathcal{F}_k]. \quad \square
\end{aligned} \tag{71}$$

D Proof of Theorem 1

The existence follows from backward induction. First, we solve the equivalent minimizing problem with respect to θ_j for $j = 1, \ldots, T-1$ in (19),(20) in Proposition 1.

$$\begin{aligned}
&\mathrm{Var}[C_{k+2}(\phi) - C_k(\phi)|\mathcal{F}_k] \\
&= \mathrm{Var}[V_{k+2} - V_k - \sum_{j=k+1}^{k+2} \theta_j \Delta\tilde{S}_j - \sum_{j=k+1}^{k+2} 2\lambda M \theta_j \Delta\theta_j + \sum_{j=k+1}^{k+2} M(\Delta\theta_{j+1})^2|\mathcal{F}_k] \\
&= \mathrm{Var}[V_{k+2} - V_k - \theta_{k+2}\Delta\tilde{S}_{k+2} - \theta_{k+1}\Delta\tilde{S}_{k+1} - 2\lambda M\theta_{k+2}\Delta\theta_{k+2} - 2\lambda M\theta_{k+1}\Delta\theta_{k+1} \\
&\quad + M(\Delta\theta_{k+3})^2 + M\theta_{k+2}^2 - 2M\theta_{k+2}\theta_{k+1} + M\theta_{k+1}^2|\mathcal{F}_k] \\
&= \mathrm{Var}[\eta_{k+2} + \theta_{k+3}(\tilde{S}_{k+2} + 2\lambda M\theta_{k+2}) - \theta_{k+2}\Delta\tilde{S}_{k+2} - \theta_{k+1}\Delta\tilde{S}_{k+1} - 2\lambda M\theta_{k+2}(\theta_{k+2} - \theta_{k+1}) \\
&\quad + M(\Delta\theta_{k+3})^2 + M\theta_{k+2}^2 - 2M\theta_{k+2}\theta_{k+1}|\mathcal{F}_k].
\end{aligned} \tag{72}$$

As we observe in (72), since $\mathrm{Var}[C_{k+2} - C_k|\mathcal{F}_k]$ is a quadratic function of θ_{k+1} by (24), there exists a minimum value.

Taking partial derivative of $R_k(\phi)$ with respect to θ_{k+1}, we have

$$\begin{aligned}\frac{\partial}{\partial\theta_{k+1}}R_k(\phi) &= \lim_{h\to 0}\frac{\mathrm{Var}(\theta+h)-\mathrm{Var}(\theta)}{h}\\ &= -2\mathrm{Cov}[\eta_{k+2}+\theta_{k+3}(\tilde{S}_{k+2}+2\lambda M\theta_{k+2})-\theta_{k+2}\Delta\tilde{S}_{k+2}-2\lambda M\theta_{k+2}^2\\ &\quad +M(\Delta\theta_{k+3})^2+M\theta_{k+2}^2-\theta_{k+1}(\Delta\tilde{S}_{k+1}-2\lambda M\theta_{k+2}+2M\theta_{k+2}),\\ &\quad \Delta\tilde{S}_{k+1}-2\lambda M\theta_{k+2}+2M\theta_{k+2}|\mathcal{F}_k]\\ &= 0. \end{aligned} \tag{73}$$

Hence

$$\begin{aligned}&\theta_{k+1}\\ &= \mathrm{Cov}[\eta_{k+2}+\theta_{k+3}(\tilde{S}_{k+2}+2\lambda M\theta_{k+2})-\theta_{k+2}\Delta\tilde{S}_{k+2}-2\lambda M\theta_{k+2}^2+M(\Delta\theta_{k+3})^2+M\theta_{k+2}^2,\\ &\quad \Delta\tilde{S}_{k+1}-2\lambda M\theta_{k+2}+2M\theta_{k+2}|\mathcal{F}_k]\\ &\quad /\mathrm{Var}[\Delta\tilde{S}_{k+1}-2\lambda M\theta_{k+2}+2M\theta_{k+2}|\mathcal{F}_k]. \end{aligned} \tag{74}$$

Similarly, $R_{T-1}(\phi)$ takes a minimum value at

$$\begin{aligned}\theta_T =&\mathrm{Cov}[\bar{\eta}_T+\bar{\theta}_{T+1}\tilde{S}_T+M\bar{\theta}_{T+1}^2,\Delta\tilde{S}_T+2M\bar{\theta}_{T+1}-2\lambda M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}]\\ &/\mathrm{Var}[\Delta\tilde{S}_T+2M\bar{\theta}_{T+1}-2\lambda M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}]. \end{aligned} \tag{75}$$

By (74) and (75), $\{\theta_{k+1}\}_{k=0,1,\ldots,T}$ in the local risk-minimizing strategy is obtained recursively. $\{\eta_k\}_{k=0,1,\ldots,T}$ is obtained by the martingale property of $\{C_k\}_{k=0,1,\ldots,T}$. Since $\mathbf{E}[C_T(\phi)|\mathcal{F}_k]=C_k(\phi)$,

$$\eta_k=\mathbf{E}[\bar{\eta}_T+\bar{\theta}_{T+1}S(T,0)-\sum_{j=k+1}^{T}\theta_j\Delta\tilde{S}_j-\sum_{j=k+1}^{T}2\lambda M\theta_j\Delta\theta_j+\sum_{j=k+1}^{T}M(\Delta\theta_{j+1})^2|\mathcal{F}_k]-\theta_{k+1}S(k,0). \tag{76}$$

Next we show the uniqueness part. Let $\phi:=\{(\theta_k,\eta_k)\}_{k=0,1,\ldots,T}$ and $\tilde{\phi}:=\{(\tilde{\theta}_k,\tilde{\eta_k})\}_{k=0,1,\ldots,T}$ be local risk-minimizing strategies. Then $\tilde{\theta}_k=\theta_k$ for $k=0,1,\ldots,T$ as both minimize $\mathrm{Var}[C_{k+2}(\phi)-C_k(\phi)|\mathcal{F}_k]$, which is a quadratic function with respect to θ_k. Then $\tilde{\eta_k}=\eta_k$ for $k=0,1,\ldots,T$ also follows from the fact that $\{C_k(\phi)\}_{k=0,1,\ldots,T}$ is a martingale. □

E Minimization of R_{T-1}

In this appendix, we solve the minimization problem of R_{T-1} with respect to θ_T in (14) and (15). By (9),

$$\begin{aligned}&C_T(\phi)-C_{T-1}(\phi)\\ &=V_T(\phi)-V_{T-1}(\phi)-\theta_T\Delta\tilde{S}_T+M(\Delta\theta_{T+1})^2-2\lambda_{T-1}M\theta_T\Delta\theta_T\\ &=\bar{\eta}_T+\bar{\theta}_{T+1}(\tilde{S}_T+\sum_{j=1}^{T}2\lambda_{j-1}M\Delta\theta_j)\\ &\quad -(\eta_{T-1}+\theta_T(\tilde{S}_{T-1}+\sum_{j=1}^{T-1}2\lambda_{j-1}M\Delta\theta_j))-\theta_T\Delta\tilde{S}_T+M(\Delta\theta_{T+1})^2-2\lambda_{T-1}M\theta_T\Delta\theta_T. \end{aligned} \tag{77}$$

Considering the $\mathcal{F}_{T-1}$-measurable terms in (77), we have

$$\begin{aligned}&\mathrm{Var}[C_T(\phi) - C_{T-1}(\phi)|\mathcal{F}_{T-1}]\\&= \mathrm{Var}[\bar{\eta}_T + \bar{\theta}_{T+1}(\tilde{S}_T + \sum_{j=1}^{T} 2\lambda_{j-1} M\Delta\theta_j) - \theta_T \Delta\tilde{S}_T + M\bar{\theta}_{T+1}^2 - 2M\bar{\theta}_{T+1}\theta_T|\mathcal{F}_{T-1}]. \qquad (78)\end{aligned}$$

Taking partial derivative with respect to θ_T, we have

$$\begin{aligned}&\frac{\partial}{\partial\theta_T}\mathrm{Var}[C_T(\phi) - C_{T-1}(\phi)]\\&= 2\mathrm{Cov}[\bar{\eta}_T + \bar{\theta}_{T+1}(\tilde{S}_T + \sum_{j=1}^{T} 2\lambda_{j-1} M\Delta\theta_j) - \theta_T \Delta\tilde{S}_T + M\bar{\theta}_{T+1}^2 - 2M\bar{\theta}_{T+1}\theta_T,\\&\quad - \Delta\tilde{S}_T - 2M\bar{\theta}_{T+1} + 2\lambda_{T-1}M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}]\\&= 0. \qquad (79)\end{aligned}$$

Therefore,

$$\begin{aligned}\theta_T =& \mathrm{Cov}[\bar{\eta}_T + \bar{\theta}_{T+1}(\tilde{S}_T - 2\sum_{j=1}^{T-1} \Delta\lambda_j M\theta_j) + M\bar{\theta}_{T+1}^2, \Delta\tilde{S}_T + 2M\bar{\theta}_{T+1} - 2\lambda_{T-1}M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}]\\&\Big/\mathrm{Var}[\Delta\tilde{S}_T + 2M\bar{\theta}_{T+1} - 2\lambda_{T-1}M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}]. \qquad (80)\end{aligned}$$

F Minimization of R_k $(k = 0, \dots, T-2)$

Next, we solve the minimization problem of R_k with respect to θ_{k+1} $(k = 0, \dots, T-2)$ in (14) and (15). By (9), we have

$$\begin{aligned}&C_T(\phi) - C_k(\phi)\\&= V_T(\phi) - V_k(\phi) - \sum_{j=k+1}^{T} \theta_j \Delta\tilde{S}_j - \sum_{j=k+1}^{T} 2\lambda_{j-1}M\theta_j\Delta\theta_j + \sum_{j=k+1}^{T} M(\Delta\theta_{j+1})^2\\&= \bar{\eta}_T + \bar{\theta}_{T+1}(\tilde{S}_T + \sum_{j=1}^{T} 2\lambda_{j-1}M\Delta\theta_j) - (\eta_k + \theta_{k+1}(\tilde{S}_k + \sum_{j=1}^{k} 2\lambda_{j-1}M\Delta\theta_j))\\&\quad - \sum_{j=k+1}^{T} \theta_j\Delta S_j - \sum_{j=k+2}^{T} 2\lambda_{j-1}M\theta_j\Delta\theta_j - 2\lambda_k M\theta_{k+1}\Delta\theta_{k+1}\\&\quad + \sum_{j=k+2}^{T} M(\Delta\theta_{j+1})^2 + M\theta_{k+2}^2 - 2M\theta_{k+2}\theta_{k+1} + M\theta_{k+1}^2. \qquad (81)\end{aligned}$$

Considering the $\mathcal{F}_k$-measurable random variables in (81), we have

$$\begin{aligned}&\mathrm{Var}[C_T(\phi) - C_k(\phi)|\mathcal{F}_k]\\&= \mathrm{Var}[\bar{\eta}_T + \bar{\theta}_{T+1}(\tilde{S}_T + \sum_{j=1}^{T} 2\lambda_{j-1}M\Delta\theta_j) - \sum_{j=k+1}^{T} \theta_j\Delta\tilde{S}_j - \sum_{j=k+2}^{T} 2\lambda_{j-1}M\theta_j\Delta\theta_j +\\&\quad \sum_{j=k+2}^{T} M(\Delta\theta_{j+1})^2 + M\theta_{k+2}^2 - 2M\theta_{k+2}\theta_{k+1}|\mathcal{F}_k]. \qquad (82)\end{aligned}$$

Taking partial derivative with respect to θ_{k+1}, we have

$$
\begin{aligned}
&\frac{\partial}{\partial\theta_{k+1}}\mathrm{Var}[C_T(\phi)-C_k(\phi)|\mathcal{F}_k]\\
&=2\mathrm{Cov}[\bar{\eta}_T+\bar{\theta}_{T+1}(\tilde{S}_T+\sum_{j=1}^{T}2\lambda_{j-1}M\Delta\theta_j)-\sum_{j=k+1}^{T}\theta_j\Delta\tilde{S}_j-\sum_{j=k+2}^{T}2\lambda_{j-1}M\theta_j\Delta\theta_j+\\
&\quad\sum_{j=k+2}^{T}M(\Delta\theta_{j+1})^2+M\theta_{k+2}^2-2M\theta_{k+2}\theta_{k+1},\\
&\quad-\Delta\tilde{S}_{k+1}-2\Delta\lambda_{k+1}M\bar{\theta}_{T+1}+2\lambda_{k+1}M\theta_{k+2}-2M\theta_{k+2}|\mathcal{F}_k]\\
&=0. \qquad (83)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\theta_{k+1}=\mathrm{Cov}[&\bar{\eta}_T+\bar{\theta}_{T+1}(\tilde{S}_T+2\lambda_{T-1}M\theta_T-2\sum_{j=1,j\neq k+1}^{T-1}\Delta\lambda_jM\theta_j)-\sum_{j=k+2}^{T}\theta_j\Delta\tilde{S}_j\\
&-\sum_{j=k+3}^{T}2\lambda_{j-1}M\theta_j\Delta\theta_j-2\lambda_{k+1}M\theta_{k+2}^2+\sum_{j=k+2}^{T}M(\Delta\theta_{j+1})^2+M\theta_{k+2}^2,\\
&\Delta\tilde{S}_{k+1}+2\Delta\lambda_{k+1}M\bar{\theta}_{T+1}-2\lambda_{k+1}M\theta_{k+2}+2M\theta_{k+2}|\mathcal{F}_k]\\
&/\mathrm{Var}[\Delta\tilde{S}_{k+1}+2\Delta\lambda_{k+1}M\bar{\theta}_{T+1}-2\lambda_{k+1}M\theta_{k+2}+2M\theta_{k+2}|\mathcal{F}_k]. \qquad (84)
\end{aligned}
$$

G First order expansion of θ_T

In this appendix, we calculate $\tilde{\theta}_T$ in (34).

The first order expansion of the numerator of θ_T in (27) is

$$
\begin{aligned}
&\mathrm{Cov}[\bar{\eta}_T+\bar{\theta}_{T+1}\tilde{S}_T+M\bar{\theta}_{T+1}^2,\Delta\tilde{S}_T+2M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}]\\
&\quad+\mathrm{Cov}[\bar{\theta}_{T+1}(-2\sum_{j=1}^{T-1}\Delta\lambda_jM\theta_j^{(0)}),\Delta\tilde{S}_T+2M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}]\\
&\quad+\mathrm{Cov}[\bar{\eta}_T+\bar{\theta}_{T+1}\tilde{S}_T+M\bar{\theta}_{T+1}^2,-2\lambda_{T-1}M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}]. \qquad (85)
\end{aligned}
$$

Similarly, the first order expansion of the denominator of θ_T in (27) is

$$\mathrm{Var}[\Delta\tilde{S}_T+2M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}]+2\mathrm{Cov}[-2\lambda_{T-1}M\bar{\theta}_{T+1},\Delta\tilde{S}_T+2M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}]. \qquad (86)$$

Thus we have

$$\begin{aligned}
\tilde{\theta}_T = & \frac{\mathrm{Cov}[\bar{\eta}_T + \bar{\theta}_{T+1}\tilde{S}_T + M\bar{\theta}_{T+1}^2, \Delta\tilde{S}_T + 2M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}]}{\mathrm{Var}[\Delta\tilde{S}_T + 2M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}]} \\
& + \Big\{\mathrm{Cov}[\bar{\theta}_{T+1}(-2\sum_{j=1}^{T-1}\Delta\lambda_j M\theta_j^{(0)}), \Delta\tilde{S}_T + 2M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}] \\
& \quad + \mathrm{Cov}[\bar{\eta}_T + \bar{\theta}_{T+1}\tilde{S}_T + M\bar{\theta}_{T+1}^2, -2\lambda_{T-1}M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}]\Big\} \\
& \quad \Big/ \mathrm{Var}[\Delta\tilde{S}_T + 2M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}] \\
& - 2\mathrm{Cov}[-2\lambda_{T-1}M\bar{\theta}_{T+1}, \Delta\tilde{S}_T + 2M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}] \\
& \quad \times \mathrm{Cov}[\bar{\eta}_T + \bar{\theta}_{T+1}\tilde{S}_T + M\bar{\theta}_{T+1}^2, \Delta\tilde{S}_T + 2M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}] \\
& \quad \Big/ \mathrm{Var}[\Delta\tilde{S}_T + 2M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}]^2.
\end{aligned} \tag{87}$$

H First order expansion of θ_{k+1}

Next, we calculate $\tilde{\theta}_{k+1}$ ($k = 0, \dots, T-2$) in (34). The zeroth order part and the first order part of the numerator of θ_{k+1} in (28) are as follows.

(i) The zeroth order part

$$\mathrm{Cov}[\bar{\eta}_T + \bar{\theta}_{T+1}\tilde{S}_T - \sum_{j=k+2}^{T}\theta_j^{(0)}\Delta\tilde{S}_j + \sum_{j=k+2}^{T}M(\Delta\theta_{j+1}^{(0)})^2 + M\theta_{k+2}^{(0)2},\ \Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}|\mathcal{F}_k], \tag{88}$$

(ii) The first order part

$$\begin{aligned}
& \mathrm{Cov}[\bar{\theta}_{T+1}(2\lambda_{T-1}M\theta_T^{(0)} - 2\sum_{j=1, j\neq k+1}^{T-1}\Delta\lambda_j M\theta_j^{(0)}) - \sum_{j=k+2}^{T}(\sum_{i=0}^{T-1}\partial_{\lambda_i}\theta_j^{(0)}\lambda_i)\Delta\tilde{S}_j - \sum_{j=k+3}^{T}2\lambda_{j-1}M\theta_j^{(0)}\Delta\theta_j^{(0)} \\
& \quad - 2\lambda_{k+1}M\theta_{k+2}^{(0)2} + 2\sum_{j=k+2}^{T}M\Delta\theta_{j+1}^{(0)}(\sum_{i=0}^{T-1}(\partial_{\lambda_i}\theta_{j+1}^{(0)} - \partial_{\lambda_i}\theta_j^{(0)})\lambda_i) + 2M\theta_{k+2}^{(0)}(\sum_{i=0}^{T-1}\partial_{\lambda_i}\theta_{k+2}^{(0)}\lambda_i), \\
& \quad \Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}|\mathcal{F}_k] \\
& + \mathrm{Cov}[\bar{\eta}_T + \bar{\theta}_{T+1}\tilde{S}_T - \sum_{j=k+2}^{T}\theta_j^{(0)}\Delta\tilde{S}_j + \sum_{j=k+2}^{T}M(\Delta\theta_{j+1}^{(0)})^2 + M\theta_{k+2}^{(0)2}, \\
& \quad 2\Delta\lambda_{k+1}M\bar{\theta}_{T+1} - 2\lambda_{k+1}M\theta_{k+2}^{(0)} + 2M(\sum_{i=0}^{T-1}\partial_{\lambda_i}\theta_{k+2}^{(0)}\lambda_i)|\mathcal{F}_k].
\end{aligned} \tag{89}$$

The zeroth order part and the first order part of the denominator of θ_{k+1} are as follows.

(i) The zeroth order part

$$\mathrm{Var}[\Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}|\mathcal{F}_k], \tag{90}$$

(ii) The first order part

$$2\mathrm{Cov}[\Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)},\ 2\Delta\lambda_{k+1}M\bar{\theta}_{T+1} - 2\lambda_{k+1}M\theta_{k+2}^{(0)} + 2M(\sum_{i=0}^{T-1}\partial_{\lambda_i}\theta_{k+2}^{(0)}\lambda_i)|\mathcal{F}_k]. \tag{91}$$

Thus, we have

$$
\begin{aligned}
&\tilde{\theta}_{k+1} \\
&= \frac{\mathrm{Cov}[\bar{\eta}_T + \bar{\theta}_{T+1}\tilde{S}_T - \sum_{j=k+2}^{T}\theta_j^{(0)}\Delta\tilde{S}_j + \sum_{j=k+2}^{T} M(\Delta\theta_{j+1}^{(0)})^2 + M\theta_{k+2}^{(0)2}, \Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}|\mathcal{F}_k]}{\mathrm{Var}[\Delta\tilde{S}_{k+1} + 2M\theta_{k+2}|\mathcal{F}_k]} \\
&+ \Big\{\mathrm{Cov}[\bar{\theta}_{T+1}(2\lambda_{T-1}M\theta_T^{(0)} - 2\sum_{j=1,j\neq k+1}^{T-1}\Delta\lambda_j M\theta_j^{(0)}) - \sum_{j=k+2}^{T}(\sum_{i=0}^{T-1}\partial_{\lambda_i}\theta_j^{(0)}\lambda_i)\Delta\tilde{S}_j - \sum_{j=k+3}^{T}2\lambda_{j-1}M\theta_j^{(0)}\Delta\theta_j^{(0)} \\
&- 2\lambda_{k+1}M\theta_{k+2}^{(0)2} + 2\sum_{j=k+2}^{T}M\Delta\theta_{j+1}^{(0)}(\sum_{i=0}^{T-1}(\partial_{\lambda_i}\theta_{j+1}^{(0)} - \partial_{\lambda_i}\theta_j^{(0)})\lambda_i) + 2M\theta_{k+2}^{(0)}(\sum_{i=0}^{T-1}\partial_{\lambda_i}\theta_{k+2}^{(0)}\lambda_i), \\
&\Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}|\mathcal{F}_k] \\
&+ \mathrm{Cov}[\bar{\eta}_T + \bar{\theta}_{T+1}\tilde{S}_T - \sum_{j=k+2}^{T}\theta_j^{(0)}\Delta\tilde{S}_j + \sum_{j=k+2}^{T}M(\Delta\theta_{j+1}^{(0)})^2 + M\theta_{k+2}^{(0)2}, \\
&2\Delta\lambda_{k+1}M\bar{\theta}_{T+1} - 2\lambda_{k+1}M\theta_{k+2}^{(0)} + 2M(\sum_{i=0}^{T-1}\partial_{\lambda_i}\theta_{k+2}^{(0)}\lambda_i)|\mathcal{F}_k]\Big\} \\
&\Big/ \mathrm{Var}[\Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}|\mathcal{F}_k] \\
&- 2\mathrm{Cov}[\Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}, 2\Delta\lambda_{k+1}M\bar{\theta}_{T+1} - 2\lambda_{k+1}M\theta_{k+2}^{(0)} + 2M(\sum_{i=0}^{T-1}\partial_{\lambda_i}\theta_{k+2}^{(0)}\lambda_i)|\mathcal{F}_k] \\
&\times \mathrm{Cov}[\bar{\eta}_T + \bar{\theta}_{T+1}\tilde{S}_T - \sum_{j=k+2}^{T}\theta_j^{(0)}\Delta\tilde{S}_j + \sum_{j=k+2}^{T}M(\Delta\theta_{j+1}^{(0)})^2 + M\theta_{k+2}^{(0)2}, \Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}|\mathcal{F}_k] \\
&\Big/ \mathrm{Var}[\Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}|\mathcal{F}_k]^2. \qquad (92)
\end{aligned}
$$

I Expression of $\tilde{\theta}_{k+1}$ $(k = 0, \dots, T)$ in the expansion form (34)

Rewriting (87) and (92) in Appendices G & H, we have the following expression for $\tilde{\theta}_{k+1}$.

$$
\begin{aligned}
\tilde{\theta}_{k+1} &= \theta_{k+1}^{(0)} + \sum_{i=1}^{T-1}\partial_{\lambda_i}\theta_{k+1}^{(0)}\lambda_i \\
&= \frac{C^{(k+1)}}{A^{(k+1)}} + \sum_{j=0}^{T-1}\frac{A^{(k+1)}D_j^{(k+1)} - B_j^{(k+1)}C^{(k+1)}}{(A^{(k+1)})^2}\lambda_j. \qquad (93)
\end{aligned}
$$

Here, $A^{(k+1)}, B_j^{(k+1)}, C^{(k+1)}, D_j^{(k+1)}$ are as follows.

I.1 The case $k = T - 1$

$$A^T = \mathrm{Var}[\Delta\tilde{S}_T + 2M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}] \qquad (94)$$

For B_j^T,

1. when $j = T - 1$,

$$B_j^T = 2\mathrm{Cov}[-2\lambda_{T-1}M\bar{\theta}_{T+1}, \Delta\tilde{S}_T + 2M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}], \qquad (95)$$

2. when $j = 0, \dots, T-2$,

$$B_j^T = 0. \tag{96}$$

$$C^T = \mathrm{Cov}[\bar{\eta}_T + \bar{\theta}_{T+1}\tilde{S}_T + M\bar{\theta}_{T+1}^2, \Delta\tilde{S}_T + 2M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}]. \tag{97}$$

For D_j^T,

1. when $j = T-1$,

$$\begin{aligned} D_j^T &= \mathrm{Cov}[\bar{\theta}_{T+1}(-2M\theta_j^{(0)}),\ \Delta\tilde{S}_T + 2M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}] \\ &\quad + \mathrm{Cov}[\bar{\eta}_T + \bar{\theta}_{T+1}\tilde{S}_T + M\bar{\theta}_{T+1}^2,\ -2M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}], \end{aligned} \tag{98}$$

2. when $j = 0, \dots, T-2$,

$$D_j^T = \mathrm{Cov}[\bar{\theta}_{T+1}(-2M\Delta\theta_{j+1}^{(0)}),\ \Delta\tilde{S}_T + 2M\bar{\theta}_{T+1}|\mathcal{F}_{T-1}]. \tag{99}$$

I.2 The case $k = 0, \dots, T-2$

$$A^{(k+1)} = \mathrm{Var}\Big[\Delta\tilde{S}_{k+1} + 2M\theta_{k+2}\Big|\mathcal{F}_k\Big]. \tag{100}$$

For $B_j^{(k+1)}$,

1. when $j \neq k, k+1$,

$$B_j^{(k+1)} = 2\mathrm{Cov}\Big[\Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)},\ 2M\partial_{\lambda_j}\theta_{k+2}^{(0)}\Big|\mathcal{F}_k\Big], \tag{101}$$

2. when $j = k$,

$$B_j^{(k+1)} = 2\mathrm{Cov}\Big[\Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)},\ 2M\bar{\theta}_{T+1}^{(0)} + 2M\partial_{\lambda_j}\theta_{k+2}^{(0)}\Big|\mathcal{F}_k\Big], \tag{102}$$

3. when $j = k+1$,

$$B_j^{(k+1)} = 2\mathrm{Cov}\Big[\Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)},\ 2M\bar{\theta}_{T+1}^{(0)} - 2M\theta_{k+2}^{(0)} + 2M\partial_{\lambda_j}\theta_{k+2}^{(0)}\Big|\mathcal{F}_k\Big]. \tag{103}$$

$$C^{(k+1)} = \mathrm{Cov}\Big[\bar{\eta}_T + \bar{\theta}_{T+1}\tilde{S}_T - \sum_{j=k+2}^{T}\theta_j^{(0)}\Delta\tilde{S}_j + \sum_{j=k+2}^{T}M(\Delta\theta_{j+1}^{(0)})^2 + M\theta_{k+2}^{(0)2},\ \Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}\Big|\mathcal{F}_k\Big]. \tag{104}$$

For $D_j^{(k+1)}$ $(k = 0, \dots, T-2)$,

1. when $j = 0, \dots, k-1$,

$$D_j^{(k+1)} \tag{105}$$

$$= \mathrm{Cov}\Big[2M\Delta\theta_{j+1}^{(0)} - \sum_{i=k+2}^{T} \partial_{\lambda_j}\theta_i^{(0)}\Delta\tilde{S}_i \tag{106}$$

$$+ 2\sum_{i=k+2}^{T} M\Delta\theta_{i+1}^{(0)}(\partial_j\theta_{i+1}^{(0)} - \partial_j\theta_i^{(0)}) + 2M\theta_{k+2}^{(0)}\partial_j\theta_{k+2}^{(0)},\ \Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}\Big|\mathcal{F}_k\Big]$$

$$+ \mathrm{Cov}\Big[\bar{\eta}_T + \bar{\theta}_{T+1}\tilde{S}_T - \sum_{i=k+2}^{T}\theta_i^{(0)}\Delta\tilde{S}_i + \sum_{i=k+2}^{T} M(\Delta\theta_{i+1}^{(0)})^2 + M\theta_{k+2}^{(0)2},\ 2M\partial_j\theta_{k+2}^{(0)}\Big|\mathcal{F}_k\Big], \tag{107}$$

2. when $j = k$,

$$D_j^{(k+1)} \tag{108}$$

$$= \mathrm{Cov}\Big[-2M\theta_j^{(0)} - \sum_{i=k+2}^{T} \partial_{\lambda_j}\theta_i^{(0)}\Delta\tilde{S}_i \tag{109}$$

$$+ 2\sum_{i=k+2}^{T} M\Delta\theta_{i+1}^{(0)}(\partial_j\theta_{i+1}^{(0)} - \partial_j\theta_i^{(0)}) + 2M\theta_{k+2}^{(0)}\partial_j\theta_{k+2}^{(0)},\ \Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}\Big|\mathcal{F}_k\Big]$$

$$+ \mathrm{Cov}\Big[\bar{\eta}_T + \bar{\theta}_{T+1}\tilde{S}_T - \sum_{i=k+2}^{T}\theta_i^{(0)}\Delta\tilde{S}_i + \sum_{i=k+2}^{T} M(\Delta\theta_{i+1}^{(0)})^2 + M\theta_{k+2}^{(0)2},$$

$$+ 2M\partial_j\theta_{k+2}^{(0)} - 2M\bar{\theta}_{T+1}\Big|\mathcal{F}_k\Big], \tag{110}$$

3. when $j = k+1$,

$$D_j^{(k+1)} \tag{111}$$

$$= \mathrm{Cov}\Big[2M\theta_{j+1}^{(0)} - \sum_{i=k+2}^{T} \partial_{\lambda_j}\theta_i^{(0)}\Delta\tilde{S}_i \tag{112}$$

$$+ 2\sum_{i=k+2}^{T} M\Delta\theta_{i+1}^{(0)}(\partial_j\theta_{i+1}^{(0)} - \partial_j\theta_i^{(0)}) + 2M\theta_{k+2}^{(0)}\partial_j\theta_{k+2}^{(0)},\ \Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}\Big|\mathcal{F}_k\Big]$$

$$+ \mathrm{Cov}\Big[\bar{\eta}_T + \bar{\theta}_{T+1}\tilde{S}_T - \sum_{i=k+2}^{T}\theta_i^{(0)}\Delta\tilde{S}_i + \sum_{i=k+2}^{T} M(\Delta\theta_{i+1}^{(0)})^2 + M\theta_{k+2}^{(0)2},$$

$$+ 2M\partial_j\theta_{k+2}^{(0)} + 2M\bar{\theta}_{T+1} - 2M\theta_{k+2}^{(0)}\Big|\mathcal{F}_k\Big], \tag{113}$$

4. when $j = k+2, \ldots, T-2$,

$$
\begin{aligned}
& D_j^{(k+1)} && (114)\\
&= \mathrm{Cov}\Big[2M\Delta\theta_{j+1}^{(0)} - \sum_{i=k+2}^{T} \partial_{\lambda_j}\theta_i^{(0)}\Delta\tilde{S}_i - 2M\theta_{j+1}^{(0)}\Delta\theta_{j+1}^{(0)} && (115)\\
&\quad + 2\sum_{i=k+2}^{T} M\Delta\theta_{i+1}^{(0)}(\partial_j\theta_{i+1}^{(0)} - \partial_j\theta_i^{(0)}) + 2M\theta_{k+2}^{(0)}\partial_j\theta_{k+2}^{(0)},\ \Delta\tilde{S}_{k+1} + 2M\theta_{k+2}^{(0)}\Big|\mathcal{F}_k\Big] \\
&\quad + \mathrm{Cov}\Big[\bar{\eta}_T + \bar{\theta}_{T+1}\tilde{S}_T - \sum_{i=k+2}^{T}\theta_i^{(0)}\Delta\tilde{S}_i + \sum_{i=k+2}^{T} M(\Delta\theta_{i+1}^{(0)})^2 + M\theta_{k+2}^{(0)2},\ 2M\partial_j\theta_{k+2}^{(0)}\Big|\mathcal{F}_k\Big]. && (116)
\end{aligned}
$$

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