CARF Working Paper

CARF-F-424

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October, 2017

CARF is presently supported by The Dai-ichi Life Insurance Company, Limited, Nomura Holdings, Inc., Sumitomo Mitsui Banking Corporation, Bank of Tokyo-Mitsubishi UFJ, Finatext Ltd., GCI Asset Management, Inc and The University of Tokyo Edge Capital Co.. This financial support enables us to issue CARF Working Papers.

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Abstract

GDP statistics have been a focus of debates, especially about whether real (constant-price) GDP figures appropriately represent the economic conditions of the economy. This paper shows that the official nominal (current price) GDP is the nominal value of utility the nation enjoys from the current consumption and future consumption that current investment enables to realize in an ideal-type economy with money, which is an intuitive interpretation. However, in this framework, an appropriate real GDP is not the official constant-price GDP but the utility or value GDP, which is the nominal GDP divided by the Consumer Price Index (most broadly defined).

The views expressed in this paper are our own and do not necessarily reflect those of the Statistics Commission of Japan and the institutions we are affiliated with. Financial supports to the first author from CARF at the University of Tokyo and JSPS KAKEN(S) #25220502 are gratefully acknowledged.

1. INTRODUCTION AND NON-TECHNICAL SUMMARY

The real economic world consists of numerous kinds of goods, services, buildings, machines, and equipment. At the same time, a wide disparity is observed in income and asset holdings among households and in capital stocks such as buildings, and equipment among firms. In order to assess economic activities in a real world in a concise way, economic statistics, such as price indexes (e.g. CPI) and quantity indexes (e.g. GDP) have been invented and constructed, based on which we grasp economic conditions and make economic forecasts. Furthermore, in the macroeconomic analysis, we have developed various “models” by combining a multitude of aggregate or macroeconomic statistics to analyze the dynamism of the economy, which have contributed to our understanding of the real world and thus enable us to make reasonable forecasts of future events.

To be more specific, what do these concepts of economic statistics such as price and quantity indexes, and especially the aggregate or macroeconomic concepts of the general price level and the Gross Domestic Product of the overall economy represent? A hint to this question may be found microeconomics, which forms the basis of economics in general. An optimal resource allocation is realized when there is no distortion in the markets, since every household maximizes its utility under a budget constraint and every firm maximizes its profit under a resource constraint. Then, because real economic activity is the result of the optimizing behaviors of households and firms, the level of households’ satisfaction (their utility level) and the amount of resources utilized by firms should also have some linkages with these economic statistics in macroeconomics.

The purpose of this paper is to show that we can fully explain the complex economic conditions by using just aggregate variables and concepts such as aggregate income and aggregate production function -- when we assume an “ideal-type” monetized economy, and to indicate that a quantity index represents total utility and a price index the unit nominal cost of the unit utility. Here, we can disregard a huge variance in income among households and in capital stocks among firms to understand the state of the macroeconomy. In other words, we can analyze a complex macroeconomic world in the same way as a simple and straightforward microeconomic model of one representative household (preferences) and one representative firm (technology).

Furthermore, we will show that macroeconomic statistics such as “household consumption” and “business investment” have intuitive interpretations based on households’ present and future utility from consumption. In the ideal-type monetized economy, aggregate consumption as an economic index is the total nominal value of utility from *current* consumption of the representative household (*i.e*., nation), and net aggregate investment is the (expected) nominal discounted present value of total utility from *future* consumption of the representative household (nation) that today's investment will create in the future. Moreover, in a closed economy version of the ideal-type monetized economy, we will demonstrate how the sum of consumption and investment in utility terms, in other words, real (utility) net domestic product, is proportional to (utility) national wealth. Here, (utility) national wealth is the (expected) total utility value of all existing capital stocks in the economy, and is equal to the (expected) discounted present value of maximal current and future utility that the capital stocks can create. In addition, in the ideal-type economy, which is a monetized economy, the quantity theory of money holds because we assume money acts solely as a medium of exchange through transaction banking and, as has often been said, "money is a veil," although we have relationship banking at the same time.

We outline the "ideal-type economy" as a perfectly competitive economy with complete markets with money in the banking system as a sole medium of exchange, in which, (1) the utility functions of households are identical and linearly homogeneous so that the marginal utility of income is constant and (2) the production functions of businesses are also identical and linearly homogeneous so that the return of technology is constant to scale. In the static version of the ideal-type economy, we do not assume the same income for each household or the same capital stocks for each firm. Instead, we allow observed variables such as "income" and "capital stocks" are varied considerably. However, we assume that unobservable utility functions and production functions are identical and linearly homogeneous, and by doing so we can disregard the huge variations in observed variables. In the dynamic closed-economy version of the ideal-type economy, we assume one representative household (that is, "nation") and one representative firm (that is, "technology"), which maximize intertemporal utility and profits. By doing so, we can grasp the meaning of aggregate concepts such as Gross Domestic Product (GDP) and relate it to National Wealth, which should be considered not as physical capital stocks but present and (expected) future utility that these capital stocks generate.

It should be noted here that the word "ideal-type" is used in this paper in a manner similar to Max Weber's *idealtypus*.[1] That is, we do not claim that the real economy satisfies the conditions of the ideal-type economy we postulate. Rather, it provides a useful frame of reference (or a yardstick) in understanding economic statistics that aggregate numerous and voluminous pieces of information. In fact, in mainstream central banking practices, the macroeconomy is often modeled

[1] See Weber (1949).

with a representative household and market clearing conditions in the long run, and the short run is formulated as the deviation from the long run caused by many rigidities and frictions. Thus, the real economy can be considered as a deviation from this ideal-type economy, and the effect of specific deviation can be explicitly analyzed. Also, specific weakness in the current practice in understanding the real economy can be identified and possibly rectified in this framework, rather than a wholesale rejection of economic statistics as irrelevant to people's needs and deficient in policy making, often found in popular presses and political documents.

Although for nominal GDP (or Current Price GDP) our framework coincides with the traditional national account framework, it differs from the official practice for "real" GDP. Our framework can be called a utility approach (or "value" approach) since it is based on the utility (or valuation) of the representative household, while the current practice of national accounts can be described as a physical approach, in which "physical outputs" are in principle constructed by reference-year constant price methods. This method tries to detach the measurement of output from utility consideration as much as possible. However, by doing so, real GDP loses the "meaning" of economic well-being (of the nation), which ordinary people including politicians and policy makers implicitly assume. Moreover, since many economic activities become virtual and non-physical, the concept of "physical outputs" is increasingly difficult to apply in many fields especially now-dominant services sectors (broadly defined). Technology switches very easily from one method to another, making identification difficult of a particular physical-output-producing "technology". In contrast, our framework virtually assumes that (current and future) utility is *the output*, so that the meaning is clear. Also, since utility is the output, it is versatile to accommodate changes and switches of virtual technology. Of course, there are problems in this approach too, with respect to specific utility (valuation) assumptions (interpersonal/intertemporal comparison of utility or valuation) and in the end, we cannot escape the measurement of "physical output" if one really wants to examine "physical productivity" as opposed to "utility (or value) productivity." Thus, this utility approach is not a substitute for the existing physical approach, but a complement to it.

The framework presented in this paper, which can be called Dynamic General Equilibrium framework to Economic Statistics, is adopted from Nishimura (1983)[2]. Although it had been largely ignored and unnoticed in the next twenty years, this approach has gotten traction in the literature

[2] Nishimura (1983) uses this framework and exploit the relationship between the optimal growth and the market economy to show aggregate investment in national accounting has a modified accelerator relationship with aggregate income.

since the turn of the twenty-first century[3]. It has been gaining support from many researchers and some practitioners (see Sefton and Weale 2006, as an example of this vastly expanding literature[4]). However, it is not yet incorporated in official statistics.

The utility approach is based on the research of the theoretical economic dynamics literature[5]. Main contributions of this paper are to extend the model explicitly incorporating monetary transaction banking and relationship banking, and to explain the basic logic behind the results in an intuitive Hamilton-Jacobi equation framework of the conventional dynamic optimization theory. In this way, we can show an appropriate way to relate the concepts implicit in economic modeling practices of policy makers such as central banks to the concepts in the national accounts.

[3] See Dasgupta and Mäler, (2000) and Weitzman (2000).

[4] Our "(utility) net national product" is close to "real income" in Sefton and Weale (2006).

[5] A seminal work is Weiztman (1976).

2. Static Ideal-Type Economy: Economic Implications of Price Indexes and Quantity Indexes

Let us begin with a static economy, in which the main results are straightforward extension of well-known ones in microeconomics. We restate them for the sake of completeness and for a precursor of a dynamic economy.

Individual Behavior of Households and Firms

Individual behavioral assumptions are utility maximization of the households subject to a budget constraint, and profit maximization of firms subject to a technological constraint.

We assume there are *n* households with the same utility functions. Household *i* decides the amounts of consumption of good *x* (x_i^D) and good *y* (y_i^D) to maximize its utility function $u(x_i^D, y_i^D)$ under the budget constraint $px_i^D + qy_i^D \leq I_i$, where I_i represents the income of household *i*, *p* the price of good *x*, and *q* the price of good *y*.

$$\max_{x_i^D, y_i^D} u(x_i^D, y_i^D) \ subject\ to\ px_i^D + qy_i^D \leq I_i$$

Further, we assume there are *m* firms that have the same technology. Consider that x_j^S is the supply of good *x* by firm *j*, y_j^S the supply of good *y* by firm *j*, and *C* a fixed cost. Profit is then expressed as $\mathrm{p}x_j^S + qy_j^S - C$. Firm *j* decides the amounts of supply of good *x* (x_j^S) and good *y* (y_j^S) to maximize its profit $\mathrm{p}x_j^S + qy_j^S - C$ under the resource constraint $\mathrm{G}(x_j^S, y_j^S) \leq K_j$, where K_j represents the amount of resources that firm *j* has, and $\mathrm{G}(x_j^S, y_j^S)$ is the amount of resources required to produce and supply (x_j^S, y_j^S). We call G the required resource function, which represents technology.

$$\max_{x_i^S, y_i^S} \mathrm{p}x_j^S + qy_j^S - C \ \ subject\ to\ \mathrm{G}(x_j^S, y_j^S) \leq K_j$$

Market Equilibrium

Equilibrium in the goods market is determined by matching total demand and total supply:

$$\sum_{n} x_i^D = \sum_{m} x_i^S$$
$$\sum_{n} y_i^D = \sum_{m} y_i^S$$

Optimum Resource Allocation and Finding Equilibrium

The fundamental theorem of welfare economics tells us that the optimal resource allocation is achieved at market equilibrium. Thus, it is possible to find market equilibrium in this line. To be specific, we can consider the equilibrium market price to be the shadow price, which helps to achieve an optimal resource allocation. We can draw the market price on a graph as a slope of a separating hyperplane that separates the set of possible consumption from the set of possible production. There is no unexploited arbitrage opportunity at market equilibrium, where "arbitrage" is to adjust the proportion of demand and supply while satisfying each budget or required resource constraint.

Additional Assumptions

We assume that the utility function *u* is concave and is a linear homogeneous function of x_i^D and y^D such that

$$\mathrm{u}\left(h \cdot x_i^D, h \cdot y_i^D\right) = h \cdot \mathrm{u}\left(x_i^D, y_i^D\right)$$
$$\frac{\partial}{\partial x_i^D} u\left(h \cdot x_i^D, h \cdot y_i^D\right) = \frac{\partial}{\partial x_i^D} \mathrm{u}\left(x_i^D, y_i^D\right)$$
$$\frac{\partial}{\partial y_i^D} u\left(h \cdot x_i^D, h \cdot y_i^D\right) = \frac{\partial}{\partial y_i^D} \mathrm{u}\left(x_i^D, y_i^D\right)$$

Also, the required resource function *G* is concave and is a linear homogeneous function of x_j^S and y_j^S, satisfying the above properties replacing *u* with *G*. In addition, households and firms are price takers without power to determine their own prices in a market. Finally, there is no uncertainty.

Indirect Utility, Price Indexes and Quantity Indexes in the Ideal-Type Economy

Let us first define an indirect utility function as follows

$$\mathrm{V}(p,q,I_i) \equiv u\left(x_i^{D^*}, y_i^{D^*}\right) = \max_{x_i^D, y_i^D} u\left(x_i^D, y_i^D\right) \, subject\ to\ px_i^D + qy_i^D \leq I_i$$

Given the above, the following equation can be derived from the linear homogeneity of the utility function *u*:

$$\mathrm{V}(p,q,I_i) = v^*(p,q) I_i$$

Here, v^* is linear homogeneous in prices.

$$v^*(h \cdot p, h \cdot q) = h \cdot v^*(p,q)$$

This yields the following equation:

$$I_i = \frac{1}{v^*(p,q)} V(p,q,I_i) ==> px_i^{D^*} + qy_i^{D^*} = P \times Q_i$$

where

$$P = \frac{1}{v^*(p,q)} \quad and \quad Q_i = \mathrm{V}(p,q,I_i) = u\left(x_i^{D^*}, y_i^{D^*}\right)$$

Using the above results, we can derive an aggregate formula for the overall economy. We define "total income" of *n* households and "total production" of *m* firms as follows:

$$I = \sum_n I_i \ and \ Q = \sum_n Q_i = \sum_n u\left(x_i^{D^*}, y_i^{D^*}\right)$$

Thus, the price of quantity index *Q*, *i.e.* "total (gross) production," is price index *P*.

$$I = P \times Q$$

To sum up, we obtain the following relationships in the static ideal-type economy:

- Quantity Index = Aggregate Total Utility

- Price Index = a Required Expense to Acquire a Unit of Total Utility

3. Dynamic Ideal-Type Economy: The implications of Gross Domestic Product and National Wealth

As we have seen above, a quantity index represents total utility and a price index the cost to acquire a unit utility in the ideal-type economy in a static environment (at one point in time). However, in a real economy, time is not completed at one point; today's actions are based on the past while at the same time taking the future into account. In this sense, the obvious difference between a static economy and a dynamic economy lies in investment. In this dynamic environment, what does economic statistics such as Gross Domestic Product (GDP) signify? This question is particularly relevant, since GDP comprises business investment as a main component alongside household consumption in the real world.

Based on the argument in the static ideal-type economy, we assume a representative household (which can be called the "nation") with linear homogeneous utility function and a representative firm with linear homogenous production technology in the dynamic ideal-type economy.

Dynamics essentially involves future uncertainty and expectations about it. In the ideal-type economy, we assume perfect foresight so that decision makers can forecast the future movement of prices and quantities. This is tantamount to assume that decision makers have the true model.

Individual Behavior of a Representative Household and a Representative Firm

As in the previous section of a static economy, let us begin with individual behavior: a representative household and a representative firm.

A representative household is assumed to maximize the discounted present value of total utility from today to the future (discounted by the rate of time preference)

$$\max_{X_t^D, \dot{K}_t} \int_{t=0}^{\infty} u(X_t^D) exp(-\rho t) dt$$

subject to a contemporary budget constraint:

$$p_t X_t^D + q_t I_t^D \leq w_t \bar{L} + \pi_t + r_t K_t$$

capital formation relationship:

$$I_t^D = \dot{K}_t + \delta K_t$$

and initial capital stocks:

$$K_0 = \overline{K_0},$$

where following notations are utilized throughout the paper:

ρ : rate of time preference

X_t^D : consumption vector of goods at time *t*

p_t: price vector of consumption goods at time *t*

I_t^D :: (gross) investment vector at time *t*

q_t : price vector of investment goods at time *t*

L : labor input vector

w_t: wage rate vector at time *t*

π_t : profit of firms at time *t*

K_t: capital stock vector at the beginning of time *t*

r_t: : capital rental price vector at time *t*

$\dot{K}_t$: net investment vector at time *t* （the time derivative of the capital stock vector）

δ : (the matrix of) the rate of depreciation

K_0 : capital stock vector at time 0

Here all "prices" are nominal prices.

In a similar way, a representative firm maximizes the discounted present value of total profits from today to the future (discounted by the market interest rate).

$$\max_{X_t^D, \dot{K}_t} \int_{t=0}^{\infty} \pi_t exp\left(-\int_{s=0}^{t} i_s ds\right) dt$$

where π_t is the contemporary profit at time *t*:

$$\pi_t = \{p_t X_t + q_t I_t - w_t L_t^D - r_t K_t^D\}$$

subject to a technological constraint:

$$X_t \leq F(I_t;\ K_t^D, L_t^D) e^{gt}$$

Here the following notations are utilized throughout the paper:

X_t: supply vector of consumption goods at time *t*

I_t: supply vector of investment goods at time *t*

K_t^D: demand vector of capital stocks at time *t*

L_t^D: demand vector of labor at time *t*

$F \times e^{gt}$: generalized production function, which is the maximum amount of consumption goods when capital (K_t^D) and labor (L_t^D) are inputted and investment goods (I_t) are produced)

g: rate of technical progress (assuming it is constant)

i_t: market interest rate

Taking account of the argument in the static ideal-type economy, we assume linear homogeneity of the utility function and production technology for the representative household and the representative firm, respectively. Specifically, we assume the utility function u is concave and linear homogeneous, where $h \cdot X_t^D$ means each component of vector X_t^D is multiplied by h,

$$\mathrm{u}(h \cdot X_t^D) = h \cdot u(X_t^D)$$

and

$$\frac{\partial}{\partial X_t^D} u(h \cdot X_t^D) = \frac{\partial}{\partial X_t^D} u(X_t^D)$$

We also assume the generalized production function F is concave and linear homogeneous.

$$\mathrm{F}(h \cdot I_t; h \cdot K_t^D, h \cdot L_t^D) = h \cdot \mathrm{F}(I_t; K_t^D, L_t^D)$$

Monetary Transactions

Finally, we assume all transactions are mediated by money held by banks. Specifically, we assume the following *instantaneous* process of monetary transactions: at the beginning of time *t*, firms borrow money from banks for wage payments ($w_t\, L$), rental payments of capital stocks (r_t ${K_t}^D$), and profit as dividends (π_t), and pay these to households. Households buy consumption goods (${X_t}^D$) and investment goods (${I_t}^D$) from firms by using the money. Firms then repay banks with the money earned $p_t X_t^D + q_t I_t^D$ at the end of time *t*. We assume no transaction costs to arise throughout this instantaneous money transaction. Thus, firms' demand for money can therefore be expressed as below:

$$M_t^D = \{w_t \bar{L} + r_t K_t + \pi_t\}.$$

Market Equilibrium and No-Arbitrage-Opportunity Conditions among Markets

There are five categories of markets in this dynamic ideal-type economy: the consumption goods markets, investment goods markets, rental markets of capital stocks, the labor markets, and money. Equilibrium is achieved when the following conditions are met.

1. The consumption goods markets: $X_t^D = X_t$
2. The investment goods markets: $I_t^D = I_t$
3. The rental markets of the capital stocks: $K_t^D = K_t$
4. The labor markets: $L_t^D = \bar{L}$
5. The (instantaneous) money market: $M_t^S = M_t^D$

In addition to the condition that demand equals supply, equilibrium requires there should not be unexploited arbitrage opportunities between markets both instantaneously and intertemporally.

The first no-arbitrage-opportunity condition is between investment goods markets and the ownership markets of capital stocks. The obvious arbitrage opportunities for speculators arise when the price of investment goods differs from that of unit ownership of corresponding capital stocks, since their physical characteristics are the same. In equilibrium, there should be no such opportunities and no speculator should earn positive profits from his/her speculation.

6. No-arbitrage-opportunity condition in the ownership market equilibrium of capital stocks.
   Price of investment goods = Price of unit ownership of corresponding capital stocks

The second possible arbitrage is a temporal arbitrage between the money market and the markets of capital stocks. A speculator can borrow money from the bank at the beginning of today, buy investment goods at the end of today to get capital stocks tomorrow, and then get rents by renting them to the firm, sell the stocks and repay the debt from the proceed to the bank at the end of tomorrow.

The speculator's maximum profit from borrowing one unit of money from time *t* to time *t*+Δ is

$$(q_{t+\Delta} + r_{t+\Delta} - q_t)z^* = Max_z \ \{(q_{t+\Delta} + r_{t+\Delta} - q_t)z \quad subject\ to\ q_t z = 1\}$$

while the speculator's borrowing cost is $i_{t+\Delta}$. Consequently, the no unexploited opportunity arbitrage condition is $(q_{t+\Delta} + r_{t+\Delta} - q_t)z^* = i_{t+\Delta}$. Letting Δ go to zero, we find the no arbitrage opportunity condition that determines the market interest rate. Moreover, since the optimization leads to a corner solution, the equilibrium requires the all capital stocks have the same rate of returns. Thus, we have the following equilibrium.

7. No-arbitrage-opportunity condition in the (temporal) money market equilibrium.

$$i_t = \frac{1}{q_t^j}\left(\dot{q}_t^j + r_t^j\right) \;\; for\ all\ capital\ stocks\ (j)$$

In a very simple one good case in which the general price level is equal to the nominal price of the consumption good and the price of investment good is the same as the consumption good by definition, it immediately follows that the market interest rate should be equal to the real rental price of the capital good plus the inflation rate (rate of change of the general price level).

Definition of GDP in this closed-economy version of a dynamic ideal-type economy

In this closed-economy version of a dynamic ideal-type economy, Gross Domestic Product is defined in the following way:

$$\begin{aligned} GDP_t &= \text{(Gross Domestic Products) } p_t X_t + q_t I_t \\ &= \text{(Gross Domestic Income) } w_t \bar{L} + r_t K_t + \pi_t \\ &= w_t \bar{L}(Employee\ Compensation) + \{r_t K_t + \pi_t\}(Operating\ Surplus) \\ &= \text{(Gross Domestic Expenditure) } p_t X_t^D + q_t I_t^D \end{aligned}$$

Optimum Resource Allocation and Finding Dynamic Market Equilibrium

As in the static ideal-type economy, we can find dynamic market equilibrium by invoking the fundamental theorem of welfare economics, namely, the property that "an optimal resource allocation is achieved at market equilibrium."[6]

[6] We simply assume there exists the optimal growth solution and its resource allocation is the same as in the dynamic market economy. For the conditions of the existence and other related properties, see the appendix of Sefton and Weale (2006) and the reference therein.

Optimal Growth and Finding Equilibrium Prices through Shadow Prices

We can achieve the optimal resource allocation at market equilibrium by using the optimal growth model as follows:

$$\mathrm{W}(\overline{K_0}, 0) \equiv \max_{X_t, \dot{K}_t} \int_{t=0}^{\infty} u(X_t) \exp(-\rho t)\, dt$$

subject to

$$X_t \leq F\big(\dot{K}_t + \delta K_t;\ K_t, \bar{L}\big) e^{gt},$$
$$K_0 = \overline{K_0}$$

The market equilibrium price can then be obtained by calculating the shadow price from the optimal resource allocation, which is derived by solving the maximization problem above.

This process is as shown below. The basic idea is to transform the above non-autonomous intertemporal optimization into an autonomous intertemporal optimization problem, and linear homogeneity of preferences and technology enables us to do so. After solving the derived autonomous optimization problem, we calculate shadow prices of the non-autonomous optimization problem above.

First, we define the following: $Z_t e^{gt} = X_t$. (Z_t is a consumption vector discounted by the constant productivity growth rate. That is, Z_t is the consumption vector equivalent of X_t which happens to be in period 0). Then, we get the following equation: $X_t \leq F(I_t;\ K_t, L_t) e^{gt} => Z_t \leq F(I_t;\ K_t, L_t)$. Because of the linear homogeneity of the utility function *u*, we obtain the two equations below:

$$u(X_t) = u(Z_t e^{gt}) = u(Z_t) e^{gt}$$

$$\int_{t=0}^{\infty} u(X_t) \exp(-\rho t) dt = \int_{t=0}^{\infty} u(Z_t) exp\{-(\rho - g)t\} dt$$

Therefore, we obtain

$$\mathrm{W}(\overline{K_0}, 0) \equiv \max_{X_t, \dot{K}_t} \int_{t=0}^{\infty} u(X_t) exp(-\rho t) dt \quad s.t. \quad X_t \leq F\big(\dot{K}_t + \delta K_t;\ K_t, \bar{L}\big)\, e^{gt}, K_0 = \overline{K_0}$$
$$= \max_{Z_t e^{gt}, \dot{K}_t} \int_{t=0}^{\infty} u(Z_t e^{gt}) exp(-\rho t) dt \quad s.t. \quad Z_t e^{gt} \leq F\big(\dot{K}_t + \delta K_t;\ K_t, \bar{L}\big)\, e^{gt}, K_0 = \overline{K_0}$$
$$= \max_{Z_t, \dot{K}_t} \int_{t=0}^{\infty} u(Z_t) exp(\{-(\rho - g)t\}) dt \quad s.t. \quad Z_t \leq F\big(\dot{K}_t + \delta K_t;\ K_t, \bar{L}\big), K_0 = \overline{K_0}$$

Consequently, applying the following definition:

$$\mathrm{V}(\overline{K_0}) = \max_{Z_t,\dot{K}_t} \int_{t=0}^{\infty} u(Z_t) exp(\{-(\rho - g)t\})dt \;\; s.t. \;\; Z_t \leq F\left(\dot{K}_t + \delta K_t;\; K_t, \bar{L}\right), K_0 = \overline{K_0}$$

we obtain the relationship below, showing $(\overline{K_0}, 0)$ is equivalent to $V(\overline{K_0})$.

$$W(\overline{K_0}, 0) = V(\overline{K_0})$$

As this relation shows, we can say that the non-autonomous problem $W(\overline{K_0}, 0)$, which includes the control variable $\left(X_t, \dot{K}_t\right)$, can be translated into the autonomous problem $V(\overline{K_0})$ with $\left(Z_t, \dot{K}_t\right)$

The Bellman equation for this autonomous problem $V(\overline{K_0})$ is:

$$\mathrm{V}(K_t) = \lim_{\Delta\to 0}\left\{\max_{Z_t,\dot{K}_t}\begin{bmatrix} u(Z_t)\Delta + e^{-(\rho-g)\Delta}V(K_{t+\Delta}) \\ where\ K_{t+\Delta} = K_t + \dot{K}_t\Delta \\ subject\ to\ Z_t\Delta \leq F\left(\dot{K}_t + \delta K_t;\; K_t, \bar{L}\right)\Delta \end{bmatrix}\right\}$$

$$= \lim_{\Delta\to 0}\left\{\max_{Z_t,\dot{K}_t}\begin{bmatrix} u(Z_t)\Delta + e^{-(\rho-g)\Delta}V\left(K_t + \dot{K}_t\Delta\right) \\ subject\ to\ Z_t\Delta \leq F\left(\dot{K}_t + \delta K_t;\; K_t, \bar{L}\right)\Delta \end{bmatrix}\right\}$$

Let us define the Lagrangean of the constraint $Z_t \leq F\left(\dot{K}_t + \delta K_t;\; K_t, \bar{L}\right)$ as γ_t. Then, letting the optimal solution be $\left(Z_t^*, \dot{K}_t^*\right)$, and the Lagrangean at the optimal solution be γ_t^* evaluating at $\Delta \approx 0$, we have the following necessary conditions for optimality (1) and (2):

$$Z_t^*: \quad \frac{\partial u}{\partial X}(Z_t^*) = \gamma_t^* \tag{1}$$

$$K_t^*: \quad \frac{\partial V}{\partial K}(K_t) = -\gamma_t^* \frac{\partial F}{\partial I}\left(\dot{K}_t^* + \delta K_t;\; K_t, \bar{L}\right) \tag{2}$$

Given that the problem $W(\overline{K_0}, 0)$ is equivalent to the problem $V(\overline{K_0})$ in the first place, the above-mentioned $X_t^* = Z_t^* e^{gt}$ and $\dot{K}_t^*$ should solve the Bellman equation for $W(\overline{K_0}, 0)$ at the same time.

$$W(K_t, t) = \lim_{\Delta\to 0}\left\{\max_{X_t,\dot{K}_t}\begin{bmatrix} u(X_t)\Delta + e^{-\rho\Delta}W(K_{t+\Delta}, t+\Delta) \\ where\ K_{t+\Delta} = K_t + \dot{K}_t\Delta \\ subject\ to\ X_t\Delta \leq F\left(\dot{K}_t + \delta K_t;\; K_t, \bar{L}\right)e^{gt}\Delta \end{bmatrix}\right\}$$

$$= \lim_{\Delta\to 0}\left\{\max_{X_t,\dot{K}_t}\begin{bmatrix} u(X_t) + e^{-\rho\Delta}W\left(K_t + \dot{K}_t\Delta, t+\Delta\right) \\ subject\ to\ X_t\Delta \leq F\left(\dot{K}_t + \delta K_t;\; K_t, \bar{L}\right)e^{gt}\Delta \end{bmatrix}\right\}$$

Then, if we define the Lagrangean of the constraint $X_t \leq F(\dot{K}_t + \delta K_t;\ K_t, \bar{L})$ as λ_t and this Lagrangean at the optimal solution $\left(X_t^*[= Z_t^* e^{gt}], \dot{K}_t^*\right)$ as λ_t^* , we obtain the following intuitive relationships

$$X_t^*: \qquad \frac{\partial u}{\partial X}(X_t^*) = \lambda_t^*$$

: a shadow price vector of consumption goods in utility terms

$$\dot{K}_t: \qquad \frac{\partial W}{\partial K}(K_t, t) = -\lambda_t^* \frac{\partial F}{\partial I}\left(\dot{K}_t^* + \delta K_t\right)$$

: a shadow price vector of captial goods in utility terms

By combining these relationships, we can characterize λ_t^* and $\frac{\partial W}{\partial K}(K_t, t)$ by using γ_t^* and $\frac{\partial V}{\partial K}(K_t)$. Since we know the following relation holds because of the linear homogeneity of the utility function u,

$$\frac{\partial u}{\partial X}(X_t^* e^{-gt}) = \frac{\partial u}{\partial X}(X_t^*)$$

we obtain

$$\lambda_t^* \left[= \frac{\partial u}{\partial X}(X_t^*)\right] = \gamma_t^* \left[= \frac{\partial u}{\partial X}(Z_t^*) = \frac{\partial u}{\partial X}(X_t^* e^{-gt}) = \frac{\partial u}{\partial X}(X_t^*)\right]$$

: a shadow price vector of consumption goods in utility terms

Furthermore, we obtain the property below by taking account of $\lambda_t^* = \gamma_t^*$

$$\frac{\partial W}{\partial K}(K_t, t) \begin{bmatrix} = -\left\{\lambda_t^* \frac{\partial F}{\partial I}\left(\dot{K}_t^* + \delta K_t;\ K_t, \bar{L}\right)\right\} e^{gt} \\ = -\left\{\gamma_t^* \frac{\partial F}{\partial I}\left(\dot{K}_t^* + \delta K_t;\ K_t, \bar{L}\right)\right\} e^{gt} \end{bmatrix}$$

$$= \frac{\partial V}{\partial K}(K_t) e^{gt} \left[= -\left\{\gamma_t^* \frac{\partial F}{\partial I}\left(\dot{K}_t^* + \delta K_t;\ K_t, \bar{L}\right)\right\} e^{gt}\right]$$

: a shadow price vector of capital goods in utility terms

Price in Utility Terms (Real Price) and Price in Money Terms (Nominal Price)

So far, we have concerned with prices in utility terms, which are *real* prices in nature. Then, how prices in money terms or *nominal* prices are found in the dynamic ideal-type economy?

First, we define the general price level of consumption goods, or Consumer Price Index, P_t^* , the number satisfying $p_t X_t^* = P_t^* u(X_t^*)$, that is, a required expense to acquire a unit of total utility from consumption. Second, as we have explained in the previous section of a static ideal-type economy, P_t^* is equal to the reciprocal of $v^*(p_t)$, $(1/v^*(p_t))$, in the ideal-type economy with linear homogeneous utility function. Consequently, a natural way to find nominal prices (prices in money terms) is to multiply prices in utility terms by Consumer Price Index, P_t^*.

$$\text{a nominal price vector } p_t \text{ of consumption goods (price in money terms)} = P_t^* \lambda_t^*$$

$$\text{a nominal price vector } q_t \text{ of capital goods (price in money terms)} = P_t^* \frac{\partial W}{\partial K}(K_t, t)$$

The general price level of consumption goods is determined in the money market, which will be discussed later.

Price of Capital Goods and Price of Investment Goods

As explained in the previous section, market equilibrium is characterized by the "no unexploited arbitrage opportunity" conditions. "No unexploited arbitrage opportunity" should be present between the investment goods market and the capital goods market as we stated before. Consequently, a nominal price vector of investment goods satisfies the following relationship:

$$\text{a nominal price vector of investment goods (price in money terms) } q_t = P_t^* \frac{\partial W}{\partial K}(K_t, t)$$

GDP (Gross Domestic Product) and Real (Utility) GDP

Now we can clarify what GDP signifies in this closed-economy version of the dynamic ideal-type economy. First, it should be noted that we have the following transformation

$$\begin{aligned} GDP_t &= p_t X_t + q_t I_t \\ &= P_t^* u(X_t^*) + q_t\left(\dot{K}_t^* + \delta K_t\right) \\ &= P_t^* u(X_t^*) + P_t^* \frac{\partial W}{\partial K}(K_t, t)\left(\dot{K}_t^* + \delta K_t\right) \end{aligned}$$

$$= P_t^* \left\{ u(X_t^*) + \frac{\partial W}{\partial K}(K_t, t)\left(\dot{K}_t^* + \delta K_t\right) \right\}$$

Thus, the natural way to define real (utility) GDP is

$$\text{real (utility) } GDP_t = \frac{p_t X_t + q_t I_t}{P_t^*}$$

$$= \mathrm{u}(X_t^*) + \frac{\partial W}{\partial K}(K_t, t)\dot{K}_t^* + \frac{\partial W}{\partial K}(K_t, t)\delta K_t$$

$$= Real\ (Utility) Ne\ Product + Real\ Depreciation\ in\ utility\ terms$$

Thus, "real" means "in utility terms" and real (utility) GDP is broken down into real (utility) net product and real (utility) depreciation. Moreover, real (utility) net product can be expressed as the sum of "the total utility of today's physical consumption" and "the (expected) discounted present value of future total utility that today's net physical investment will create in the future."

It should be noted that real (utility) GDP is different from the real GDP in national accounts. In the national accounts, real consumption and real investment expenditure are calculated as a constant price measurement procedure, and then added to get real GDP. Price indexes are then calculated as a Paache index by dividing nominal GDP (consumption, investment) by real GDP (consumption, investment).

In contrast to these practices, our utility-based approach implies that *an appropriate deflator of both nominal consumption and investment expenditure is the price index of current consumption*, which may correspond to the Broadly-based Consumer Price Index or Personal Consumption Expenditure (PCE) deflator.

Determination of the General Price Level of Consumption Goods P_t^*

Finally, the money market determines the general price level of consumption goods P_t^*

$$M_t^S = \{w_t \bar{L} + r_t K_t + \pi_t\} = p_t X_t + q_t I_t$$

$$= P_t^* \left\{ u(X_t^*) + \frac{\partial W}{\partial K}(K_t, t)\left(\dot{K}_t^* + \delta K_t\right) \right\}$$

$$= P_t^* \{Real\ GDP_t\}$$

Real (utility) GDP is determined by the solution of the optimal growth problem (optimal allocation of resources). The above equation therefore suggests that the price level P_t^* is determined by the money supply M_t^S.

It is clear from this construct that an increase in money supply simply increases the general price level, and has no effect on the real economy

National Wealth and Real (Utility) GDP: Intuitive interpretation

A natural definition of "national wealth" is the present value of the (expected) maximum utility of the representative household (that is, nation) that today's available resources will create. Thus, we define (utility) national wealth in the following way:

$$\mathrm{W}(\overline{K_0},0) \equiv \max_{X_t,\dot{K}_t} \int_{t=0}^{\infty} u(X_t) exp(-\rho t) dt \; s.t. X_t \leq F\left(\dot{K}_t + \delta K_t;\ K_t, \overline{L}\right) e^{gt}, K_0 = \overline{K_0}$$

First, we derive the following equation from the Bellman equation of the problem $\mathrm{V}(\overline{K_0})$

$$\mathrm{V}(K_t) = u(Z_t^*)\Delta + e^{-(\rho-g)\Delta} V\left(K_t + \dot{K}_t^*\Delta\right)$$

Then, we apply the Taylor series expansion to the second term of the right-hand side of the above equation with respect to Δ, and evaluate the equation at Δ= 0

$$e^{-(\rho-g)\Delta} V\left(K_t + \dot{K}_t^*\Delta\right) = V(K_t) + \left\{-(\rho-g)V(K_t) + \frac{\partial V}{\partial K}(K_t)\dot{K}_t^*\right\}\Delta$$

Then, by using these expressions, we have the following transformations.

$$\mathrm{V}(K_t) = u(Z_t^*)\Delta + V(K_t) + \left\{-(\rho-g)V(K_t) + \frac{\partial V}{\partial K}(K_t)\dot{K}_t^*\right\}\Delta$$

$$\Longrightarrow 0 = u(Z_t^*)\Delta + \left\{-(\rho-g)V(K_t) + \frac{\partial V}{\partial K}(K_t)\dot{K}_t^*\right\}\Delta$$

$$\Rightarrow (\rho-g)V(K_t) = u(Z_t^*) + \frac{\partial V}{\partial K}(K_t)\dot{K}_t^*$$

The last equation is the *Hamilton-Jacobi equation.*

We can obtain another Hamilton-Jacobi equation for W from the two relationships $Z_t^* = X_t^* e^{-gt}$ and $u(X_t^* e^{-gt}) = u(X_t^*)e^{-gt}$ (linear homogeneity of utility function):

$$(\rho - g)V(K_t) = u(X_t^*)e^{-gt} + \frac{\partial V}{\partial K}(K_t)\dot{K}_t^*$$

Next, we can construct the following relationship from the necessary condition of optimality (1) and (2):

$$\frac{\partial W}{\partial K}(K_t, t) = \frac{\partial V}{\partial K}(K_t)e^{gt},$$

which implies

$$(\rho - g)V(K_t) = u(X_t^*)e^{-gt} + \left(\frac{\partial W}{\partial K}(K_t, t)e^{-gt}\right)\dot{K}_t^*$$

Because $\mathrm{V}(\overline{K_0}) = W(\overline{K_0}, 0)$, we get the equation below:

$$(\rho - g)W(\overline{K_0}, 0) = u(X_t^*) + \frac{\partial W}{\partial K}(\overline{K_0}, 0)\dot{K}_0^*$$

This automatically implies the following:

$$\mathrm{W}(\overline{K_0}, 0) = \frac{u(X_0^*) + \frac{\partial W}{\partial K}(\overline{K_0}, 0)\dot{K}_0^*}{(\rho - g)}$$

Here, the numerator of the right-hand side of the above equation is equal to the net part of real (utility) GDP (i.e., real (utility) Net Domestic Product) which is real (utility) Gross Domestic Product minus the amount of depreciation in utility terms. This consists of the utility of today's consumption and the present value of utility from future consumption that today's investment produces. Besides, the relationship between real (utility) GDP and (utility) national wealth is very simple. Real (Utility) Net Domestic Product is proportional to (Utility) National Wealth, which is equal to the sum of the today's utility and the present value of future utility that people can enjoy.

What is significant here is that real (utility) GDP (real (utility) Net Domestic Product, to be precise) reveals the value of (utility) national wealth (the discounted present value of maximized utility that today's capital stock will bring).

In conclusion, we have demonstrated the following properties in the ideal-type dynamic closed-economy. Firstly, "real" means "in utility terms". Secondly, real (utility) Gross Domestic Product is broken down into real (utility) Net Domestic Product and real Depreciation in utility terms. Thirdly, real (utility) Net Domestic Product can be expressed as the sum of "the utility of today's consumption" and "the discounted present value of total future utility that today's net investment will create. Fourthly, real (utility) net product and (utility) national wealth are directly proportional to each other.

4. Further Extensions and Interpretations

Relationship Banking and FISIM

In the previous section, we have introduced banking and monetary transactions in a very simple way. In the literature, two types of banking activities are distinguished and examined extensively: one is transaction lending and the other relationship lending (Bolton et al 2016). Transaction lending is to simply facilitate monetary transactions among firms and households, which has been formulated in the above setting in the simplest form. In contrast, relationship lending is to increase overall efficiency of the economy to provide insurance, monitoring and screening. Although relationship banking is often formulated under uncertain environment, it can be reinterpreted as an activity of the banks to raise the long-run average production efficiency of their customer firms by appropriately advising and screening them based on gathered information about them through constantly monitoring their activities.

Relationship lending activities can easily be incorporated in the representative firm framework by introducing a new type of capital stocks, called relationship capital. The representative firm's production function now includes the relationship capital services as its inputs in the same way as other capital services (that is, linear homogeneity is maintained). The representative household has this relationship capital to rent to the banks, and in turn the banks rent it to the representative firm. The representative firm uses the relationship capital as inputs improving production efficiency and thus pay competitive rents to the banks. Then, the banks pay the rent to the representative household. The only complication here is relationship capital should first be rented to the banks and then the banks rent them to the firm, and rent payments go just in the opposite way.

In the presence of relational banking, monetary transactions involve the banks in the following way. At the beginning of time t, the representative firm borrows money for wage payments and rental payments excluding payments on relationship capital, to pay them to the representative household, while the relationship capital rental payments are paid directly from the banks to the household. The household then supply labor and capital stocks to the firm and relationship capital to the bank, which in turn rent to the firm immediately. At the end of time t, the firm pays back the debt they made at the beginning of time t, which exceeds the original debt by rental payments on relationship capital. Thus, the banks get more money than they first lend to the firm, as a form of extra interest on the debt. Thus, in the presence of relationship banking, the banks' within-time lending rate is higher than within-time borrowing rate (which is zero in our setting). This corresponds to FISIM (Financial Intermediation Services Indirectly Measured) in the System of National Accounts.

Ideal-Type Economy GDP as Long-run Trend GDP or Potential GDP

In mainstream central banking practices, the macroeconomy is often modeled with a representative household and market clearing conditions in the long run, and the short run is formulated as the deviation from the long run caused by many rigidities and frictions. The ideal-type economy formulation of this paper is congenial to these practices.

In fact, the ideal-type economy can be considered as a long-run trend economy and its GDP as Potential GDP, to which short-run dynamics of the economy plagued with rigidities and frictions eventually converges. A natural way to incorporate this idea into practices is to use error correction mechanism incorporating long-run relationship and short-run slow adjustment with other purely short-run factors[7].

The insights in the utility approach may be helpful in this regard. Real (utility) GDP has a long run relationship with its components, such as a (modified) multiplier mechanism between real (utility) net domestic products and physical investment[8]. Similar relationship in the ideal-type economy aggregates may be utilized in the error correction estimation.

In addition, the framework of ideal-type economy incorporates relationship banking. Even though central banks' primary concerns include the efficiency of financial intermediation through banking institutions and the stability of the financial systems, these issues are elusive in their modeling of the macroeconomy, and at best limited to ad hoc treatment (such as "shocks" of some kind. There are on-going attempts to rectify the problem, for example, in the analysis of macroeconomic stress tests. However, this is not yet properly incorporated in the national account framework. Though rudimentary, the ideal-type economy may provide a framework to incorporate financial institutions in the macroeconomic modeling based on national accounts.

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[7] An example is found in the Bank of Japan's Quarterly Model. See Fukunaga et al (2011).

[8] See Nishimura (1983).

Utility Approach and "Virtual Outputs"

One of the most difficult problems to implement current national accounting practices of getting "real GDP" is to identify and measure "physically" virtual outputs. Many services have the property of virtual outputs to some extent. An obvious example of which one of the authors know well is "services" provided by a recipe exchange service provider.[9] Many services on the Internet are literally virtual services which is hard to quantify physically. The problem is rampant: for example, it is extremely difficult to identify exact quantity (of any) that the relationship banking produces.

One advantage of the utility approach is that we do not have to know exact quantity data to calculate real (utility) GDP. To get real (utility) GDP, it is sufficient to deflate, in the case of relationship banking, FISIM (nominal output) by the current consumption price index.

Of course, if we are concerned with "physical productivity" of these services providers, we should know the physical quantity or its reasonable proxy. In contrast, if we concerned only with "value productivity", the amount of utility they produce per unit input, it is sufficient to know real (utility) output.

[9] Nishimura had been an independent director of a company managing the largest recipe exchange site on the Internet in Japan until the spring of 2017.

5. Concluding Remarks

The GDP statistics have been the focus of debates both politically and economically in many economies, especially Japan. Market participants and politicians alike express their frustration with the "accuracy" of real GDP statistics on which economic policies are based. They often complain that the official real GDP figure deviate wildly from their own assessment of economic conditions (that is, whether the economy is stagnating or not in the case of Japan). When the real GDP is reported to increase steady in the past several years, a typical reaction is that although the official real GDP is growing but the economic conditions they see is not improving as such. At the same time, many market participants claim that the official *nominal* GDP is much better to track their assessment.

This paper has shown that the official nominal (current price) GDP is the nominal value of utility the nation (that is, the representative household of the economy) enjoys from the current consumption and future consumption that current investment enables to realize, in an ideal-type economy with money. So long as the actual economy is not deviated much from this ideal-type economy, the nominal (current price) GDP has an intuitive interpretation.

However, it has also been shown that an appropriate real GDP is not the official constant-price GDP but the utility or value GDP, which is the nominal GDP divided by the Consumer Price Index (most broadly defined). The official real constant-price GDP is essentially trying to estimate "physical outputs", since the constant price method presupposes that one can identify physical outputs in every category of "production" which we are concerned, which is becoming increasing difficult in the age of "virtual output". In contrast, the appropriate real GDP in the ideal-type economy with money is utility of the nation or the representative household. In this sense, "output" is not physical outputs, but utility (or "value") that the nation or representative agent places on the current and future consumption. Thus, the appropriate deflator to get real (utility or value) GDP is the current consumer price index (CPI as broadly as possible defined or Personal Consumption Expenditure deflator).

Other major results of this paper can be summarized as follows. In the ideal-type dynamic closed-economy with a representative household and linear homogeneous preferences and technology with standard convexity assumptions, we have demonstrated firstly that term "real" should be interpreted as "in utility terms". Secondly, real (utility) Gross Domestic Product is broken down into real (utility) Net Domestic Product and real Depreciation in utility terms. Thirdly, real (utility) Net Domestic Product can be expressed as the sum of “the utility of today’s consumption” and “the discounted present value of total future utility that today’s net investment will create. Fourthly, real (utility) net domestic product and (utility) national wealth are directly proportional to each other.

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