Intertemporal Information Loss and Asset Price Cycles*

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Abstract

I develop a tractable model of asset price cycles with a new mechanism for financial accelerator. Heterogeneous traders with different quality of private information adopt inaction strategy due to ambiguity aversion, making the price signal increasingly imprecise. In the intertemporal chain reaction, escape and loss of private information reinforce each other. The model finds that an informationally efficient economy is more vulnerable to aggregate shocks on the price signal than an informationally inefficient economy due to the adverse selection of poorly informed traders. Moreover, the relation between the degree of irrationality and economic instability is hump-shaped. The correlation of the private signals among irrational traders helps determine the sign of the feedback and suggests that endogenous financial crises can occur without trigger shocks.

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1. Introduction

Most modern finance and macroeconomics models have had difficulty predicting the financial crises that have occurred repeatedly. This is because these crises can be attributed to irrationality of decision making, heterogeneity, informational friction, and the global properties of the model–all factors that most mainstream models ignore.[1] Specifically, markets can collapse without significant aggregate news. After Black Monday in 1987, we could not identify the news that was significant enough to trigger the huge drop in the market. This was also the case during the Great Depression (Brady Report 1988).

Only a few works on market microstructure provide models for market crashes that occur without significant news (Gennotte and Leland (1990), Caplin and Leahy (1994), Lee (1998), Hong and Stein (2003)). Also, Morris and Shin (2004) and Brunnermeier and Pedersen (2005) investigate the chain reaction of sales (escape) by traders in the context of predatory trading. In these models, market participants trade strategically and exploit price swings, anticipating the selling behavior of other traders.[2]

The other framework to explain the disproportionate effect of a small shock is called the financial accelerator. The macro-finance models with financial friction and heterogeneous agents, originally investigated by Bernanke and Gertler (1989), Kiyotaki and Moore (1997) and Bernanke, Gertler and Gilchrist (1999), focus on imperfections in the financial market, such as limited enforcement and asymmetric information. The second stream of the literature, including works by Gromb and Vayanos (2002) and Lorenzoni (2008), analyzes the consequences of fire-sale phenomena due to the pecuniary externality.[3]

My model also represents financial acceleration, even though it does not involve financial friction or the pecuniary externality, and traders do not engage in strategic trading. Instead, it considers the heterogeneous qualities of information and the irrationality of traders in the market microstructure framework. These factors can generate huge fluctuations if they are combined with several key components, such as past price reference due to information constraints. Specifically, this paper seeks to answer two important questions regarding irrationality and finance:[4] How does irrationality cause a tremendous drop in the market with the acceleration of a small shock, and how does the degree of irrationality affect the magnitude of the fluctuations in asset prices?

The paper is founded on the presence of irrational or behavioral agents. I make use of ambiguity aversion, which Gilboa and Schmeidler (1989) axiomatize as the representation of Knightian uncertainty (Knight (1921)). It was first shown by Dow and Werlang (1992), and subsequently argued in several works, such as Easley and O'Hara (2009), that ambiguity averse agents adopt Ss-type inaction behavior as a result of optimization.[5] They dislike an unknown probability distribution over random results and prefer a lottery with a known objective measure. If the asset return is risky as well as ambiguous, the agents expose themselves to ambiguity just by holding a non-zero position. Therefore, they will decide to hold the asset only if the possible return outweighs the psychological cost of being

[1] Stiglitz (2017) argues that the issue with modern DSGE modeling "is not that it involves simplification: all models do. It is that it has made the wrong modeling choices, choosing complexity in areas where the core story of macroeconomic fluctuations could be told using."

[2] For models of crises based on multiple equilibria and the global game, see Morris and Shin (1998), Atkeson (2000), Goldstein and Pauzner (2004), Goldstein and Pauzner (2005), Angeletos and Werning (2006), and Ozdenoren and Yuan (2008).

[3] See also Jeanne and Korinek (2013), Moore (2013), and Dávila and Korinek (2017).

[4] Over the past decades, behavioral economics has demonstrated how irrationality can explain various economic activities which look puzzling and bizarre from a rational expectations perspective. See DellaVigna (2007) for a comprehensive survey.

[5] Inaction or Ss-type behavior is well documented and analyzed in the financial market. Most authors, such as Vissing-Jorgensen (2003) and Stokey (2008), argue that fixed cost is a fundamental factor that causes inaction, and there is some evidence for this (see Guiso, Haliassos and Jappelli (2002)). However, as Gouskova, Juster and Stafford (2004) claim, it is not plausible that the non-participation of most of the population comes solely from the fixed cost of trading.

exposed to the ambiguity.[6] A couple of papers (Illeditsch (2011), Condie, Ganguli and Illeditsch (2015), Yu (2013)) propose inefficient information aggregation of the price due to existence of ambiguity averse agents. My paper goes one step further and analyzes dynamic feedback between the price information and irrational behavior.

In my model, ambiguity averse traders are continuously heterogeneous in the quality of their private signals. They participate in the market if, and only if, their information set is precise. Since the information set consists of the (exogenous) private signal and the (endogenous) past price information, a less precise price signal in the current trading round makes them reluctant to trade in the next trading round. Given that the price information endogenously reflects the private signals of the market participants, the tremendous price movements stem from the feedback mechanism between the participation behavior of ambiguity averse traders and the change in the price information. It will be shown that this feedback mechanism brings about the multiple steady states.[7]

Once some of the participants stop trading (escape), the price information no longer reflects their private signals and can be less precise. Upon reviewing this less precise price information, even more traders will decide to stay away from the market in the next period, making the price signal even less precise. In this way, the escape of some traders triggers the escape of others through the intertemporal change in the price information. Therefore, even a small shock can create a chain reaction of escape. This results in a tremendous reduction in the total demand and causes a price drop disproportionate to the original shock. Moreover, whether this chain reaction occurs depends entirely on the information structure: the covariance (correlation) of private signals across traders matters a lot.[8] The model has a strong descriptive power for asset price movements and can explain many real-world phenomena simply by changing the information structure.

As well, the model allows us to investigate the relationship between irrationality (ambiguity aversion) and market instability. Besides the work on "irrational exuberance" by Shiller (2000), some papers, such as Epstein and Schneider (2008), Leippold, Trojani and Vanini (2008), and Mele and Sangiorgi (2010), incorporate irrationality in the form of ambiguity aversion into the finance model to illustrate the creation of excess volatility and other market anomalies. In their framework, the existence of ambiguity averse traders generates economic instability.

As for the effect of irrationality on equilibrium, the debate goes back to Friedman (1953). He claims that an evolutionary mechanism must eventually weed out the irrational traders from the market because they must pay for their irrationality. He argues that speculations by rational traders stabilize the financial market.[9] The more irrational traders become, the more stable the economy will be since they will be eliminated more easily.

In this paper, these two seemingly opposing arguments will be reconciled by demonstrating that irrationality has a nonlinear effect on economic (in)stability. Suppose that $\delta \in [0, 1]$ represents the degree of irrationality (a higher δ means more irrational traders, and $\delta = 0$ means that traders are rational).

[6] As for the other expression of irrationality, Ang, Bekaert and Liu (2005) show that agents with disappointment aversion, which is axiomatized by Gul (1991), also exhibit inaction behavior.

[7] A long list of papers look at the positive feedback between information in the public price and investment behavior, although the sources of the strategic complementarity and the positive feedback are different from my model. See, for example, Froot, Scharfstein and Stein (1992), Hirshleifer, Subrahmanyam and Titman (2006), and Goldstein, Ozdenoren and Yuan (2013).

[8] In Cespa and Vives (2015), the determinants of multiplicity are completely different from my model. To make the feedback positive, they use persistent liquidity traders and "retrospective inference." In my model, in contrast, whether the feedback becomes positive depends on the covariance structure of private signals, which is not highlighted in their paper. Also, the stability of two steady states naturally emanates from the finite heterogeneity in my model, while they rely on the exogenous behavior of long-term liquidity traders.

[9] Irrational traders cannot make the economy unstable "since the speculation can be destabilizing, in general, only if speculators on average sell when the commodity is low in price and buy when it is high."

First, given that the economy fluctuates as a result of irrational traders who frequently switch between being active and inactive, the effect of irrational traders on the asset price is limited when δ is extremely high. The reason is the Ss-type inaction strategy of irrational agents. As δ becomes elevated, the inaction region becomes very large. This implies that the switch between active and inactive becomes less frequent and the irrational traders tend to steer clear of trading. This is in line with the perspective of Friedman (1953). Hence, when the degree of irrationality is high, the market share of rational traders increases and the economy converges toward a stable (and rational) equilibrium because the irrational traders escape from the market.

As irrationality becomes intermediate, traders tend to exhibit more frequent switches between being active and inactive. Compared to rational traders, who show no inaction region, irrationality opens the door for inaction and disturbs the market. In this situation, intertemporal information loss begins to occur, and the economy becomes vulnerable to shocks. This is reminiscent of Shiller (2000) since irrationality precipitates the fluctuation of the economy through the feedback mechanism.

However, the market grows resilient again when the degree of irrationality is extremely low, i.e., when the traders are almost or completely rational. This is intuitive because the inaction region of each trader is very narrow and the switch across the inaction threshold is less frequent. The share of rational *behavior* becomes large again because irrational traders adopt an almost rational trading strategy. The upshot is that the irrationality of traders affects the (in)stability of the market equilibria in a non-monotonic way; the model yields a hump-shaped instability curve if we take δ as the horizontal axis.[10]

Furthermore, the model finds that the endogenous adverse selection mechanism is at play when the economy exhibits dramatic fluctuations. In contrast to the existing research starting with Akerlof (1970),[11] my model shows that traders with imprecise private signals are adversely selected when the economy is efficient.[12] The participation of poorly informed[13] traders is "adverse" because it makes the economy more vulnerable to shocks.

First, traders with a low-quality (imprecise) private signal rely more heavily on the price signal. This implies that (i) the behavior of these poorly informed traders is sensitive to shocks on the price signal; and (ii) in an informationally efficient state, even traders with low-quality private signals can participate in the market because the high-quality price signal compensates for the low-quality private signal. Because of (i) and (ii), an efficient economy is full of sensitive traders and becomes vulnerable to aggregate news shocks on the price signal. The symmetric arguments imply that an inefficient economy is more resilient to shocks on the public signal because only traders with precise private signals are active. Therefore, even if the positive shock hits an inefficient economy, it is difficult for it to escape from the inefficient situation. In such a case, adverse selection can be helpful to avoid the inefficiency trap and to move toward an efficient state.

[10]Even though this paper explores the within-period inaction behavior of irrational agents, we can also discuss the disappearance of irrational traders in the market from an intertemporal point of view. It is well known that the arguments based on Friedman (1953) and Fama (1965) imply that irrational traders' market share tends to diminish and they end up being eliminated by rational traders in the long run. In other words, as traders get more irrational, their contribution to price formation lasts only into the short term, while they have a limited effect when their irrationality low. This may explain the time-series interpretation of the non-linear effect of irrationality that is proposed in this paper.

[11]In the existing literature, adverse selection applies to poor-quality assets. When the economy is booming, low-quality assets are accumulated and the economy becomes vulnerable to a negative shock. See, for example, Kurlat (2013) and Fukui (2016).

[12]In line with Grossman and Stiglitz (1980), the word "(in)efficient" in this paper refers to an economy with (im)precise price information.

[13]The term "poorly (or well) informed" refers to the relative precision of the information set. Since the price is common among traders, the term "poorly informed" is independent of the quality of the price signal and refers to traders with low-quality private information.

Based on these premises, this model provides counterintuitive insights on the effect of irrationality. When the economy is in an efficient equilibrium, the price level, informativeness of the price, and the market participation rate are all high. However, it also shows higher volatility/vulnerability, and the market turns out to be irrational in the sense that the proportion of irrational traders is relatively high. On the other hand, even though an inefficient equilibrium represents a low price level, less precise price information, and weak market activity, she tends to be robust, since the market is rational (i.e., the proportion of rational traders is high). This differs from the traditional findings on multiple equilibria with "discretionary liquidity traders," such as those by Admati and Pfleiderer (1988), Pagano (1989), and Han, Tang and Yang (2016).[14] They show that the participation of liquidity traders (who are irrational in the context of REE) makes the market deeper and the price more resilient to changes in the news. Even though my model does not analyze liquidity traders as do Glosten and Milgrom (1985) and Kyle (1985), the greater participation of irrational traders brings about a more vulnerable price structure.

In independent works, Van Nieuwerburgh and Veldkamp (2006) and Fajgelbaum, Schaal and Taschereau-Dumouchel (2017) propose similar positive feedbacks between information precision and agents' activity. In the model by Van Nieuwerburgh and Veldkamp (2006), more production makes it easier to infer the fundamentals, since the noisy part of the output becomes insignificant compared to the fundamental part of the production. This explains sharp declines and slow booms (asymmetric growth rate) in business cycles. The model of irreversible investment by Fajgelbaum, Schaal and Taschereau-Dumouchel (2017) is closer to mine. When uncertainty is low (or high), each investor tends to conduct (or postpone) investment projects, and this behavior transmits an exogenous signal regarding the fundamental return to all other investors. Even though their models exhibit the same feedback as mine, the precision of the aggregated public signal is always increasing with the number of investing agents or total activity. For example, the public signal in Fajgelbaum, Schaal and Taschereau-Dumouchel (2017) linearly sums up the signals of investors, which induces the law of large numbers and eliminates the i.i.d. noise components. This method of aggregating signals is crucial to generating the complementarity and multiplicity of steady states, but is exogenously assumed in their model. Furthermore, both of these models cannot analyze the effect of information allocation (covariance across the signals), and making the type of traders continuous eliminates uncertainty due to the law of large numbers in the latter model.

In contrast to the exogenous media (public signal) that aggregates individual behavior in the models above, this paper focuses on the endogenous means of information aggregation, i.e., the equilibrium price, which provides richer insights than exogenous signal aggregation. The active market behavior of each trader affects the price information not only through the transmission of her private signal, but also through the change in the total trading volume. As one trader becomes more intensive, the relative weight (share) of other traders and their private signals becomes less important in price formation, which means that active market participation does not always improve the price signal. The result also depends on allocation of the private information, and the covariance across signals has a significant role in determining the dynamics as well as the multiplicity of steady states.

In the final section, I show that the model also suggests a new source for financial crises. As the literature on adverse selection makes clear, busts follow booms in my model. Surprisingly, however, the crashes happen without any external shocks. Once a boom starts, the positive feedback brings about sequential market participation, which sustains the boom and the high/efficient price. If the boom

[14] The multiplicity in these models comes from coordination motives of discretionary liquidity traders, while it appears due to the covariance structure in my model, where traders do not have to behave strategically. Section 5. provides a more detailed discussion of this point.

goes too far and the price becomes too efficient, it can attract harmful traders who have a detrimental effect on the price signal. At this point, the feedback turns significantly negative and the price signal quickly becomes imprecise. If the incumbent traders cannot absorb this negative effect, the advent of the harmful traders triggers a crisis because all the incumbent traders stop trading based on the contaminated price information. In this scenario, the advent of harmful traders can be seen as the shock that triggers the crisis, yet this is endogenously induced. We call this a "self-organized market crash" since the economy moves toward it in a self-organized manner and since it is reminiscent of the "self-organized criticality" investigated by Bak, Tang and Wiesenfeld (1987).[15]

In this paper, Section 2. characterizes the model's environment and equilibrium. In Section 3., we investigate the underlying mechanisms of economic dynamics, and Section 4. shows the resulting evolution of the key variables. Section 5. explores the effect of covariance, and Section 6. analyzes instability due to irrationality. Section 7. proposes the concept of self-organized market crash, while Section 8. concludes the discussion.

2. The Model

Consider a three-periods overlapping generations economy in which a number of auctions (or rounds of trading) take place sequentially for a risky/ambiguous investment opportunity. Note that the use of the term "generation" has no implications for the length of the time span between trades.

The supply of the asset, $x \geq 0$, is fixed. The asset pays out v_t per investment if the trader holds it from date $t-1$ to t. The fundamentals of the economy follow a mean-reverting AR(1) process,

$$\begin{aligned} v_{t+1} &= \rho v_t + (1-\rho)\mu + u_{t+1}, \\ u &\sim N(0, \sigma_u^2), \end{aligned} \tag{1}$$

where I denote $\sigma^2 \equiv Var(v_t)$. Let P denote the price *level* of this asset and q be the *information content* of P.

Each trader lives for three periods. For instance, suppose that a trader is born on date $t-1$. In the first phase (youth), she looks for market information and obtains the price signal q_{t-1} at date $t-1$. In the next period, she becomes middle-aged and learns a private signal, $s_t(\omega)$, which is characterized by the parameter ω. During this phase, she arrives at the market and invests based on the set of information she obtained during her young and middle periods, which is denoted by $y_t(\omega) = (s_t(\omega), q_{t-1})$. In the final phase, she consumes and passes away. The private signal takes the form of

$$s_t(\omega) = v_{t+1} + e_{t+1}(\omega),$$

where the structure of $\{e(\omega)\}$ will be specified later. Thus, the traders who arrive at the market at date t try to infer v_{t+1} based on the information set $y_t(\omega)$.[16]

The assumption that traders can extract information only from the public signal in the previous trading round but not from the current price is based on the lagged price reference, which is explicated by Dubey, Geanakoplos and Shubik (1987).[17] They argue that rational expectations equilibria are

[15] See Section 7. for more literature on this topic.

[16] Note that only the combination $v_{t+1} + e_{t+1}(\omega)$ is privately observable for ω-traders at date t. They will be separately public at the end of date $t+1$.

[17] As for allowing traders to use the current price information, see Appendix C. If the cutoff equilibrium still holds under the current price reference, I can show that the model has similar (but more complicated) properties as in the past price reference case. However, the current price reference makes it almost impossible to solve for the cutoff analytically and to show that the

unnatural because traders can gain information from the current equilibrium price even though it is formed as a result of the market clearing. Thus, assuming that it is possible to elicit information means that traders can interpret the price signal before the price is constructed in the market.[18] Therefore, I assume that it takes a bit of time for traders to interpret the *information content* of the price after they see it. We can also think of it as a time lag that stems from the limited speed of information pooling by the price or information disclosure by the exchange.

Moreover, if the assumption that traders cannot interpret the current price at all seems strained, we can view the private signal, which traders can obtain in real time, as a noisy version of the current price signal, where the noise comes from the heterogeneous information processing constraint in the sense of Sims (2003) and Van Nieuwerburgh and Veldkamp (2010).

Romer (1992) conceptualizes the past information reference in a similar way. He argues that the immediate processing of information is more costly than delayed processing. To save on processing costs, the time of trading is delayed compared to the time of information acquisition, and the price may react after the change in the information. In my model, however, the interaction of the past price signal with individual behavior makes the reaction persistent: the past price reference alone cannot explain the propagation.

This assumption makes it possible to analyze the dynamic properties of the economy. Otherwise, it becomes difficult since we need to consider within-period multiple equilibria,[19] the selection of which depends on the coordination motives of traders. By assuming the past price reference, we can stay away from this problem, while still being able to analyze the dynamic implications of strategic complementarity. That is, the multiplicity of the dynamic steady states arises without the coordination game and the traders are perfectly competitive. The equilibrium path is unique and depends on the initial condition.

Key Idea

The main idea behind the model can be explained by using two agents ($i = 0, 1$) with signals $s_i = v + e_i$. To grasp the intuition, focus on the simple (and exogenous) noise-aggregation by the price; the price linearly sums up the private signals of traders. If only one agent is trading in the market (say, $i = 0$), the price signal reflects s_0, so that $q \sim s_0$. Then, consider an additional participant, $i = 1$, who makes the price signal $q' = v + \beta_0 e_0 + \beta_1 e_1$, where coefficients βs reflect trading shares of traders. If e_i are independent, having more traders reduces the noise variance by the LLN. If they are not independent, as in my model, this is not the case. Specifically, whether q becomes noisier than q' depends on the variance of $\{e_i\}_{i=0,1}$, their weights βs, and the covariance between them. To incorporate this idea, we introduce a specific form of the signal that has common components. After analyzing that structure, we move on to a more general one.

Information Structure

The noise, $e(\omega)$, consists of two components:

$$e_{t+1}(\omega) = \sqrt{\frac{\omega}{2}}(\hat{e}_{t+1} + e^{\omega}_{t+1}) \tag{2}$$

where $\hat{e} \sim N(0, 1)$ is common across traders, and $e^{\omega} \overset{i.i.d}{\sim} N(0, 1)$ is an idiosyncratic factor specific to ω. The coefficient, $\sqrt{\omega/2}$, controls the precision of $s(\omega)$ so that $Var(e(\omega)) = \omega$.

cutoff strategy is the equilibrium. The complication comes mainly from the fact that the price information will be used twice: once during the current period and once during the next. This makes us keep track of the two state variables.

[18]See Hellwig (1982), Blume and Easley (1984), and Geonka (2003) for related discussions.

[19]The properties of the multiple equilibria are the same as those of the multiple steady states in this model.

In Section 5., we will deal with a more general class of noise by assuming that $\boldsymbol{e} = \{e(\omega) : \omega \in \Omega\}$ is jointly normal and

$$\begin{aligned} Var(e(\omega)) &= \omega, \\ Cov(e(\omega), e(\omega')) &= l(\omega, \omega'), \end{aligned} \tag{3}$$

for any $\omega, \omega' \in \Omega$.[20]

The flow of the events at date t is as follows. First, traders in their middle period obtain the private signal. Then, they make the optimal investment given their information, and, finally, the random variables with subscript t become public.[21]

The parameter ω can be seen as a "type" of trader, which has a support $\omega \in \Omega = [\underline{\omega}, \bar{\omega}]$ and the measure $f(\omega)$. Suppose that the economy also has sophisticated traders with measure λ and type ω_r. Let the cumulative measure be $F(y) = \int^y f(x)dx$ and $\underline{\omega} \geq \omega_r$ to make the model analytically solvable.

The irrational traders are ambiguity averse in the sense of Gilboa and Schmeidler (1989) and have a set of multiple prior $\mathcal{M} = [\mu_L, \mu_H]$ regarding μ. Specifically, the bounds of $\mathcal{M}$ are denoted by

$$\mu_L = (1-\delta)\mu, \ \mu_H = (1+\delta')\mu.$$

Thus, $\delta \geq 0$ represents the severity of ambiguity or ambiguity aversion.[22] I also assume that irrational (and sophisticated) traders feel ambiguity with respect to $\boldsymbol{e} = \{e(\omega) : \omega \in \Omega\}$ and have the set of possible prior $\mathcal{Z}$.

2.1. Behavior of Traders: Ss-type Strategy

As shown by Gilboa and Schmeidler (1989), the optimization problem of irrational traders with type ω is to maximize the worst-case expected utility:

$$x_t(\omega) = \arg\max_x \min_{\mu \in \mathcal{M}, \boldsymbol{e} \in \mathcal{Z}} E_{(\mu, \boldsymbol{e})} \left[u(w_{t+1}) | y_t(\omega) \right], \tag{4}$$

$$\text{s.t. } w_{t+1} = \bar{w} + (v_{t+1} - P_t)x_t(\omega),$$

with $u(w) = -e^{-\tau w}$. $\bar{w}$ represents the endowment of wealth, which can be normalized to zero because the optimal portfolio does not depend on this variable in the CARA-Normal world.

Following Veldkamp (2006) and Mele and Sangiorgi (2009), trading is described as a sequence of one-shot auctions. Thus, traders do not intend to gain from the swing of the price. This assumption is somewhat unrealistic for the security market, but makes the model analytically tractable. In Appendix A3., I allow traders to resale their asset with an assumption regarding the stationary belief and atomless property of an individual trader to show that the resulting equations do not change.[23]

[20] The function l is determined so that the covariance matrix is positive semi definite. It also satisfies the regular properties of covariance, such as symmetricity and stationarity.

[21] This implies that the trader who makes an investment at date t is unaware of the realized value of v_t, e_t, and u_t, but has a signal with respect to v_t. Also, assume that she knows the realization v_{t-1} at date t. This is equivalent to knowing the history $\{v_j\}_{j=0}^{t-1}$ by the Markov property of AR(1) process. Given v_{t-1}, the unconditional variance of v_{t+1} (without knowing v_t) is $Var(v_{t+1}|v_{t-1}) = Var(\rho v_t + u_{t+1}|v_{t-1}) = (1+\rho^2)\sigma_u^2 \equiv \sigma^2$. If there is no fear of confusion, I omit the conditioning representation and denote $Var(v_t|v_{t-2}) = Var(v_t)$.

[22] We cannot distinguish between the degree of ambiguity aversion and ambiguity itself. See Klibanoff, Marinacci and Mukerji (2005) and Nishimura and Ozaki (2006) for more details. Also, Brenner and Izhakian (Forthcoming) provide empirical research that measures the degree of ambiguity (aversion).

[23] See Spiegel (1998) and Watanabe (2008) for the overlapping generations model with CARA-Normal setting.

Pseudo-Stochastic Environment

The final assumption relates to the randomness and ambiguity of e. This model is hard to solve because of the complexity stemming from multi-dimensions of heterogeneity: different types, Ω, and the realized values of $e(\omega)$. Thus, based on the theory of ambiguity aversion, we introduce the notion of "pseudo-stochastic" environment to eliminate this complexity. The theoretical justification and its intuitions are as follows.

Suppose that there is an ambiguous normal random variable $z \sim N(\bar{z}, \sigma_z^2)$. For ambiguity averse agents, the distribution is uncertain, and let the ambiguity be represented by the set of multiple priors such as

$$\mathcal{N} = \{N(\bar{z}, \hat{\sigma}_z^2) : \hat{\sigma}_z \in \mathcal{S}_z\}.$$

As Gilboa and Schmeidler (1989) and Max-Min expected utility suggest, the agent evaluates her expected utility by choosing $\hat{\sigma}_z$ which achieves the worst-case scenario. Now we can define the pseudo-stochastic variables.

Definition 1. Let $\hat{\sigma}_z^* = \arg\min_{\hat{\sigma}_z \in \mathcal{S}_z} E_{\hat{\sigma}_z}[u(w)]$. An ambiguous normal random variable z is pseudo-stochastic if and only if its true (objective) distribution is degenerate and given by

$$z \sim N(\bar{z}, \sigma_z^2) \text{ s.t. } \sigma_z = 0,$$

while ambiguity averse traders believe that $\hat{\sigma}_z^* > 0$.

Definition 1 means that the random variable z is a constant random variable, i.e., it is degenerate and has $\sigma_z = 0$. Then we have $z = \bar{z}$ a.s. However, for irrational ambiguity averse agents, z seems stochastic. In other words, if traders are rational, they notice that z is constant over time a.s., while irrational traders fear its random realization because this is the worst case they can subjectively imagine.[24] Hereafter, I assume that the constant set $\mathcal{Z}$ is common across the traders and $\mathcal{Z} \subset \mathbb{R}_{++}$.[25]

In a pseudo-stochastic environment, we can analytically solve the model because (almost) all the random variables are degenerate and equal to their means (for example, $v_t = \mu$ over time a.s.), while traders behave as if these variables are stochastic. The other important benefit from this assumption is that we can focus on the fundamental driving force of the economic dynamics by making the aggregate shocks, which are not our interest, silent. (For $z = e(\omega)$, let the subjective variance that achieves the worst-case scenario be denoted by ω to save notations).

Inaction Strategy

In the pseudo-stochastic environment, the optimal portfolio of the agent with type ω in her middle phase at date t is characterized by the Ss-type inaction behavior (see Appendix A1. for the proof):

$$x_t(\omega) = \begin{cases} \frac{R_t^L(\omega) - P_t}{\tau Var(v_{t+1}|y_t(\omega))} & \text{if } R_t^L > P_t \\ 0 & \text{if } P_t \in [R_t^L, R_t^H] \\ \frac{R_t^H(\omega) - P_t}{\tau Var(v_{t+1}|y_t(\omega))} & \text{if } R_t^H < P_t, \end{cases} \tag{5}$$

[24] The pseudo-stochastic environment with the heterogeneous $e(\omega)$ can also be interpreted as follows: even if all traders observe the same realization of the signal, the subjective belief regarding the distribution of the signal is different across traders. One set of traders belief that the signal emanates from $\mathcal{N}(0, \omega)$, while other traders think that it comes from $\mathcal{N}(0, \omega')$. Under this setting, the distribution of ω over the traders is the common knowledge.

[25] This implies that we abstract away from the learning about the ambiguous distribution, i.e., update of $\mathcal{Z}$ over time. The update of belief under the ambiguity aversion is discussed by Chapter 14 of Nishimura and Ozaki (2017) under the representation by ε-contamination.

with

$$R_t^j(\omega) = \begin{cases} \inf_{\hat{\mu}\in\mathcal{M}} E_{\hat{\mu}}[v_{t+1}|y_t(\omega)] & \text{if } j = L \\ \sup_{\hat{\mu}\in\mathcal{M}} E_{\hat{\mu}}[v_{t+1}|y_t(\omega)] & \text{if } j = H. \end{cases}$$

Note that the standard filtering problem gives

$$E[v_{t+1}|y_t(\omega)] = \mu + \beta_t^s(\omega)(s_t(\omega) - \mu) + \beta_t^q(\omega)(q_{t-1} - E[q_{t-1}]) \tag{6}$$

where β_ts are the filtering coefficients given in Appendix A2..

As previously explained, the inaction region in (5) comes from aversion toward exposure to an ambiguous situation by holding the asset. If she believes that she will outperform even in the worst case, she will take a non-zero position. Otherwise, she will not trade.[26]

As for sophisticated traders, their set of multiple priors is a singleton and unbiased, i.e., they know the true μ. Their optimal behavior is characterized by the typical MV portfolio:

$$x_t^r(\omega_r) = \frac{R_t(\omega_r) - P_t}{\tau Var(v_{t+1}|y_t(\omega_r))},\ R_t = E[v_{t+1}|y_t(\omega_r)]. \tag{7}$$

Finally, the equilibrium price is determined by the market clearing condition:

$$x = \int_\Omega x_t(\omega)dF(\omega) + \lambda x_t^r(\omega_r). \tag{8}$$

2.2. Equilibrium

The equilibrium of the model is defined by the sequence of quantities $\{x_t(\omega), X_t : \omega \in \Omega\}_{t=0}^\infty$ and the price $\{P_t\}_{t=0}^\infty$ which maximize the expected utility of each generation of traders and clear the market given the sequence of information/signals $\{s_t(\omega) : \omega \in \Omega\}_{t=0}^\infty$. Hereafter, we focus on the equilibrium with pseudo-stochastic random variables, which we call the pseudo-stochastic equilibrium.

2.3. Cutoff Equilibria

We guess that the equilibrium price signal constructed at date $t-1$ takes the form of

$$q_{t-1} = v_t + m_t \tag{9}$$

and let

$$M_t \equiv Var(m_t)$$

be the *noise variance* of the price signal constructed at the trading round $t-1$. In the sequel, I show by induction that this guess is correct, and M_t turns out to be the (single) endogenous state variable of the economy.

Partial Equilibrium

In the pseudo-stochastic equilibrium, the worst and best conditional expected return $R_t^j(\omega)$ can be expressed as a function of M_t and ω because each trader relies on the signals with precision M_t^{-1} and

[26] The inaction strategy, as well as most of the results in this paper can be obtained by using a fixed cost model or disappointment aversion, as discussed in the introduction. I use ambiguity aversion because it yields richer intuitions and allows us to use the pseudo-stochastic environment as the derived result from MEU rather than just an assumption.

ω^{-1}. Thus, we get

$$R_t^j \equiv R^j(M_t, \omega) = \mu_j + B(M_t, \omega)(\mu - \mu_j),\ j \in \{L, H\}$$

where $B_t = \beta_t^s + \beta_t^q$ or equivalently (see Appendix A2.),

$$B(M_t, \omega) = \sigma^2 \frac{(1-\rho^2)\sigma^2 + M_t + \rho\omega}{(1-\rho^2)\sigma^4 + \sigma^2(M_t + \omega) + M_t\omega}. \tag{10}$$

Intuitively, this coefficient represents how traders revise their beliefs after seeing the set of information. By looking at the signal, they realize the bias $\mu - \mu_j$ imperfectly and update their prior belief μ_j. When the information is precise, i.e., M_t and/or ω is low, traders can revise more toward the true mean μ. That is, the weight B should be higher when the information is precise.

Lemma 1. *(i) $B_t(M_t, \omega)$ and R_t^L are decreasing in M_t and ω. (ii) R_t^H is increasing in M_t and ω.*

This is intuitive given that the worst (best)-case conditional expected return is biased downward (upward) compared to the true return μ.

Lemma 1 shows that $\min_\omega R^H(\omega) > \max_\omega R^L(\omega)$, $\partial R^H/\partial\omega > 0$, and $\partial R^L/\partial\omega < 0$. Thus, if there is a $\omega \in \Omega$ such that $R^L(\omega) > P$, then $R^H(\omega) > P\ \forall \omega \in \Omega$. That is, if one trader takes a long (short) position, nobody else will take a short (long) position. Then, the possible equilibria would be (i) all traders either take a long position or stay inactive and (ii) all traders either take a short position or stay inactive. There is no possibility for an equilibrium in which some traders take a long position while others take a short position (see Fig.18 and Appendix A4. for more details).

Given this property, I will focus on an equilibrium in which some irrational traders take a long position in each period. This selection is natural; otherwise, all irrational traders (including sophisticated ones) potentially take a short position, and no traders take a long position, which is inconsistent with $x \geq 0$. Therefore, we can focus on the behavior of R^L and disregard R^H.

In this situation, it is optimal for each trader to set a cutoff as a threshold for being active or inactive.

Lemma 2. *Given P_t and M_t, an individual equilibrium involves a unique cutoff ω_t^L such that*

$$R_t^L(M_t, \omega) \gtreqless P_t \Leftrightarrow \omega_t^L \gtreqless \omega. \tag{11}$$

Proof. Immediate from Lemma 1. □

Therefore, each trader is willing to trade if her private information is precise (ω is low) or the noise variance of the previous trading price is precise (M_t is low).

General Equilibrium

Given the cutoff strategy and M_t, the market clearing condition at date t is

$$\int_{\underline{\omega}}^{\omega_t^L} \frac{R_t^L(M_t, \omega) - P_t}{\tau Var(v_{t+1}|y_t(\omega))} dF(\omega) + \lambda \frac{R(M_t, \omega_r) - P_t}{\tau Var(v_{t+1}|y_t(\omega_r))} = x, \tag{12}$$

where the cutoff is defined by the type ω which makes a trader indifferent between trading and being inactive:

$$R^L(M_t, \omega_t^L) = P_t. \tag{13}$$

Furthermore, by letting

$$Var(v_{t+1}|y_t(\omega)) = \sigma^2[1 - B_t(\omega) + (1-\rho)\beta_t^q(\omega)] = \sigma^2\gamma_t(\omega),$$

(12) can be rewritten in terms of the price (I omit $\underline{\omega}$ as the lower bound of the integrals in the following expressions):

$$P_t = \left(\int^{\omega_t^L} \frac{dF}{\gamma_t(\omega)} + \frac{\lambda}{\gamma(\omega_r)} \right)^{-1} \left[\int^{\omega_t^L} \frac{R_t^L(\omega)}{\gamma_t(\omega)} dF(\omega) + \lambda \frac{R(\omega_r)}{\gamma(\omega_r)} - \sigma^2 \tau x \right]. \tag{14}$$

To understand the cutoff determination, let $P_t = P(\omega_t^L)$ and consider the effect of the change in the cutoff evaluated at some $\hat{\omega}$,

$$\frac{\partial P_t(\hat{\omega})}{\partial \omega_t^L} = \left(\int^{\hat{\omega}} \frac{dF}{\gamma_t(\omega)} + \frac{\lambda}{\gamma(\omega_r)} \right)^{-1} \frac{f(\hat{\omega})}{\gamma(\hat{\omega})} [R^L(\hat{\omega}) - P_t]. \tag{15}$$

Thus, the additional participants with $\hat{\omega}$ push the price up (or down) if the valuation R^L is higher (lower) than the market valuation P_t. Intuitively, the additional participants at $\hat{\omega}$ replace the market share of the incumbents who, in aggregate, have the valuation P_t. Thus, when the additional participants have a higher valuation, $R^L > P$, then the market clearing price increases because the marginal traders have a higher valuation than that of the market (and vice versa). Since R^L is decreasing in ω, this positive effect of newcomers on the price becomes weaker as $\hat{\omega}$ rises.

Together with the cutoff's definition (13), the cutoff occurs at the point where all traders who evaluate the asset higher than the current price P (determined by the incumbents) partake in the market, so that $\partial P(\hat{\omega})/\partial \omega^L = 0$ at $\hat{\omega} = \omega_t^L$. Traders with type $\omega > \omega_t^L$ have a negative effect on the price, but they do not engage in trading because they find it unprofitable.

Next, we can verify that the cutoff property in (11) holds in the general equilibrium.

Proposition 1. *Given the noise variance M_t of the previous trading round, a unique pair of the cutoff ω_t^L and the price level P_t simultaneously solves (12) and (13), or, equivalently, the following equation with respect to ω^L.*

$$\frac{\tau \sigma^2 x}{\delta \mu} = \sigma^2 \frac{M_t + (1-\rho)\sigma^2}{\xi(M_t, \omega^L)} \int^{\omega^L} \frac{\omega^L - \omega}{\omega} dF(\omega) + \lambda \frac{1 - B_t(\omega^L)}{\gamma_t(\omega_r)}. \tag{16}$$

Proof. See Appendix B1.. □

From Proposition 1, we can define the function that relates M_t to the new cutoff ω_t^L, which I denote as

$$\omega_t^L = \phi_A(M_t).$$

One of the main properties of this *attraction function* is the negative relationship between ω_t^L and M_t.

Proposition 2. *ω_t^L monotonically decreases in M_t.*

Proof. See Appendix B2.. □

An imprecise price signal induces a lower R_t^L ($\partial R_t^L / \partial M_t < 0$) and makes it more unlikely for a trader with a given type ω to participate in the market. Thus, the cutoff declines, and only traders with precise private signals (low-ω) will participate.

We also have a negative monotonic relationship between M_t and P_t.

Proposition 3. *The price level P_t monotonically decreases in M_t.*

Proof. See Appendix B3.. □

M_t influences P_t via two channels. It directly affects the valuation R^L of participants, and it also changes the number of participating traders ω_t^L. We know that the indirect effect of M_t through the change in ω^L does not affect the price because, by the definition of the cutoff, the marginal traders always have $R^L = P_t$. On the other hand, the direct effect is negative because each trader lowers her valuation R^L after seeing the imprecise signal M_t. Put differently, in the continuous ω economy, the demand effect (or the extensive margin) is always muted, while the valuation effect (intensive margin) reduces the price level.

Corner Solutions

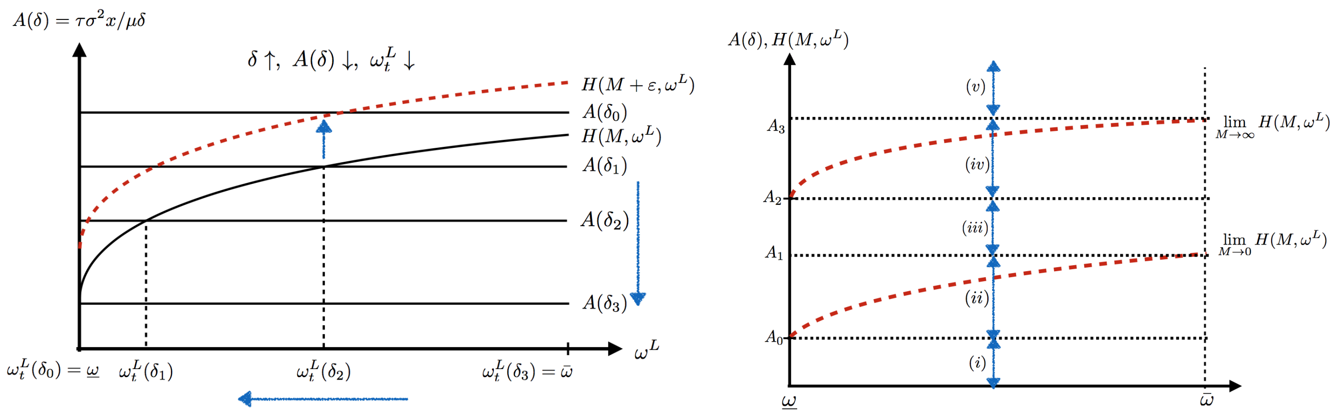
The discussion up to this point has considered the interior solution $\omega_t^L \in (\underline{\omega}, \bar{\omega})$. However, Proposition 2 implies that M_t can be small (large) enough to make $\omega_t^L = \phi_A(M_t) \geq (\leq)\bar{\omega}$. Fig.1 shows that the cutoff is determined by the intersection of the line represented by the LHS of (16), denoted by $A(\delta)$, and the curve given by the RHS of (16), denoted by $H(M_t, \omega)$. We can check that $\partial H/\partial \omega^L > 0$ and $\partial H/\partial M > 0$. Thus, by changing the severity of irrationality δ, or the noise variance M_t, we can investigate how the equilibrium cutoff behaves.

In Fig.1, δ increases from δ_0 to δ_3, and the cutoff diminishes as long as it is an interior solution (left panel). If $\omega^L < \underline{\omega}$, all irrational traders do not trade (only rational traders will participate). Two factors cause this situation. A huge ambiguity with a larger inaction region makes all the irrational traders reluctant to trade. Also, even if δ is not too large, an imprecise price signal (too high M_t) makes all the irrational traders inactive. We call this the *no-traders equilibrium.*

On the other hand, if the ambiguity or noise variance is extremely small, the cutoff will be extremely high so that all the irrational traders are willing to trade, $\omega_t^L > \bar{\omega}$. In this case, the traders are rational or well informed enough to eliminate the inaction region. We call this the *full-participation equilibrium*.

Otherwise, the cutoff is an interior solution and traders with relatively low ω are active, while those with relatively high ω are not. This can be called the *mixed equilibrium.*

Figure 1: Cutoff Determination as δ changes



We can also check that H is bounded with respect to $M \in [0, \infty)$, and can categorize five cases according to the existence of the interior cutoff, as in the right panel of Fig.1.

Proposition 4. *There are four critical values of $A(\delta)$ denoted by $A_0 < \cdots < A_3$ that separate the possible results as follows.*
(i) If δ is extremely high s.t. $A(\delta) < A_0$, only the no-traders equilibrium exists.
(ii) If δ is relatively high s.t. $A_0 \leq A(\delta) < A_1$, the no-traders and mixed equilibrium exist. When M_t is high

(low), the no-traders equilibrium (mixed equilibrium) materializes.
(iii) If δ is intermediate s.t. $A_1 \leq A(\delta) < A_2$, all three equilibria are possible. The no-traders equilibrium materializes with a higher M_t, the mixed equilibrium with an intermediate M_t, and the full-participation equilibrium with a lower M_t.
(iv) If δ is relatively small s.t. $A_2 \leq A(\delta) < A_3$, the mixed and full-participation equilibria exist. When M_t is high (low), the mixed equilibrium (full-participation equilibrium) materializes.
(v) If δ is extremely low s.t. $A_3 \leq A(\delta)$, only the full-participation equilibrium exists.

Proof. See Appendix B4., which also provides the explicit formulas for $\{A_k\}_{k=0}^3$.. □

This result indicates that an extremely high (low) δ cannot admit the interior solution even if the price signal, M_t, is extremely precise (imprecise). When δ is not extreme, whether $\omega_t^L \in \Omega$ or not depends on the level of M_t. Note that, in (ii), (iii), and (iv), we can define the lower or upper bound of the noise-variance that separates the full-participation, mixed, and no-traders equilibria.

Proposition 5. *In cases (ii) and (iii), there is a unique $M_t = \bar{M}$ that achieves $\omega_t^L = \phi_A(\bar{M}) = \underline{\omega}$, which is given by*

$$\bar{M}(\delta) = \frac{A(\delta) - \frac{1}{1+\rho}}{1 - A(\delta)}(1-\rho^2)\sigma^2 > 0.$$

In (iii) and (iv), there is a unique $M_t = \underline{M}$ that achieves $\omega_t^L = \phi_A(\underline{M}) = \bar{\omega}$ given by the positive root of

$$a_2(\delta)M^2 + a_1(\delta)M + a_0(\delta) = 0,$$

where the explicit forms of $a_2 < 0$, a_1, and $a_0 > 0$ are provided in Appendix B5..

Proof. Plugging $\omega^L = \underline{\omega}$ into (16) and solving for M_t yield $\bar{M}$. From $A_0 \sim A_2$ that characterize (ii) and (iii) (provided in Appendix B4.), we can see that $A \in (\frac{1}{1+\rho}, 1)$. As for $\underline{M}$, see Appendix B5.. □

Corollary 1. *$\bar{M}$ and $\underline{M}$ are decreasing in δ.*

Proof. It is immediate from $\partial H(M_t, \omega^L)/\partial M_t > 0$ and $dA(\delta)/d\delta < 0$. □

As ambiguity aversion worsens, it gets more difficult for each trader to obtain a sufficiently upward revision of R^L to attend the market. Therefore, to attract all traders, the price signal must be more precise, and the noise-variance that achieves the full-participation equilibrium, $\underline{M}$, must be smaller. Also, it becomes easier for the price signal to make all traders reluctant to participate when δ is high. Thus, the level of M_t that achieves the no-traders equilibrium, $\bar{M}$, becomes lower.

Also, in (ii) and (iii), the cutoff cannot reach its upper and lower bounds ($\bar{\omega}$ and $\underline{\omega}$) regardless of M_t. Namely, there is a limit of the cutoff that realizes when the previous state is extremely high ($M_t = \infty$) and extremely low ($M_t = 0$). As long as δ is not extreme, the economy cannot attract (kick out) all the irrational traders even if the price signal of the previous trading round is very precise (imprecise). The following proposition provides the maximum (minimum) possible participation rate when δ is intermediate.

Proposition 6. *(a) In case (ii), the highest possible cutoff, ω_h, is lower than $\bar{\omega}$ and is given by the solution of*

$$A(\delta) = \frac{\sigma^2(1-\rho)\int^{\omega^L} \frac{\omega^L - \omega}{\omega} dF + \frac{\lambda\omega^L}{\omega_r(1+\rho)}((1-\rho^2)\sigma^2 + \omega_r)}{(1-\rho^2)\sigma^2 + \omega^L}.$$

(b): In (iv), the lowest possible cutoff, ω_l, is greater than $\underline{\omega}$ and is given by the solution of the following equation.

$$A(\delta) = \frac{\sigma^2 \int^{\omega^L} \frac{\omega^L - \omega}{\omega} dF(\omega) + \lambda \frac{\omega^L}{\omega_r}(\omega_r + \sigma^2)}{\omega^L + \sigma^2}.$$

Proof. Based on Lemma 2, the highest (lowest) possible cutoff is achieved by $M_t = 0$ ($M_t = \infty$). Thus, by substituting these M_t into (16), we obtain the equations above. Also, we can check that ω_h and ω_l cannot achieve the boundaries by substituting the conditions of $A(\delta)$ that characterize the region (ii) and (iv). □

In each case in Proposition 4, the price level and the cutoff are given by the following proposition.

Proposition 7. *The price level and the cutoff are given by*

$$P_t = \begin{cases} \frac{\left[E[R_t^L(M_t,\omega)\gamma_t(M_t,\omega)^{-1}] + \lambda \frac{R(M_t,\omega_r)}{\gamma(M_t,\omega_r)} - \sigma^2 \tau x\right]}{\left(E[\gamma_t^{-1}(M_t,\omega)] + \frac{\lambda}{\gamma(M_t,\omega_r)}\right)} & \text{if } M_t < \underline{M} \text{ or in (v)} \\ \frac{\left[\int^{\omega_t^L} \frac{R_t^L(M_t,\omega)}{\gamma_t(M_t,\omega)} dF(\omega) + \lambda \frac{R(M_t,\omega_r)}{\gamma(M_t,\omega_r)} - \sigma^2 \tau x\right]}{\left(\int^{\omega_t^L} \frac{dF}{\gamma_t(M_t,\omega)} + \frac{\lambda}{\gamma(M_t,\omega_r)}\right)} & \text{if } M_t \in [\underline{M}, \bar{M}] \\ R(M_t,\omega_r) - \frac{\sigma^2 \tau x}{\lambda}\gamma(M_t,\omega_r) & \text{if } M_t > \bar{M} \text{ or in (i),} \end{cases} \tag{17}$$

$$\omega_t^L = \begin{cases} \bar{\omega} & \text{if } M_t < \underline{M} \text{ or in (v)} \\ \text{given by (16)} & \text{if } M_t \in [\underline{M}, \bar{M}] \\ \underline{\omega} & \text{if } M_t > \bar{M} \text{ or in (i).} \end{cases}$$

Based on propositions 2 and 3, a more (in)efficient price signal, which is represented by a lower (higher) M_t, is associated with a higher (lower) participation rate and higher (lower) price level. This monotone property is strict only if the cutoff lies between $[\underline{\omega}, \bar{\omega}]$, and it is shunted aside once the cutoff reaches the corner solution.

2.4. Adverse Selection

The last property worth mentioning in this section is the adverse selection of poorly informed traders. I define adverse selection in terms of the share of active high-ω traders compared to the share of low-ω traders in the market.[27]

Why is having more high-ω traders problematic? The reason is that traders with a low-quality private signal are more likely to modify their behavior when the public (price) signal changes. Formally, the weight on each signal, β^j, increases with the precision of the signal j. Since the same holds with respect to the level of M_t, and βs sum up to 1, we can show that

$$\frac{\partial \beta^q(M_t,\omega)}{\partial \omega} > 0, \frac{\partial \beta^q(M_t,\omega)}{\partial M_t} < 0.$$

[27]We do not specify the critical value of ω^* to say that the traders with $\hat{\omega} \in \{\omega : \omega > (<)\omega^*\}$ are poorly(well) informed. Instead, we consider the possibility of multiple steady states and define one state as suffering from the adverse selection problem if it attracts more high-ω traders than other states. In other words, adverse selection reduces the share of low-ω traders in the market and makes the share of high-ω traders larger. Thus, the economy with a higher cutoff ω^L adversely selects a larger number of poorly informed traders than a low-ω^L economy.

Given (6), the reaction to the shock on q is more sensitive for high-ω traders than for low-ω traders. This makes the economy that involves a higher ω^L more reactive to the shock on q.

For instance, consider a shock on q_{t-1},

$$q_{t-1} = v_t + m_t = \mu + \theta$$

with a small $\theta > 0$. This represents an information perturbation in the price signal. After looking at this, a trader modifies her conditional expectation to

$$R_t^L(\omega) = \mu_L + \delta\mu B(M_t, \omega) + \beta^q(M_t, \omega)\theta.$$

The market clearing condition and, equivalently, the attraction equation are modified as

$$A(\delta) = \sigma^2 \frac{M_t + (1-\rho)\sigma^2}{\xi(M_t, \omega^L)} \left(1 - \tilde{\theta}\right) \int^{\omega^L} \frac{\omega^L - \omega}{\omega} dF(\omega) + \lambda \frac{1 - B_t(\omega^L)}{\gamma_t(\omega_r)} - \lambda \frac{\tilde{\theta}}{\xi(M_t, \omega^L)} \frac{\omega^L - \omega_r}{\omega_r}, \tag{18}$$

with $\tilde{\theta} \equiv \theta\rho\sigma^2/\delta\mu$. For the small shock ($\theta \to 0$), the effect on the cutoff is expressed by

$$\Gamma(M_t, \omega^L) \equiv \left.\frac{d\omega_t^L}{d\theta}\right|_{\theta=0} = \frac{\xi(M_t, \omega^L)}{\sigma^2(M_t + (1-\rho)\sigma^2)} \frac{\sigma^2 \int^{\omega^L} \frac{\omega^L - \omega}{\omega} dF + \lambda \frac{\omega^L - \omega_r}{\omega_r}}{\int^{\omega^L} \frac{\xi(\omega)}{\omega} dF + \lambda \frac{\xi(\omega_r)}{\omega_r}} > 0. \tag{19}$$

This is intuitive because a more favorable signal improves the conditional expected return. Thus, each trader becomes more willing to trade, and the cutoff increases. We wish to know how much.

The effect of adverse selection on the aggregate level can be phrased as $d\Gamma(M_t, \omega^L)/dM_t < 0$. That is, a lower M_t makes the economy more vulnerable. As mentioned, there are two effects: the direct effect of M_t and the effect through the cutoff, which are represented by the first and the second term in the total derivative:

$$d\Gamma/dM_t = \frac{\partial\Gamma}{\partial M} + \frac{d\omega^L}{dM_t} \frac{\partial\Gamma}{\partial\omega^L}. \tag{20}$$

Lemma 3. *The direct effect of M_t on Γ is negative, i.e., $\frac{\partial\Gamma}{\partial M} < 0$.*

This lemma can be proved simply by taking a partial derivative. The negative partial derivative implies that a more efficient economy makes the reaction of the market to the θ-shock more sensitive through the direct effect of M_t, since with a lower M_t, traders rely more heavily on the public signal.

Since we have not specified the form of f, it is not analytically proven that the second term is also negative. In Appendix D, we analyze the indirect effect (second term) numerically and show that, in most cases, the sign of the indirect effect is indeed negative.

When adverse selection works, the shock on the public signal strongly influences the behavior of traders in an efficient market. The reason is that those who are sensitive to the change in the price signal (as well as the noise variance M_t) tend to be attracted to the market due to a lower M_t (the indirect effect). Moreover, this lower M_t itself also makes the reaction of each trader sensitive to the shock (direct effect).

The opposite arguments suggest that an inefficient economy with a less precise price signal is more resilient to shocks on the public signal because it attracts low-quality private signals and can kick out traders who are sensitive to the shock. We will revisit the dynamic implications of adverse selection in Section 6.3., which will highlight the differences from the existing literature.

3. Dynamics: The Intertemporal Information Effect

In this model, the lagged information reference is an important feature that makes the dynamics of the economy persistent. It will be shown, however, that the lagged reference alone cannot explain how the economy evolves over time.

The second key factor in depicting rich and realistic dynamics is the inaction behavior of (continuously) heterogeneous irrational traders. Without inaction and heterogeneity, the cutoff level is obviously constant regardless of the content of the previous price signal M_t. As a result, the price level and price information would also be constant and the economy would not evolve (see the cases with $M_t \notin [\underline{M}, \bar{M}]$ in Proposition 7).

Hence, the inaction property of the irrational agents makes the information content of the price signal in the previous trading round different from that in the current one. That is, the price information *bridges* the information and behavior from date t to date $t+1$; the noise variance M_t which attracts the traders up to ω_t^L is different from the noise variance of the price, which aggregates the private signals of the traders up to ω_t^L.

In the previous section, we derived the attraction function $\omega_t^L = \phi_A(M_t)$. Next, we analyze the level of the noise variance of the price signal at date t (M_{t+1}), determined by the current cutoff ω_t^L to get the dynamics. We call this the *noise aggregation function* and denote it by $M_{t+1} = \phi_N(\omega_t^L)$.

3.1. Noise Aggregation by Price

M_{t+1} is determined by the information content of the price (17), which is given by the following equations. Proof for Lemma 4 and Proposition 8 is provided in Appendix B6..

Lemma 4. *The information content of the price P_t is given by $q_t = v_{t+1} + m_{t+1}$ with*

$$m_{t+1} = \begin{cases} \hat{e}_{t+1} \int^{\bar{\omega}} \pi(\bar{\omega}, \omega) \sqrt{\frac{\omega}{2}} d\omega + \pi_r(\bar{\omega}) e_{t+1}(\omega_r) & \text{if } M_t < \underline{M} \text{ or in (v)} \\ \hat{e}_{t+1} \int^{\omega_t^L} \pi(\omega_t^L, \omega) \sqrt{\frac{\omega}{2}} d\omega + \pi_r(\omega_t^L) e_{t+1}(\omega_r) & \text{if } M_t \in [\underline{M}, \bar{M}] \\ e_{t+1}(\omega_r), & \text{if } M_t > \bar{M} \text{ or in (i).} \end{cases}$$

where

$$\pi(\omega_t^L, \omega) \equiv \frac{g(\omega)}{\lambda \omega_r^{-1} + G(\omega_t^L)}, \ \pi_r(\omega_t^L) = 1 - \int^{\omega_t^L} \pi(\omega_t^L, \omega) d\omega,$$

$$g(\omega) \equiv f(\omega)/\omega, \ G(k) = \int^k g(\omega) d\omega.$$

The price aggregates the private signals of active traders. Each noise component is weighted by the relative trading share, π, and summed up. The trading share contains the adjusted measure g rather than the original measure f. This is due to the CARA preference and MV portfolio, which make the trading intensity of each participant proportional to the precision of the signal, ω^{-1}.

Note that the idiosyncratic components, e^{ω}, disappear almost surely because of the law of large numbers, and only the common component remains effective.[28] In other words, the disturbances in the price information emanate from the covariance (correlation) structure across the private signals. The following proposition provides simple forms of the noise aggregation function ϕ_N.

[28] The law of large numbers with a continuum of random variables requires a different set of assumptions from discrete cases. See Judd (1985) and Vives (2010) for more details.

Proposition 8. *The noise variance, M_{t+1}, of the current price P_t is given by*

$$M_{t+1} = \phi_N(\omega_t^L) = \begin{cases} \left(\lambda\omega_r^{-1} + G(\bar{\omega})\right)^{-2}\left[\left(\int^{\bar{\omega}} g(\omega)\sqrt{\frac{\omega}{2}}d\omega\right)^2 + \lambda^2\omega_r^{-1}\right] & \text{if } M_t < \underline{M} \text{ or in (v)} \\ \left(\lambda\omega_r^{-1} + G(\omega_t^L)\right)^{-2}\left[\left(\int^{\omega_t^L} g(\omega)\sqrt{\frac{\omega}{2}}d\omega\right)^2 + \lambda^2\omega_r^{-1}\right] & \text{if } M_t \in [\underline{M}, \bar{M}] \\ \omega_r & \text{if } M_t > \bar{M} \text{ or in (i).} \end{cases} \tag{21}$$

If the market is completely devoid of irrational traders, the price signal has a noise variance of ω_r because the price reflects the signals of rational traders only (see the third line). In this case, the state variable does not evolve, i.e., M_t stays at the constant level ω_r. Also, as long as the economy is under the full-participation equilibrium, there is no disturbance in the dynamics of M_t because the number of active traders and price information do not change (see the first line). Only if the cutoff is the interior solution does the economy evolve over more than one period due to the dependence of M_{t+1} on ω_t^L (the middle line).

Now, given some initial conditions regarding the information, the mathematical induction completes and yields the following proposition.

Proposition 9. *With the set of initial information $\{y_0(\omega) = (s_0(\omega), v_0 + M_0)\}_{\omega\in\Omega\cup\{\omega_r\}}$, the dynamic pseudo-stochastic equilibrium exists and is defined by the sequence of price, information-related variables $\{P_t, q_t, M_t, (y_t(\omega),)_{\omega\in\Omega\cup\{\omega_r\}}\}_{t=0}^{\infty}$, and quantities $(\{x_t(\omega)\}_{\omega\in\Omega\cup\{\omega_r\}})_{t=0}^{\infty}$ given by (5)~(21).*

Property of the Noise-Aggregation Function: ϕ_N

Consider $\omega^L \in (\underline{\omega}, \bar{\omega})$. The cutoff ω_t^L has the following effect on the noise-variance, M_{t+1}.

$$\frac{dM_{t+1}}{d\omega^L} = \frac{2g(\omega^L)}{(G(\omega^L) + \lambda\omega_r^{-1})^3}\Bigg\{ \overbrace{\sqrt{\frac{\omega^L}{2}}\left(\int^{\omega^L} g(\omega)\sqrt{\frac{\omega}{2}}d\omega\right)\left(G(\omega^L) + \lambda\omega_r^{-1}\right)}^{\text{Covariance Effect}} \underbrace{- \left[\left(\int^{\omega_{t+1}^L} g(\omega)\sqrt{\frac{\omega}{2}}d\omega\right)^2 + \lambda^2\omega_r^{-1}\right]}_{\text{Replacement Effect}}\Bigg\}. \tag{22}$$

First, a higher cutoff pushes M_{t+1} up and the price signal becomes less precise. This is due to the positive correlation between $e(\omega)$ of newcomers and incumbents, which stems from the common component $\hat{e}$. This effect is represented by the first positive term in (22), and we denote it as the *covariance effect.* [29] On the other hand, the increase in the cutoff reduces the noise-variance because the market share of each trader declines. This constitutes the negative terms in (22) and is called the *replacement effect* because the additional traders replace the incumbents in terms of their market share.

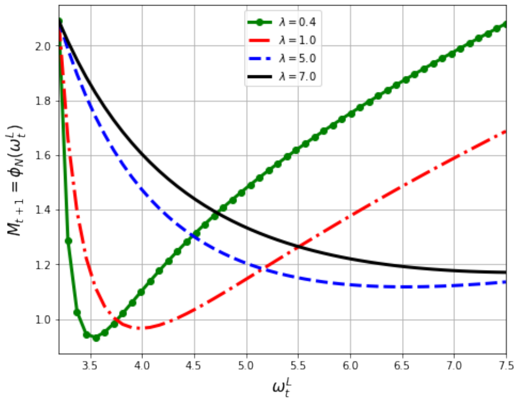
Fig.2 plots ϕ_N with different values of λ to illustrate how changes in the relative share affect the noise aggregation.[30] Note that the covariance effect brought by an additional noise, say, $e(\omega^L)$ is

[29] Fajgelbaum, Schaal and Taschereau-Dumouchel (2017), assume that $Var(\frac{1}{N}\sum_{j\in\text{Active}} e(\omega))$ is decreasing in the number of active traders, which corresponds to the cutoff ω^L in my model. This comes from the assumption that the noise components are i.i.d. across discrete types of traders and the law of large numbers, which leads to $dM_{t+1}/d\omega^L < 0$ for all ranges of the cutoff. My model could provide a richer dynamics of the state M_{t+1} since we consider a more general setting regarding $e(\omega)$ and the aggregation of private signals.

[30] Fig.2 uses a benchmark F equal to the uniform distribution.

stronger when it has many noise components to correlate with, $\{e(\omega) : \omega < \omega^L\}$, which is measured by the share of the irrational incumbents compared to that of sophisticated traders.

Figure 2: Noise-Aggregation Function



Remarks: The replacement effect dominates (ϕ_N is downward sloping) when the share of the sophisticated traders is large ($\lambda\omega_r^{-1}$ is high) compared to the share of irrational incumbents. This is because the addition of noise by newcomers has a positive correlation with the irrational incumbents, while it is independent of the noise of the sophisticated traders. The newcomers make the price signal more precise when the share of the sophisticated traders is large since the replacement effect is stronger than the covariance effect. As the number of irrational traders increases, however, these newcomers reduce the relative share of the sophisticated traders and weaken the replacement effect. As the relative share of the irrational traders becomes large enough, the covariance effect and the replacement effect balance out ($\phi_N^{'} = 0$). If the number of irrational traders increases further, their participation behavior contaminates the price signal because of the dominating covariance effect (ϕ_N is upward sloping).

Since Ω is bounded, the global configuration of ϕ_N depends on the parameters $(\underline{\omega}, \bar{\omega}, \lambda\omega_r^{-1})$.

Proposition 10. *There is a unique $\omega^N(\geq \underline{\omega})$ such that*

$$\omega_t^L \gtrless \omega^N \Leftrightarrow \phi_N^{'}(\omega_t^L) \gtrless 0. \tag{23}$$

(ii) There is a unique $\lambda\omega_r^{-1}$, denoted by $\tilde{\lambda}$, such that

$$\lambda\omega_r^{-1} \gtrless \tilde{\lambda} \Leftrightarrow \omega^N \gtrless \bar{\omega}.$$

Proof. See Appendix B7.. □

Therefore, the replacement effect globally dominates the covariance effect when $\lambda\omega_r^{-1}$ is sufficiently large.

Finally, the noise variance of the current period, M_{t+1}, will never be able to achieve its lower or upper bound when $\phi_N^{'} < 0$ under the mixed-equilibrium. ω_l and ω_h in the following proposition are given by Proposition 6.

Corollary 2. *Given that $\phi_N^{'} < 0$, the lowest possible M_{t+1} cannot be lower than the full-participation noise-variance under case (iv), that is,*

$$M_{t+1}(\omega_h) > \left(\lambda\omega_r^{-1} + g(\bar{\omega})\right)^{-2} \left[E[\omega^{-1}f(\omega)] + \lambda^2\omega_r^{-1}\right].$$

Also, the highest possible M_{t+1} under case (ii) cannot be higher than the no-traders noise-variance, that is,

$$M_{t+1}(\omega_l) < \omega_r.$$

4. Phase Dynamics

4.1. Determinants of the Dynamics: Intuitions

Consider $\omega^L \in \Omega$. Suppose that the state variable M_t increases to $M_t + \varepsilon$. This ε makes traders less willing to trade, and the cutoff declines below the level with no shock: $\omega_t^L(\varepsilon) = \phi_A(M_t + \varepsilon) < \phi_A(M_t) = \omega_t^L(0)$. Then, the price information (M_{t+1}) is affected since the noise components brought by the traders $\{\omega : \omega \leq \omega_t^L(\varepsilon)\}$ is different from that brought by the traders $\{\omega : \omega \leq \omega_t^L(0)\}$. This, in turn, changes the cutoff in the next period through ϕ_A.

Let the phase dynamics of $\{M_t\}_{t=1}^{\infty}$ be denoted by

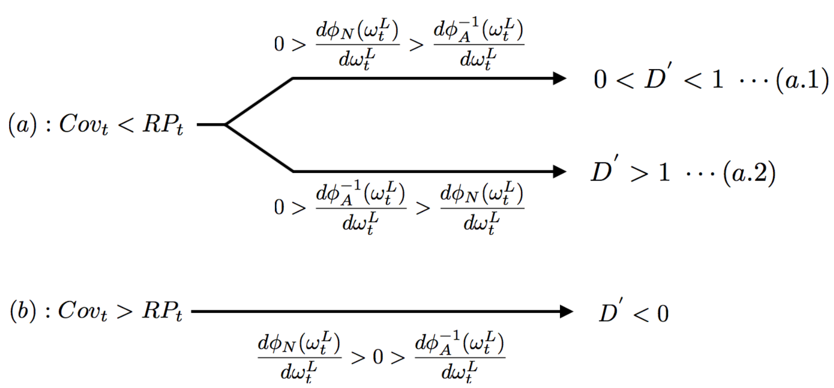
$$M_{t+1} = D(M_t) \equiv \phi_N(\phi_A(M_t)). \tag{24}$$

Namely, the dynamics of $\{M_t\}_{t=0}^{\infty}$ must be determined by the combination of the attraction function ϕ_A and the noise-aggregation ϕ_N, which is further decomposed into the covariance and replacement effects:

$$\begin{aligned} D^{'}(M_t) &= \frac{d\omega_t^L}{dM_t}\frac{dM_{t+1}}{d\omega_t^L} \\ &= \frac{d\phi_A(M_t)}{dM_t}\frac{d\phi_N(\omega_t^L)}{d\omega_t^L} \\ &= \frac{d\phi_A(M_t)}{dM_t}(Cov_t - RP_t) \end{aligned}$$

where Cov_t and RP_t represent the first and second terms in (22).

Figure 3: Determinants of $D^{'}$



D' is establish by the relative slopes of ϕ_A and ϕ_N. Furthermore, the slope of the noise-aggregation function is determined by the covariance vs. the replacement effect. Fig. 3 summarizes the possible cases.

(a.1): Convergence

When $0 > d\phi_N/d\omega_t^L > d\phi_A^{-1}/d\omega_t^L$, the effect of ε-change in M_t diminishes over time, and M_t converges to its steady state monotonically. These inequalities mean that, if ε-change in M_t is required to attract 1% more traders, then the change in the M_{t+1} caused by this 1% increase in the cutoff is smaller than the original change ε. That is, $M_{t+1}(\varepsilon) - M_{t+1}(0) < M_t(\varepsilon) - M_t(0)$, which implies that M_t converges.

Intuitively, both the noise-aggregation and attraction functions have a negative relationship with the cutoff, while the *noise aggregation function is less sensitive* to the change in the cutoff than the attraction function. Any change in the state, M_t, that induces the change in the cutoff will be absorbed by the insensitive noise-aggregation function. As mentioned in the Remarks, this is more likely when the replacement effect weakly dominates the covariance effect. Since $\phi_N' < 0$ increases the number of irrational participants, the replacement effect becomes weaker. At some point, the replacement and covariance effects balance out so that the economy converges to the steady state.

(a.2): Amplification

If $0 > d\phi_A^{-1}/d\omega_t^L > d\phi_N/d\omega_t^L$, we have $dM_{t+1}/dM_t > 1$. In this case, the noise-aggregation function is more sensitive than the (inverse) attraction function. Thus, the ε-increase in M_t induces a greater change in M_{t+1}, that is, $M_{t+1}(\varepsilon) - M_{t+1}(0) > \varepsilon$. This implies that the phase dynamics of M_{t+1} will be exploding as long as M_t is in this region.

Intuitively, when M_t rises by ε, the cutoff decreases as traders become less confident. Then, the lower ω_t^L increases the current M_{t+1} through the (negative) replacement effect. The inequality above implies that this increment is greater than the original change in the state variable, ε, because the noise-aggregation function is more sensitive to the change in the cutoff. Then, the ε-shock induces a greater surge in M_{t+1} which, in turn, induces an even greater increase in M_{t+2}, and so on.

This phase is more likely when the replacement effect is much stronger than the covariance effect and continues as long as the relative share of irrational traders is small enough to keep the economy out of the region of (a.1).

(b): Oscillating Phase

What if $d\phi_N/d\omega_t^L > 0 > d\phi_A^{-1}/dM_t$? This happens when the covariance effect dominates the replacement effect, i.e., the relative share of irrational traders is much larger than that of sophisticated traders. Then, $M_{t+1} = D(M_t)$ is downward sloping.

First, an ε-increase in M_t decreases the cutoff. Then, this decline *improves* the noise variance because it reduces the relative share of irrational traders and weakens the covariance effect. That is, the price does not have to sum up the extra noise, and M_{t+1} decreases. This reduction in the noise variance attracts additional traders at date $t+1$, leading to a higher $M_{t+2}, \cdots$, and the economy oscillates. Whether the dynamics converge or not completely depends on how sensitive ϕ_A is, as in the previous case. If $|\phi_N'| > |(\phi_A^{-1})'|$, then the economy does not converge and vice versa.

4.2. Benchmark Phase Diagram

Figure 4: $M_{t+1} = D(M_t)$ with $\phi_N^{'} < 0$

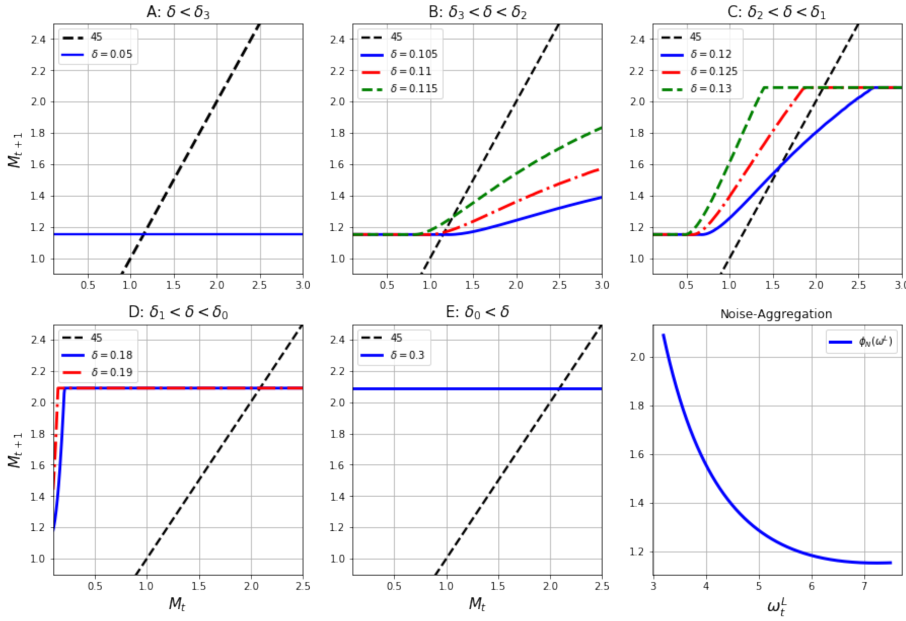
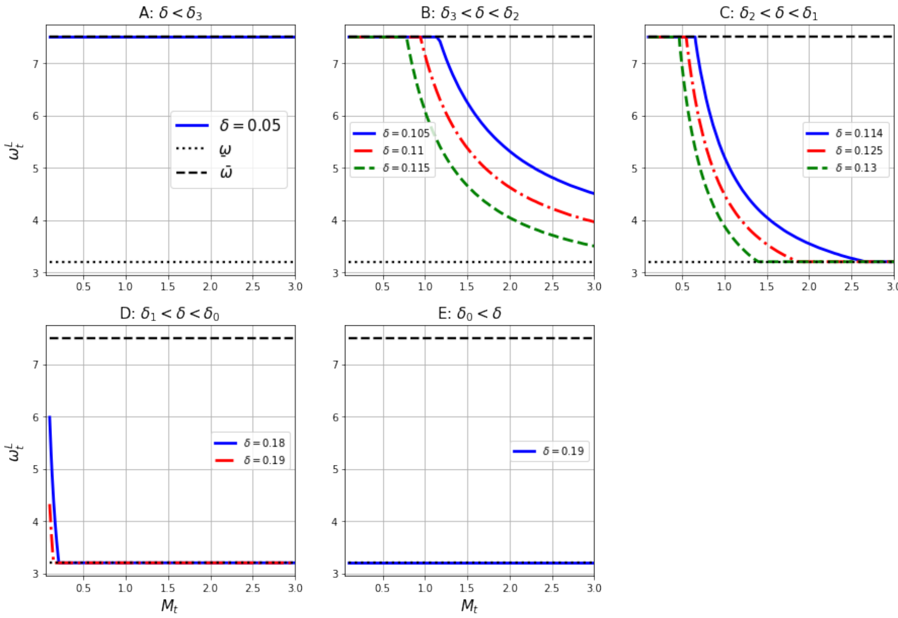


Figure 5: $\omega_t^L = \phi_A(M_t)$ with $\phi_N^{'} < 0$



In this subsection, we assume that F is uniform over Ω and set this as a benchmark. As Proposition 4 predicts, the evolution of the noise variance can be separated into five cases.

Case with $\phi_N^{'} < 0$

First, consider the monotonically converging phase dynamics as in Figs.4 and 5. As plotted in Panel F, the noise-aggregation function is globally downward sloping as a result of the dominating replacement effect. In the figures, the five cases provided in Proposition 4 are described in accordance with the value of δ, where δ_k is defined by $A(\delta_k) = A_k$. If δ is extremely small (Panel A) or large (Panel E), then we have only the full-participation or no-traders equilibrium, respectively, regardless of the level of M_t. In these regions, there are no changes in the participants because ω^L is not reactive (see Fig.5), and the level of M_{t+1} is also constant over time.

As irrationality grows, the level of the previous M_t works to determine the cutoff. As in panels B and C of Figs.4 and 5, the traders decide not to participate when M_t is high, which results in a higher M_{t+1}. This is because of the downward sloping ϕ_N. If the degree of irrationality becomes more severe, as in Panel D, the full-participation equilibrium region disappears, and an even higher δ eliminates the mixed equilibrium.

The economy undergoes "dynamic evolutions" only if the value of δ is intermediate, i.e., $\delta \in [\delta_3, \delta_0]$. In other cases with $\delta \notin [\delta_3, \delta_0]$, the market is trapped in a constant steady state with the full participation or no traders equilibria. Note that the slope of D within the mixed equilibrium region becomes steeper as δ increases. This is due to the low cutoff, which leads to a weaker covariance effect and a steeper ϕ_N.

Remarks: When $\phi_N^{'} < 0$ and $\omega^L \in \Omega$, the phase diagram, D, must be concave as M expands, which is the result of adverse selection. As M_t becomes large, the cutoff declines. First, the higher M_t lowers β^q because traders put less weight on the imprecise price signal. Second, the lower cutoff also lowers β^q since well-informed traders put more weight on $s(\omega)$. As a result, the attraction function flattens in the high-M_t region, and the effect of M_t on M_{t+1} is muted due to the insensitive behavior of traders to the change in M_t. This guarantees the existence of the upper stable steady state and convergence to it.

Case with $\phi_N^{'} > 0$

Figure 6: $M_{t+1} = D(M_t)$ with $\phi_N' > 0$

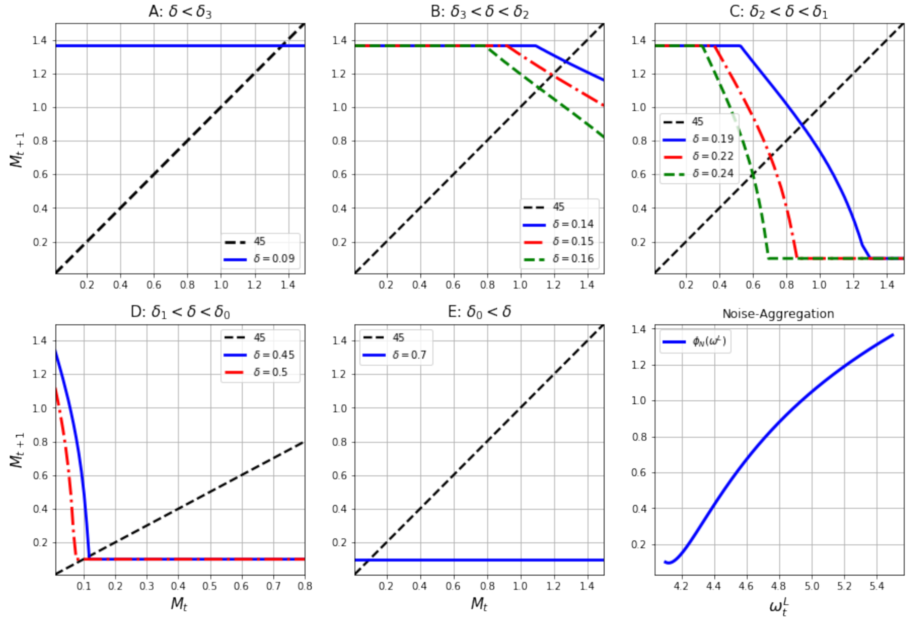
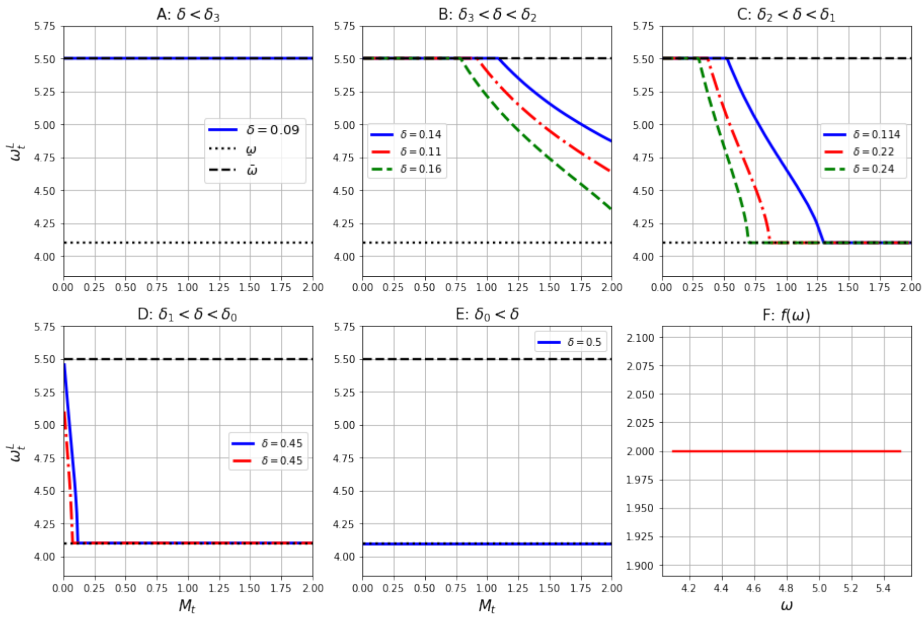


Figure 7: $\omega_t^L = \phi_A(M_t)$ with $\phi_N' > 0$



The second possible case is the downward sloping ϕ_N due to the dominating covariance effect as in Panel F in Fig.6. Once the state variable is trapped in the oscillating region, a higher noise-variance attracts a smaller number of irrational traders (ω_t^L becomes lower). This lower cutoff, however, leads to a lower noise variance in the next trading round because of the dominating covariance effect. This precise signal, in turn, can attract more traders (ω^L becomes higher), but the additional traders can make the signal worse (M_{t+1} becomes higher). In this situation, not only does the intermediate value of δ make the economic evolution persistent but it can also make the economy infinitely volatile.

5. General Covariance Structure

Before we move on to the stability analysis, we explore the effect of covariance structure and its implications under a generalized setting. Different covariance structures will induce different evolutions of the economy by changing the noise-aggregation function without touching the attraction function.

Specifically, we analyze the general form of the covariance structure given by (3), which provides the following form of the noise-aggregation function and the covariance term (see Appendix B6.):

$$\phi_N^C(\omega_t^L) = C(\omega_t^L) + \frac{\lambda^2}{(G(\omega_t^L) + \lambda\omega_r^{-1})^2}\omega_r^{-1}, \tag{25}$$

with

$$C(\omega^L) \equiv \frac{1}{\left(G(\omega^L) + \lambda\omega_r^{-1}\right)^2}\int^{\omega^L}\int^{\omega^L} l(\omega,\omega')g(\omega)g(\omega')d\omega d\omega'.$$

The first term in (25) stems from the covariance across irrational traders, and the second is the noise from the sophisticated traders. As in the previous case, the change in the cutoff has a generalized covariance effect (first term) and a replacement effect (second term):

$$\begin{aligned}\frac{d\phi_N^C(\omega_t^L)}{d\omega^L} =& C'(\omega_t^L) - \frac{2\lambda^2\omega_r^{-1}}{(G(\omega_t^L) + \lambda\omega_r^{-1})^3}g(\omega_t^L) \\ =& \frac{2g(\omega^L)}{(G(\omega_t^L) + \lambda\omega_r^{-1})^3}\Bigg\{\overbrace{(G(\omega^L) + \lambda\omega_r^{-1})\int^{\omega^L} l(\omega^L,\omega')g(\omega')d\omega'}^{\text{Covariance Effect}} \\ & \underbrace{-\left[\int^{\omega^L}\int^{\omega^L} l(\omega,\omega')g(\omega)g(\omega')d\omega d\omega' + \lambda^2\omega_r^{-1}\right]\Bigg\}}_{\text{Replacement Effect}}.\end{aligned} \tag{26}$$

The competition between the covariance effect and the replacement effect pins down the evolution, D. In the previous case, the sign and the magnitude of two effects are mechanically determined by $l(\omega,\omega') = \sqrt{\omega\omega'}/2$, while they are now arbitrarily determined by the functional form of $l(\omega,\omega')$.[31] It is worth mentioning that we are interested in the financial acceleration given by $D' > 1$, and this is more likely to be achieved by a sufficiently negative $(\phi_N^C)'$.

In the analysis with $e = \sqrt{\omega/2}(\hat{e} + e^{\omega})$, ϕ_N could be steeply downward sloping only if the cutoff is small. The reason is that the replacement effect can be stronger than the covariance effect only if the relative share of irrational traders is small. Thus, even if amplification occurs, a larger number of irrational traders mutes the replacement effect by the stronger covariance effect in the small-M region.

The general form of l provides a higher degree of freedom. First, the additional participants around the cutoff, denoted by $d\omega^L$, add the covariance effect since these traders' noise has a covariance with the incumbents $\omega \in [\underline{\omega}, \omega^L]$. This is captured by the first term of (26), whose sign and magnitude depend on $\{l(\omega^L,\omega)\colon \omega \in [\underline{\omega},\omega^L]\}$. Second, $d\omega^L$ lowers the share of each trader as well as the weight on the covariance across incumbents. This effect is represented by the second term of (26) and, again, depends on the form of $l(\omega,\omega')$.

The discussion above implies that we can establish a sufficiently negative $(\phi_N^C)'$ when one of (or both of) the following two statements are true. First, the noise terms of additional participants with $\omega \in$

[31] The choice of l is not completely arbitrary since the covariance function must be symmetric. Also, l must induce the positive semidefiniteness of covariance space, which is briefly explained in Appendix B8..

$[\omega^L, \omega^L + d\omega^L]$ have a small (or non-positive) covariance with incumbent traders with $\omega \in [\underline{\omega}, \omega^L)$. This is given by a small (or non-positive) $l(\omega^L, \omega^{'})$ for $\omega^{'} \in [\underline{\omega}, \omega^L]$. Second, the covariance across the incumbents is sufficiently great, that is, $l(\omega, \omega^{'})$ is large for $\omega, \omega^{'} \in [\underline{\omega}, \omega^L)$. Both of them make the replacement effect stronger than the covariance effect, or even makes the covariance effect help the replacement effect. The following arguments show that, for the multiplicity of the steady state, it is enough for these effects to work only when $\omega^L \approx \bar{\omega}$.

Multiple Steady States

If we have $(\phi_N^C)^{'} < 0$, the additional participants create an information *gain*. The opposite argument also holds: when marginal traders withdraw due to some shocks, their escape behavior causes information *loss*, which, in turn, makes a different set of traders reluctant to trade, and the price signal deteriorates even more. This positive feedback generates multiple steady states. When a large number of irrational traders are active, M stays at a lower level and makes it easier for them to participate in the market. Also, a small number of irrational traders keeps itself stable by making M higher.

Figure 8: Multiple Steasy States

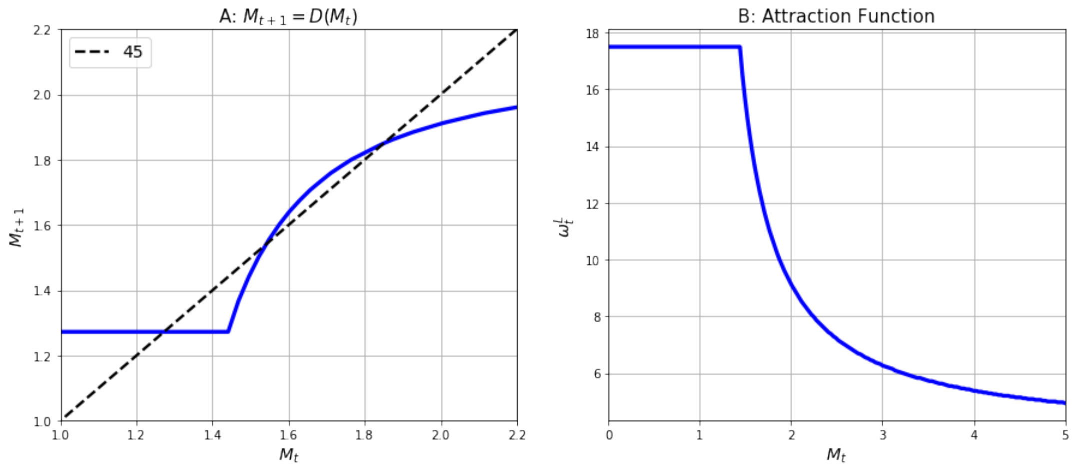


Fig.8 illustrates the multiple steady states induced by the locally negative covariance, $C^{'}(\omega^L) < 0$, for relatively high cutoffs.[32] The information-gain effect by the additional participants does not diminish even though the cutoff rises. This makes the slope of D around $\underline{M}$ sufficiently steep.

With interior cutoffs, we have one stable steady state and one unstable saddle point. The unstable saddle point reflects the amplification scenario, and the additional newcomer generates even more participation by reducing the risk, M_t.

The stability of the upper steady state arises from the weak reaction of ϕ_A around the high-M region. This is because of the adverse selection discussed in subsection 2.4. and the second remark in subsection 4.2.. When M is large, traders' behavior depends more on the private signal and less on the price signal. Thus, even if M_t changes, each trader does not modify her behavior that much, leading to the insensitive reaction of the cutoff (and M_{t+1}) to M_t.

Finally, there is another stable steady state in the full-participation region. It is stable because there are no more potential traders who can participate in the market, even with the additional information gain. Technically, the economy cannot explode because Ω is bounded from above. In the sequel, these three steady states are denoted by $M_l < M_m < M_h$.

[32] Appendix B8. provides an example of $l(\cdot, \cdot)$ that describes Fig.8 together with the approximation of the continuum of random variables $\{e(\omega) : \omega \in \Omega\}$ by the limit of a set of discrete random variables.

The source of the multiplicity is reminiscent of Admati and Pfleiderer (1988), Pagano (1989), Han, Tang and Yang (2016), and other literature on complementarity discussed in the introduction. In these models, the market becomes deeper (shallower) as each discretionary liquidity trader anticipates more (less) intensive activity by other liquidity traders. Thus, it comes from the coordination motive of liquidity traders, which is absent in my model. Instead, my model highlights irrationality and the covariance of private signals with heterogeneous qualities as a source of multiple steady states in a dynamic setting.

This model also differs from that of Fajgelbaum, Schaal and Taschereau-Dumouchel (2017). In both models, the multiplicity stems from the positive feedback (strategic complementarity) between the active behavior of agents and the quality of the public (price) signal. The source of the feedback, however, is completely dissimilar. In Fajgelbaum, Schaal and Taschereau-Dumouchel (2017), a larger number of active agents improves the public signal because of the law of large numbers; traders give out i.i.d. signals and the noise components disappear as the number of signals increases. In contrast, the feedback occurs as a result of the covariance and replacement effects in my model. Specifically, my model proposes the importance of the allocation of private information across traders as a determinant of the dynamics.

Remarks: The two stable steady states can be compared in terms of their key economic characteristics. The efficient steady state with the lower $M_{SS} = M_l$ exhibits a high market participation rate (cutoff) and price level. The market is, however, occupied by a larger set of irrational traders and is vulnerable to shocks due to adverse selection. On the other hand, the inefficient steady state can only transmit a less precise signal ($M_{SS} = M_h$), and the price level is low, but it is free from adverse selection and resilient to shocks because the share of rational traders is large.

6. Irrationality and Instability

This section focuses on two important aspects of the disturbance in the economy around the steady state: the propagation and amplification of shocks. Two types of shocks are considered. The first is the disturbance in M around the steady state, which is specified by

$$M_{SS} \rightarrow M_{SS} + \varepsilon$$

at date $t = 0$ with $\varepsilon > 0$. The second is on the distribution f, which will be formally defined later. Note that the following analyses do not restrict our attention on the local dynamics around the steady states, but capture the global evolution.

6.1. Recovery after a Shock

First, we focus on an economy that monotonically converges to its original steady state after a shock. This is illustrated by the first benchmark example in subsection 4.2.. The instability is measured by the variation of the main variables, such as M_t, ω_t^L, and P_t. Since we do not have $dM_{t+1}/dM_t > 1$ around the steady state, the shock is not amplified. The shock, however, can have a persistent effect even if it is transitory. The formal definition of the variation of the variable z_t is given by $V(z) = \sum_{t=1}^{T} \frac{|z_t - z_{t-1}|}{T}$ with a sufficiently large T so that the economy converges to the steady state.

Figure 9: Variation of the Economy

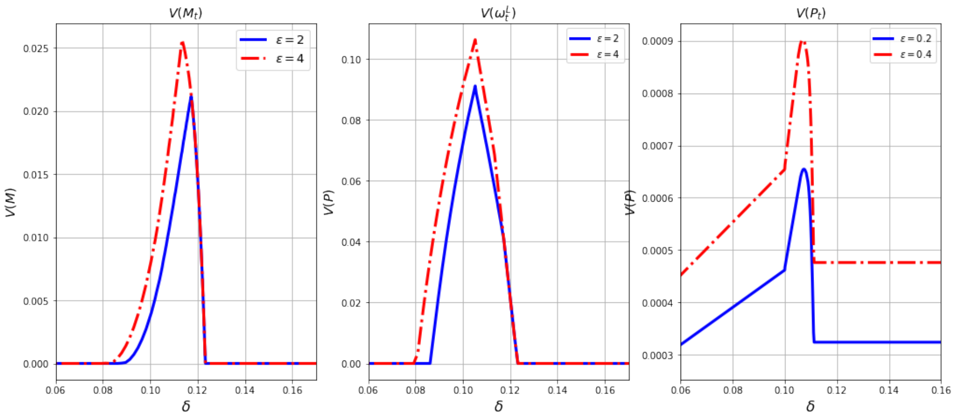


Fig.9 plots this measure of the variation against δ. When the economy is trapped in the full-participation or no-traders equilibrium, the shock cannot perturb it because the cutoff does not change even if there is a shock on M. As δ becomes intermediate, the mixed equilibrium appears. If the steady state is around the mixed region, the effect of the shock does not disappear for a while because the cutoff is reactive to the change in M_t, and the asset price cycles explained in the preceding sections start working.

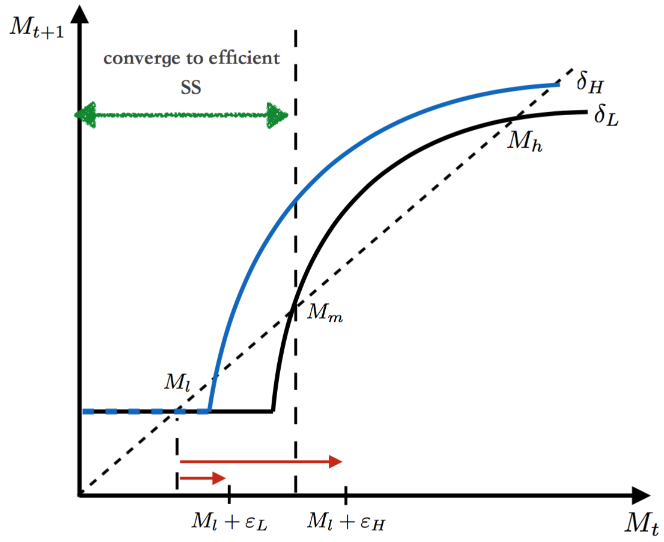
$V(M)$ is smaller than $V(\omega^L)$ because in this benchmark case, the initial shock is absorbed by the change in the cutoff so that the economy converges to the steady state. Note that the difference between these two variations comes from the reaction of ϕ_N to the change in ω^L. Thus, if δ is relatively small and the cutoff is high, ϕ_N is less sensitive since the replacement effect is weak.

The variation of P_t takes a different configuration from the other two variables. Particularly, the price is sensitive to the shock even if δ is extremely small (and thus $\omega^L = \underline{\omega}$). This comes from $\partial P / \partial M < 0$. That is, the noise variance has a direct effect on the price through the change in the valuation, $R(M, \omega) = (1 - \delta)\mu + \delta \mu B(M, \omega)$, even if the cutoff does not change. As δ increases, the effect of M on P becomes larger since the bias of irrational traders is more significant (see the second term). If the economy attains the no-traders equilibrium, δ no longer changes $V(P)$ since there are no biased traders. In the middle-range of δ, the price fluctuates more as the economy reaches the mixed equilibrium. Even though the change in the cutoff does not affect the price directly, the change in the noise variance does.

6.2. Amplification and Stagnation

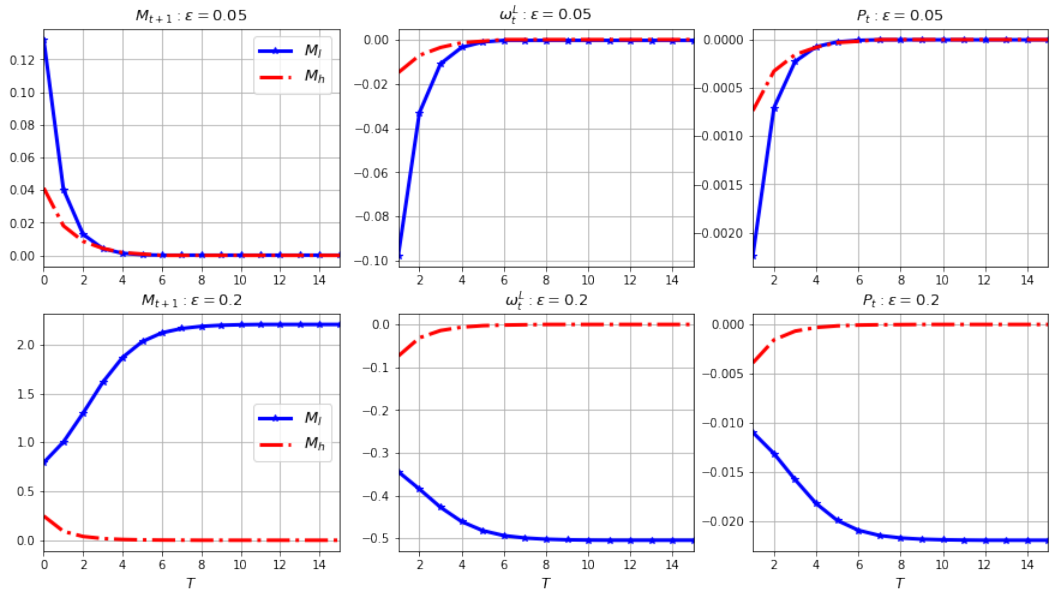
In the real world, most crises are characterized by a significant drop and subsequent stagnation. This transpires if the economy consists of multiple steady states and some of them disappear due to the shock. Thus, we consider the example with multiple steady states provided by the generalized covariance in Section 5.. Fig.10 shows the shifts of D caused by the shock.

Figure 10: Effect of Shock on D



The temporary disappearance of multiple steady states is effected by the large shock, denoted by ε_H, that pushes the state variable out of the convergence region of its original steady state. This is more likely to occur when (i) the temporary shock is big enough or (ii) the convergence region toward the efficient steady state, $M \in [0, M_m]$, is small. Even though (i) is purely exogenous, (ii) is strongly affected by the degree of ambiguity δ, and we can examine whether irrationality raises the likelihood of financial acceleration.

Figure 11: $\delta = 10\%$: Deviation from SS



In the first experiment, small and large ε-shocks hit the economy. Fig.11 shows the impulse responses of the variables as the deviations from the original steady state.[33]

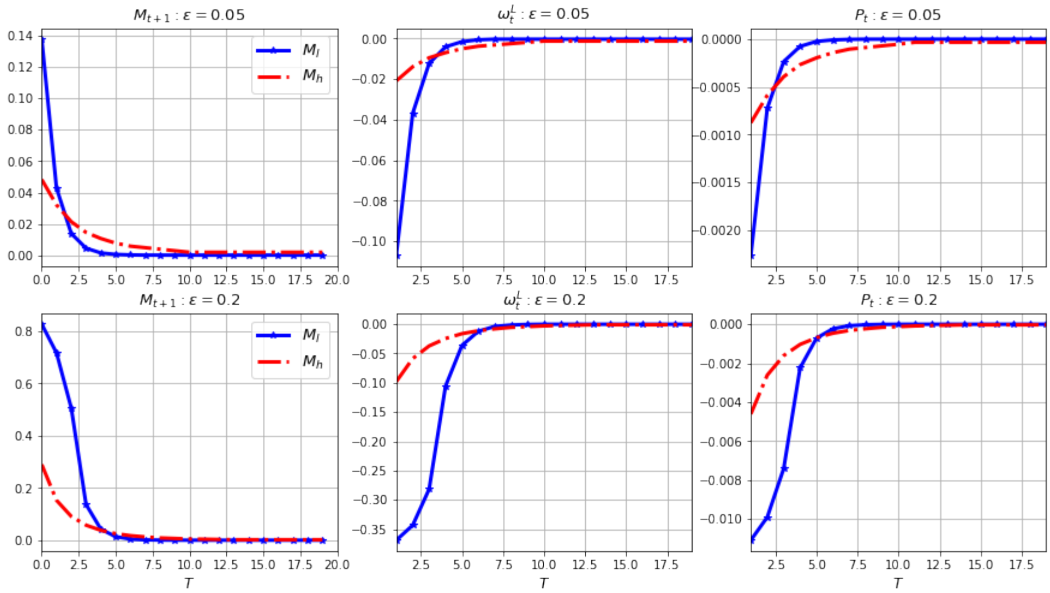
If the shock is small ($\varepsilon = 0.05$), both of the stable steady states exhibit recovery after the shock. However, if it is large ($\varepsilon = 0.2$), the lower stable steady state (M_l) converges to the upper steady state

[33] I omit the response of the middle steady state, M_m, because this is the saddle-point path and has zero-measure

(M_h). In this case, the shock is too great for the efficient equilibrium to go back to its original level, and it has to converge toward the inefficient equilibrium. Note that the original shock is amplified and has a persistent effect, i.e., the financial market accelerates the shock.

The interaction of traders triggers the amplification. In this case, with $D' > 1$, the first-round escape by the participants, which is directly due to the shock, makes the price signal less precise because of the information loss. Then, based on this imprecise signal, even more traders decide to escape in the next period. Therefore, escape will be followed by further rounds of escape. This chain reaction remains in effect until the number of traders becomes small so that (negative) adverse selection sorts out the insensitive traders.

Figure 12: $\delta = 9.5\%$: Deviation from SS



Furthermore, we know that an efficient economy can be more resilient to a shock when the region $[0, M_m]$ is large, and this could be the case when δ is smaller (see the line with δ_L compared to δ_H). This comes from the shifts in the phase diagram due to the change in δ shown in Proposition 4. Intuitively, a smaller δ makes traders more willing to trade, and the economy tends to converge to the steady state with a high participation rate and small M.

Fig.12 shows the reaction of the economy to the same shocks when δ is lower than in the previous example. In this case, even if the shock is large, it cannot push the efficient steady state out of the stable region since each trader is more eager to trade. As a result, the economy converges to the efficient steady state even after the large shock (the blue line in the lower panels). As previously discussed, if δ becomes even lower, the inefficient steady state with a low participation rate disappears and the efficient steady state becomes globally stable. In contrast, when δ is extremely high, the convergence region $[0, M_m]$ is shut down, and the two lower steady states disappear since each trader becomes reluctant to participate.

In summary, within the middle range, a higher δ makes the efficient steady state more vulnerable and the inefficient steady state more resilient to a bad shock ($\varepsilon > 0$), and vice versa.

6.3. Adverse Selection Revisited

Figure 13: Adverse Selection Effect

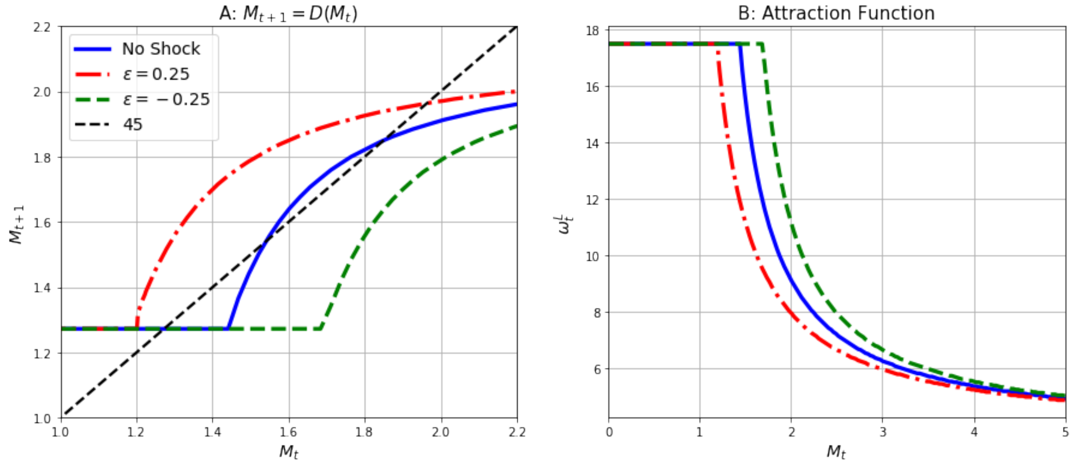


Fig.13 shows how the phase diagram of M changes after the shock $M_t \rightarrow M_t \pm \varepsilon$ for each M_t. First, M_l is sensitive to bad shocks ($\varepsilon > 0$), but insensitive to good ones ($\varepsilon < 0$). Also, the reaction of M_l to the positive shock is larger than that of M_h to the same magnitude of negative shock. The latter effect comes from the concavity of D.

As in the right panel, even if the shock affects M_t uniformly ($|\varepsilon|$ does not rely on the level of M_t), the reaction of ω_t^L is not uniform and depends on M_t and ω_t^L. For a smaller M_t and a higher ω^L, the shock has a disproportional effect because a lower M_t induces adverse selection as discussed in subsection 2.4..

In my model, adverse selection has asymmetric effects due to the insensitive cutoff at the corner solution. This is similar to the literature, such as Kurlat (2013) and Fukui (2016). In their models, adverse selection relates to poor-quality assets accumulated during good economic times. If a bad shock hits, such an economy is vulnerable, while the good shock has no amplifying effects.[34]

Our findings, however, are very different. Recall that the inefficient steady state (M_h) cannot move significantly even if a good shock (negative ε) hits the economy. This is due to the smaller adverse selection effect, and it implies that the inefficient economy is stuck in the inefficient steady state. In the literature (Kurlat (2013), Fukui (2016)), an exogenous variable, such as TFP, completely governs the state of the economy. Thus, if the level of TFP is low, the economy is in a bad state, while if it happens to be high, a potentially bad economy can easily move to the good state.

In contrast, the transition from an inefficient state to an efficient state is not easy in my model because the aggregate state variable is endogenous, and the steady state is endogenously stuck in the inefficiency trap. This is clear from the left panel in Fig.13. Comparing the blue line and the green line for different Ms around M_h, the inefficient states (higher M) do not move even if a shock hits, because such inefficient economies can attract only well informed traders who exhibit an insensitive reaction to the shock on the price signal.

In other words, adverse selection in the previous literature always works against the economy when it is hit by a shock since it makes the transition from the good to the bad state more likely. Conversely,

[34] I use the terms "good" and "bad" instead of positive and negative to avoid confusion. The negative (positive) shock that makes M_t lower (higher) and more (in)efficient would be good (bad) in my model, while the opposite is true in the traditional literature in which the state variable is usually the aggregate technology.

my model suggests that adverse selection also works to help an inefficient economy escape from the inefficiency trap.

In addition, my model shows that the inefficiency trap is stronger than the "uncertainty trap" of Fajgelbaum, Schaal and Taschereau-Dumouchel (2017). In their model, the steady state with higher uncertainty is said to be a "trap" since it is stable. In mine, by contrast, the adverse selection of traders with low-quality information makes the inefficient steady state not only stable but also robust to the aggregate information shock.

6.4. Exclusion Shock

Finally, we consider a temporary shock on f that makes $f(\omega) = 0$ for some ω. We can think of it as a government regulation that directly affects traders' participation behavior; it could be an exogenous constraint such as the margin requirement constraint, taxation, or even an information constraint that shuts down the flow of information for a specific set of traders.

To describe this exclusion shock formally, let $\tilde{\omega} \in (\underline{\omega}, \omega^L)$ be the "ground zero" of the shock and assume that the traders in the ε-ball of $\tilde{\omega}$ must stop trading for an exogenous reason.[35]

Definition 2. For any set of parameters $\epsilon \in \{(\varepsilon, \tilde{\omega}) | \tilde{\omega} \in (\underline{\omega}, \omega_t^L), \varepsilon \leq \min\{\underline{\omega} - \tilde{\omega}, \omega^L - \tilde{\omega}\}\}$, define the ε-*exclusion* around $\tilde{\omega}$ by the new cutoff value $\omega_t^L(\epsilon)$, as determined by the solution of the following equation with respect to ω^L.

$$\frac{\tau\sigma^2 x}{\delta\mu} = \sigma^2 \frac{M_t + (1-\rho)\sigma^2}{\xi(M_t, \omega^L)} \int_{\Omega(\omega^L, \epsilon)} \frac{\omega^L - \omega}{\omega} dF(\omega) + \lambda \frac{1 - B_t(\omega^L)}{\gamma_t(\omega_r)}. \tag{27}$$

with

$$\Omega(\omega^L, \epsilon) \equiv [\underline{\omega}, \omega^L] \backslash (\tilde{\omega} - \varepsilon, \tilde{\omega} + \varepsilon).$$

(27) is the version of (16) where the market participants are limited to $\omega \in \Omega(\omega^L, \epsilon)$.

First, we consider the intra-temporal effect of this shock. The ε-exclusion affects the new cutoff $\omega_t^L(\epsilon)$ in the following way.

Proposition 11. *A larger exclusion (higher ε) induces a higher $\omega_t^L(\epsilon)$ and lower P_t.*

Proof. See Appendix B9.. □

The result in Proposition 11 is intuitive. As the traders with type $(\tilde{\omega} - \varepsilon, \tilde{\omega} + \varepsilon)$ are exogenously excluded by the shock, new traders with $\omega_t^L(\tilde{\omega}, 0) \leq \omega \leq \omega_t^L(\tilde{\omega}, \varepsilon)$ must cover the reduction in the demand to clear the market. Therefore the new cutoff is an increasing function in ε.

Moreover, the definition of the cutoff implies that the price is decreasing in ε. Because of the shock, excluded traders with relatively precise information (and a higher R^L) are replaced by traders with relatively less precise information (and a lower R^L). To make the latter traders willing to demand the risky asset, the price must become lower than the ex-ante value.[36]

The uniform benchmark provides further analytical insights (proofs are omitted as they are straightforward from definitions):

[35] We set $\varepsilon > 0$ small enough and $\omega^L - \tilde{\omega}$ large enough so that the new cutoff defined by (27) is not included by the ε-Ball of the ground zero, that is, we make $\omega^L(\epsilon) > \tilde{\omega} + \varepsilon$ hold.

[36] This effect is identical to the one discussed in Yu (2013) but, in contrast to his model, the chain reaction continues to work in this model.

Corollary 3. *Given M_t and the uniform distribution, the cutoff is obtained via the solution of (28), and the noise variance is given by (29).*

$$\frac{\tau x}{\delta \mu} = \frac{1}{(\bar{\omega} - \underline{\omega})} \frac{[M_t + (1-\rho)\sigma^2]\left\{\omega^L \left[\log\left(\frac{\tilde{\omega}-\varepsilon}{\tilde{\omega}+\varepsilon}\frac{\omega^L}{\underline{\omega}}\right) - 1\right] + 2\varepsilon + \underline{\omega}\right\}}{\xi(M_t, \omega^L)} + \lambda \frac{1 - B(\omega^L)}{\gamma(\omega_r)}, \tag{28}$$

$$M_{t+1} = \phi_N(\omega^L(\epsilon)) = \frac{\left(\frac{\sqrt{2}}{\bar{\omega}-\underline{\omega}}[\sqrt{\omega^L(\epsilon)} - \sqrt{\tilde{\omega}+\epsilon} + \sqrt{\tilde{\omega}-\epsilon} - \sqrt{\underline{\omega}}]\right)^2 + \lambda^2 \omega_r^{-1}}{\left[\frac{1}{\bar{\omega}-\underline{\omega}} \log\left(\frac{\tilde{\omega}-\varepsilon}{\tilde{\omega}+\varepsilon}\frac{\omega^L(\epsilon)}{\underline{\omega}}\right) + \lambda \omega_r^{-1}\right]^2}. \tag{29}$$

We can check that a higher ε increases ω_t^L as hypothesized in Proposition 11. If we look at the (29) in Proposition 3, we see that the effect of the exclusion on the noise variance depends on how $\omega_t^L(\tilde{\omega} - \varepsilon)/(\tilde{\omega} + \varepsilon)$ changes as ε or $\tilde{\omega}$ fluctuates. The term $(\tilde{\omega} - \varepsilon)/(\tilde{\omega} + \varepsilon)$ is decreasing in ε and captures the information loss due to the ε-exclusion, while the increase in ω^L changes the noise variance through the traditional replacement and covariance effects. The total effect is determined by the relative powers of these three components.

Also, the following result shows that it matters "which traders are excluded".

Corollary 4. *In the benchmark with a sufficiently high $\bar{\omega}$, we have $d\omega_t^L(\epsilon)/d\tilde{\omega} < 0$.*

A higher value of the ground zero must be associated with a lower value of the ex-post threshold. The excluded traders, who must be replaced, have relatively less precise signals. Hence, the reduction of the demand is small, and the replacement by the new traders does not require as many participants as in the case of a lower $\tilde{\omega}$ to maintain the market clearing condition.

6.5. Amplification and Stagnation: Exclusion Shock

Now we consider the dynamic behavior of the economy after an exclusion shock. As in the case with the information shock, an exclusion shock can also be amplified and have a permanent effect.

Figure 14: Impulse Response

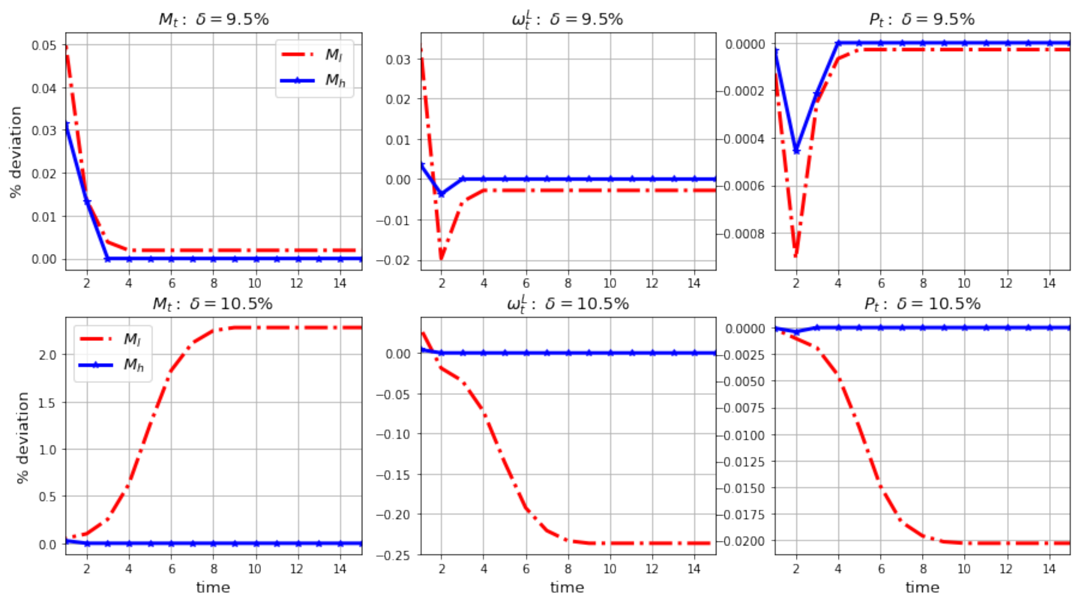


Fig.14 shows the impulse responses (% deviation from the original steady state). When the irrationality is not severe ($\delta = 9.5\%$), the economy can recover after an exclusion shock. However, in a more irrational economy ($\delta = 10.5\%$), the efficient steady state (M_l) becomes more vulnerable. The logic is the same as in the analyses with the information shock; a higher δ makes the convergence region toward M_l smaller because each trader is more reluctant to participate. As a result, even though the rational economy (low-δ) can stand the shock and recover, the irrational economy (high-δ) amplifies the shock and tends to stagnate. As we can imagine, if we make δ even higher, the efficient steady state completely disappears and the economy consists of only a unique inefficient steady state. At this point, the market becomes stable again.

7. Self-Organized Market Crashes

One of the most prominent features of my model is the covariance structure, which allows it to explain various scenarios in the real world, including a huge economic movement which occurs even with insignificant news shocks. It goes, however, beyond the amplification of a shock (financial accelerator) to propose a new source of crashes.

The literature on adverse selection and the "efficient economy" in this paper explain why market crashes commonly follow the booms: the share of low-quality assets (poorly informed/sensitive traders) grows during good periods, which makes the economy vulnerable to a negative shock. Thus, the magnitude of the effect of the trigger shock is deeply related to the historical state of the economy.

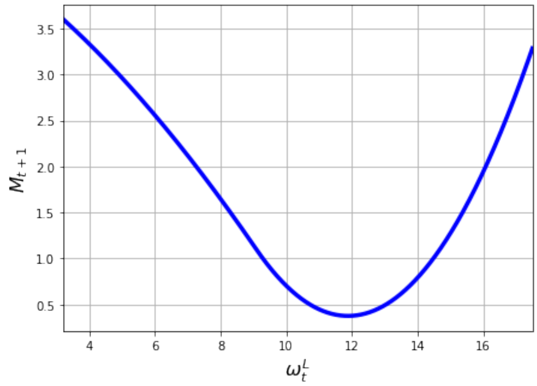
The arrival of trigger shocks is, however, mostly independent of the preceding state of the economy, and it is hard to identify why they occur. In other words, if we are in a steady state, there must be two shocks to create booms and busts: one is supposed to start a boom, and the other is supposed to trigger a crash. Moreover, in most of the literature, these shocks have an aggregate feature, such as a shock on TFP.

As a new explanation, I propose "self-organized market crashes (SOMCs)." In an SOMC, one shock pushes a button to start a boom. Unlike in the traditional models, however, the boom itself can initiate the bust. Namely, we do not need any extraneous shocks to trigger the crash, and only one initial shock explains the entire boom and bust cycle. Furthermore, this one-shot initial shock can be idiosyncratic in my model.

The backdrop of SOMCs is the interaction of heterogeneous traders with externality. The "self-organized" nature of this critical phenomenon was originally investigated by Bak, Tang and Wiesenfeld (1987) and applied to explain economic fluctuations by Bak, Chen, Scheinkman and Woodford (1993), Scheinkman and Woodford (1994), and Nirei (2006). Even though the definition of self-organized criticality (SOC) in their works is slightly different from the one discussed here, the fact that a booming economy spontaneously moves toward a critical threshold for a crash and exceeds it via endogenous interactions is reminiscent of SOC. [37]

[37] Gabaix (2011) and Acemoglu, Carvalho, Ozdaglar and Tahbaz-Salehi (2012) also argue that aggregate fluctuations stem from the interactions of micro-motived individuals.

Figure 15: $M_{t+1} = \phi_N(\omega_t^L)$ in SOMC

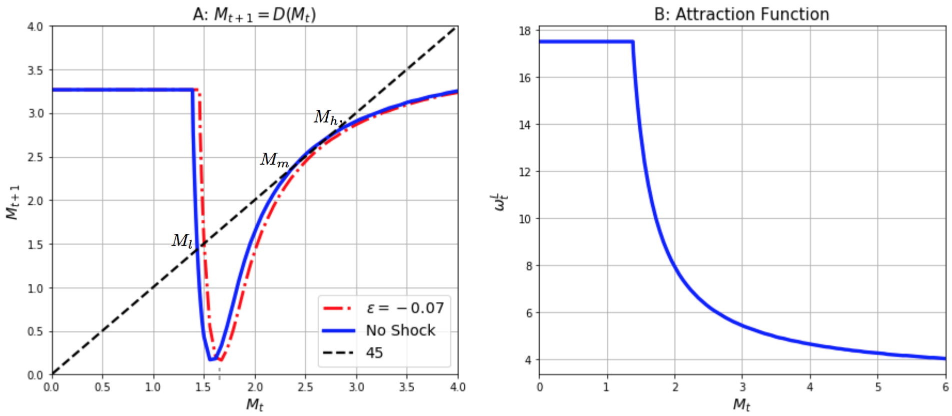


Consider the noise-aggregation function in Fig.15. The intuition for ϕ_N is almost the same as the U-shaped ϕ_N in the benchmark case provided in Fig.2, while ϕ_N for SOMC has a positive and *convex* slope for the high-ω^L region. That is, when the cutoff is small (roughly $\omega^L < 12$ in this example), the feedback is positive; additional market participants improve the price information and lead to further participation. However, the feedback becomes strongly *negative* once the cutoff ω^L gets into a higher region (roughly $\omega^L \geq 12$ in this example). Suppose that at $\omega^L = \omega_{SOMC}$ the feedback switches from positive to negative.

One possible interpretation for this switch is as follows. The threshold ω_{SOMC} separates traders into two groups, and the those with $\omega > \omega_{SOMC}$ have a strongly positive correlation within the group. I believe we can find this situation in the real economy; there are groups of traders, each of which contains individuals with a different set of information (such as reading more or fewer newspapers), generating a covariance structure depending on the information sources used and how well they are interpreted.

When the market is not informationally efficient, only traders with $\omega < \omega_{SOMC}$ participate. This makes the replacement effect dominate the covariance effect since the share of irrational traders is not sufficiently large. However, as the positive feedback (downward-sloping ϕ_N) makes the price information more precise, traders with $\omega > \omega_{SOMC}$ start trading. Once these traders participate in the market, the strong correlation between them can overturn the replacement effect. From a noise aggregation's perspective, these traders with $\omega > \omega_{SOMC}$ are not only poorly informed but also harmful, since they significantly contaminate the price information.

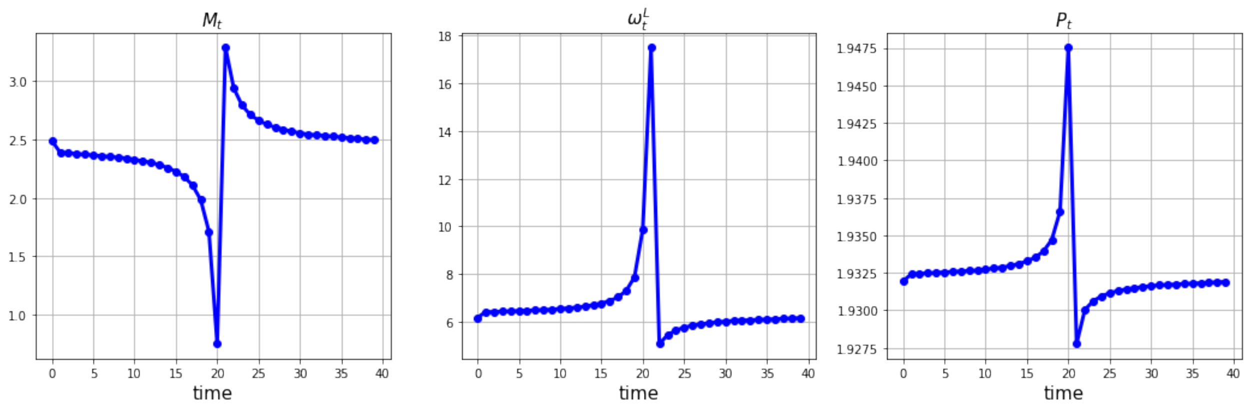
Figure 16: Phase Diagram in SOMC



The noise aggregation function in Fig.15 draws the phase dynamics of M_t in Fig.16. Let the multiple steady states once again be denoted by $\{M_l, M_m, M_h\}$. Then, SOMC is the situation given by M_h with the blue line. In this case, a small aggregate shock on M_h makes the economy evolve through $M_h = M_0 > M_1 > \cdots > M_T$ for some T that is not large enough. During this phase, the economy is pushed out of the inefficient, low-price, and low-activity steady state and becomes increasingly efficient. This is due to the positive feedback, and the more numerous market participants induce a more efficient and higher price. We can think of this as a boom phase.

However, once M dips below the threshold, say, $M_{SOMC} = M_l$, the precise price signal attracts too many harmful traders with $\omega > \omega_{SOMC}$. At this point, the harmful market participants contaminate the price signal too much, and M dramatically increases, $M_T \ll M_{T+1}$. Then, the significantly contaminated price information weeds out many traders from the market, and the total demand, price level, and market participation rate all crash. Fig.17 describes the evolution of an economy hit by a small ε-shock at date $t = 1$. After that, no external shocks arrive, but the economy crashes after the boom.

Figure 17: SOMC



It is worth mentioning that the price level (and other variables) overshoots after the crash. This is because the information quality of the price signal becomes lower than in the initial condition. The harmful traders who trigger the crash make the ex-post price signal even worse than before the boom. If the negative effect of the harmful traders is less significant, the economy does not overshoot. Moreover, the initial shock ε does not have to be an aggregate shock and can be idiosyncratic as long as the measure of traders hit by the shock is not zero (see Appendix E for details).

The upshot of this is that we can analyze the SOMCs caused by the interaction of micro-motived traders. In this mechanism, the covariance structure brought into the market by the competitive (non-strategic) traders is pooled by the price, which induces a tremendous drop in the signal quality. Even if the price signal deteriorates locally ($\omega > \omega_{SOMC}$), this is enough to confound the traders and trigger a crisis. The economy experiences a boom when the feedback is positive, but the positive feedback can go too far and push itself into the negative feedback region. This dynamic change in the feedback is not captured by Fajgelbaum, Schaal and Taschereau-Dumouchel (2017), since their model dismisses the importance of the medium of information aggregation, and the i.i.d nature of private signals always results in positive feedback. In an SOMC, the trigger is the advent of the harmful traders with $\omega > \omega_{SOMC}$. The "self-organized" property of the crash refers to the fact that these traders switch from being inactive to active not because of the exogenous shock, but because of the endogenous change in the state variable.

With the results of this model in mind, the allocation of information across traders and the covariance structure can be seen as relevant predictors of the asset price dynamics. When the information structure is one that induces multiple steady states, we must prepare for the trigger shocks. In an SOMC, the trigger shock is predictable by looking at the information structure of active traders, since it is endogenously attracted by the booming economy. This paper provides theoretical and positive analyses and does not investigate empirical tests or policy problems, but these predictions should be explored in further research.

8. Concluding Remarks

This paper is a theoretical study into how irrationality and information with heterogeneous qualities interact with each other and how they cause asset prices to fluctuate. Specifically, the model explains the presence of financial accelerator mechanism in the absence of traditional driving forces.

Whether traders are active or inactive is completely determined by their private signals and the common price information. Traders decide to participate if, and only if, the information set transmits precise information about the asset. Since the private signals are continuously heterogeneous in terms of their precision, we can define the cutoff that separates the active and inactive traders.

Once a portion of traders escape from the market due to a shock (first-round escape), their private signal will no longer be reflected by the price, and the price signal can become less precise. As a consequence, more traders become reluctant to participate, and the cutoff declines to kick out the marginal traders who have relatively imprecise signals, leaving only well-informed traders in the market. Then, the escape of the marginal traders (second-round escape) further contaminates the price information, and even more traders decide to quit (third-round escape). This chain reaction causes a huge decline in the participation rate, informativeness of the price signal, and price level.

We also investigate how the degree of irrationality affects the likelihood of tremendous fluctuations. The second novel contribution of this paper is the hump-shaped relationship between irrationality and the instability of the economy. When irrationality is extremely high, the inaction region of each trader becomes very wide and the switch between active and inactive does not occur, even if the economy is hit by the shock, i.e., the economy is resilient. Also, when irrationality is low, the behavior of irrational traders becomes similar to that of rational traders; the inaction region shuts down and the economy becomes stable again. Only in the middle range of irrationality does the chain reaction explained above start working and make the economy vulnerable to shocks.

Furthermore, the model illustrates the adverse selection of low-quality information. When the price signal is efficient, even traders with an imprecise private signal can participate in the market. Since the behavior of poorly informed traders depends more on the public price signal, they are more sensitive to shocks on the price signal. Therefore, an efficient economy, with a higher participation rate and a higher price level, becomes more vulnerable to aggregate shocks on the public signal. This also implies that an informationally inefficient economy with a low price is more resilient to shocks since it attracts only traders with precise private signals. We can think of this as an inefficiency trap because it is difficult for this type of economy to escape from the inefficient state even after a good shock hits. For an inefficient economy, adverse selection helps to escape from the trap.

Finally, the model can explain market crashes that happen without external trigger shocks. When the economy experiences a boom, the positive feedback sustains the high/efficient price. However, once the price information becomes too efficient, harmful traders with detrimental correlations start trading. Then, these traders contaminate the price signal significantly, and most of the active traders withdraw, leading to a price crash. The arrival of the harmful traders is caused by the endogenous change in the state variables. This is the trigger shock for the crash, but is not exogenously induced.

Even though the model has a strong descriptive power, the dynamic economy, in aggregate level, is deterministic. The next critical question we should investigate is to what extent the aggregate risk affects the probability of the cascading chain reaction. In most of the literature regarding market crashes and asset price bubbles, the probability of the crash is exogenously given (see Weil (1987)), and we have not endogenized the collapse of the market in a tractable manner. Since the behavior of each trader in my model considers the stochastic realization of random variables, making aggregate risks stochastic by relaxing the pseudo-stochastic assumption can open the way to viewing market crashes from a stochastic perspective.

1

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A Appendix A:

A1. Optimal Portfolio with Ambiguity Aversion

CARA-Normal assumption makes the optimization problem equivalent to solve

$$\begin{aligned}
&\max_x \min_{\hat{\mu}} E_{\hat{\mu}}[v_{t+1} - P_t | y_t(\omega)] - \frac{\tau}{2} x^2 Var(v_{t+1}|y_t(\omega)) \\
&= \max_x \min_{\hat{\mu}} \left(E_{\hat{\mu}}[v_{t+1}|y_t(\omega)] - P_t\right) x - \frac{\tau}{2} x^2 Var(v_{t+1}|y_t(\omega)).
\end{aligned}$$

Depending on the long or short position, the objective function is written as

$$= \begin{cases} \min_{\hat{\mu}} \left(E_{\hat{\mu}}[v_{t+1}|y_t(\omega)] - P_t\right) x - \frac{\tau}{2} x^2 Var(v_{t+1}|y_t(\omega)) & \text{if } \min_{\hat{\mu}} \left(E_{\hat{\mu}}[v_{t+1}|y_t(\omega)] - P_t\right) > 0 \\ 0 & \text{o/w} \\ \max_{\hat{\mu}} \left(E_{\hat{\mu}}[v_{t+1}|y_t(\omega)] - P_t\right) x - \frac{\tau}{2} x^2 Var(v_{t+1}|y_t(\omega)) & \text{if } \max_{\hat{\mu}} \left(E_{\hat{\mu}}[v_{t+1}|y_t(\omega)] - P_t\right) < 0. \end{cases}$$

By using the notations introduced in the main text,

$$= \begin{cases} (R_t^L(\omega) - P_t)x - \frac{\tau}{2} x^2 Var(v_{t+1}|y_t(\omega)) & \text{if } R_t^L > P_t \\ 0 & \text{if } P_t \in [R_t^L, R_t^H] \\ (R_t^H(\omega) - P_t)x - \frac{\tau}{2} x^2 Var(v_{t+1}|y_t(\omega)) & \text{if } P_t > R_t^H \end{cases}$$

and solving this provides the optimal portfolio in (**??**).

A2. Explicit Formulas for Filtering Problem

Consider to infer v_{t+1} based on the signals $s_t(\omega)$ and $q_{t-1} = v_t + m_t$ with $M_t \equiv Var(m_t)$. The normality assumption provides the following Kalman filer arguments.

$$
\begin{aligned}
\begin{pmatrix} \beta^s(q_{t-1}, \omega) \\ \beta^q(q_{t-1}, \omega) \end{pmatrix} &= \begin{pmatrix} Var(s_t(\omega)), & Cov(s_t(\omega), q_{t-1}) \\ Cov(s_t(\omega), q_{t-1}), & Var(q_{t-1}) \end{pmatrix}^{-1} Cov(v_{t+1}, y_t(\omega))^T \\
&= \begin{pmatrix} \sigma^2 + \omega, & \rho\sigma^2 \\ \rho\sigma^2, & \sigma^2 + M_t \end{pmatrix}^{-1} \begin{pmatrix} \sigma^2 \\ \rho\sigma^2 \end{pmatrix} \\
&= \frac{\sigma^2 \begin{pmatrix} (1-\rho^2)\sigma^2 + M_t \\ \rho\omega \end{pmatrix}}{(1-\rho^2)\sigma^4 + \sigma^2(M_t + \omega) + M_t\omega}, \\
B(q_{t-1}, \omega) &= \frac{(1-\rho^2)\sigma^2 + M_t + \rho\omega}{(1-\rho^2)\sigma^4 + \sigma^2(M_t + \omega) + M_t\omega}\sigma^2. \qquad (30)
\end{aligned}
$$

Note that q has effects only through its variance. Thus, I suppress the element q and denote the function as $B(M, \omega)$. Note that

$$
\gamma(\omega, M_t) = \frac{\omega(M_t + (1-\rho^2)\sigma^2)}{(1-\rho^2)\sigma^4 + \sigma^2(M_t + \omega) + M_t\omega}.
$$

Also, let

$$
\xi(M_t, \omega) = \xi(M_t, \omega) \equiv (1-\rho^2)\sigma^4 + \sigma^2(M_t + \omega) + M_t\omega
$$

for later uses.

A3. The Model with Resale

In order to introduce the resale of the asset in the old period, we need to fix the supply of the asset constant x and assume that the risk free asset earns a positive (net) interest payment r. Then the trader maximizes the objective in (4) subject to

$$
w_{t+1} = [V_{t+1} - (1+r)P_t]x_t
$$

where $V_{t+1} = v_{t+1} + P_{t+1}$ is the gross return.

For simplicity, let $\lambda = 0$ and focus on the case with an interior cutoff. Given the cutoff strategy, the market clearing price is

$$
P_t = \frac{\int^{\omega_t^L} \frac{\inf_\mu E_t[V_{t+1}|y_t(\omega)]}{\tau\sigma^2\gamma(\omega)} dF(\omega) - x}{\int^{\omega_t^L} \frac{1}{\sigma^2\tau\gamma(\omega)} dF(\omega)}.
$$

This is not analytically solvable because we have to calculate the expectation of an integral which contains the random cutoff ω_{t+1}^L as the integral boundary.

In order to proceed, we make use of the fact that each individual trader does not take into account the effect of their behavior on the price level, the price information, and the future cutoff. This is natural since we consider the infinite number of traders and each individual has zero measure. Under this specification, we need to define the belief of each trader regarding the price one period ahead. Suppose that the conditional stationarity for each individual level holds, that is $E_t[P_t|y_t(\omega)] = E_t[P_{t+1}|y_t(\omega)]$. Then, we have

$$
rP_t = \frac{\int^{\omega_t^L} \frac{\inf_\mu E_t[v_{t+1}|y_t(\omega)]}{\tau\sigma^2\gamma(\omega)} dF(\omega) - x}{\int^{\omega_t^L} \frac{1}{\sigma^2\tau\gamma(\omega)} dF(\omega)},
$$

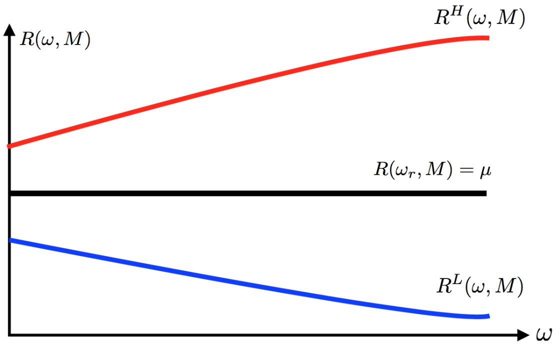
which is exactly the same as the equation (14) except r is a coefficient.

Also, the cutoff definition is $(1+r)P_t = \inf E_t[v_{t+1} + P_{t+1}|y_t(\omega^L)]$ and this reduces to $rP_t = \inf E[v_{t+1}|y_t(\omega^L)]$, which corresponds to (13). As a result, the properties of the model do not change except that the price level is scaled by r.

A4. No possibility of the short selling under $x \geq 0$

From Lemma 1, we can plot R^k for irrational traders and sophisticated traders as in Fig. 18.

Figure 18: Best and Worst Conditional Expected Return



If there is one trader who takes a short position, then we can see that there is a unique cutoff ω^H such that $R^H(\omega^H, M) = P_t > \mu$. That is, all traders with $\omega > \omega^H$ are taking a short position, while traders with $\omega \leq \omega^H$ and the sophisticated traders are inactive. Then, the market clearing condition with $x \geq 0$ implies that $P_t \to -\infty$ because nobody wants to demand the asset to cover the short selling (as well as the positive supply). This is a contradiction to $P > \mu$. On the other hand, the equilibrium price such that $P < \mu$ is plausible because, at least, the sophisticated traders demand a positive position. This argument shows that we can focus on the "taking long or being inactive" decision of irrational traders.

B Proof

B1. Proof of Proposition 1

By plugging $R_t^L(M_t, \omega^L) = P_t$ into (12) gives us:

$$\int_{\underline{\omega}}^{\omega_t^L} \frac{R_t^L(M_t, \omega) - R_t^L(M_t, \omega^L)}{\tau Var(v_{t+1}|y_t(\omega))} dF(\omega) + \lambda \frac{R(M_t, \omega_r) - R_t^L(M_t, \omega^L)}{\tau Var(v_{t+1}|y_t(\omega_r))} = x \tag{31}$$

where

$$\begin{aligned}
\frac{R_t^L(M_t, \omega) - R_t^L(M_t, \omega^L)}{\tau Var(v_{t+1}|y_t(\omega))} &= \frac{\mu - \mu_L}{\tau\sigma^2\gamma_t(M_t, \omega)} \left[B(M_t, \omega) - B(M_t, \omega^L)\right] \\
&= \frac{\delta\mu}{\tau} \frac{\xi(M_t, \omega)}{\omega[M_t + (1-\rho^2)\sigma^2]} \left[\frac{(1-\rho^2)\sigma^2 + M_t + \rho\omega}{\xi(M_t, \omega)} - \frac{(1-\rho^2)\sigma^2 + M_t + \rho\omega^L}{\xi(M_t, \omega^L)}\right] \\
&= \frac{\delta\mu/\tau}{\omega\xi(M_t, \omega^L)[M_t + (1-\rho^2)\sigma^2]} \\
&\times \left[[(1-\rho^2)\sigma^2 + M_t + \rho\omega]\xi(M_t, \omega^L) - [(1-\rho^2)\sigma^2 + M_t + \rho\omega^L]\xi(M_t, \omega)\right] \\
&= \frac{\frac{\delta\mu}{\tau}[M_t + (1-\rho)\sigma^2]}{\xi(M_t, \omega^L)} \frac{\omega^L - \omega}{\omega},
\end{aligned}$$

and

$$\begin{aligned}
\frac{R_t(M_t, \omega_r) - R_t^L(M_t, \omega^L)}{\tau Var(v_{t+1}|y_t(\omega))} &= \frac{\mu - [\mu_L + (\mu - \mu_L)B(M_t, \omega^L)]}{\tau\sigma^2\gamma(M_t, \omega_r)} \\
&= \frac{\delta\mu}{\tau\sigma^2} \frac{1 - B(M_t, \omega^L)}{\gamma(M_t, \omega_r)}.
\end{aligned}$$

These formulas directly give (16). Let

$$A(\delta) = \frac{\tau\sigma^2 x}{\delta\mu}$$

and

$$H(M_t, \omega^L) = \sigma^2 \frac{M_t + (1-\rho)\sigma^2}{\xi(M_t, \omega^L)} \int^{\omega^L} \frac{\omega^L - \omega}{\omega} dF(\omega) + \lambda \frac{1 - B(M_t, \omega^L)}{\gamma(M_t, \omega_r)}.$$

Note that, for $\omega^L \geq \underline{\omega} > 0$,

$$\frac{\partial}{\partial \omega_t^L} \left(\frac{\int^{\omega^L} \frac{\omega^L - \omega}{\omega} dF(\omega)}{\xi(M_t, \omega^L)} \right) = \frac{\left[\sigma^2(M_t + (1-\rho^2)\sigma^2) \int^{\omega^L} \frac{1}{\omega} dF + (M_t + \sigma^2) F(\omega^L)\right]}{\xi^2(M_t, \omega^L)} > 0$$

and

$$\frac{\partial(1 - B(M_t, \omega^L))}{\partial \omega^L} > 0$$

from Lemma 1. Therefore, $dH/d\omega_t^L > 0$ for any distribution f and $\omega^L \geq \underline{\omega}$ and there is a unique solution for $A = H(M_t, \omega^L)$ with respect to ω^L. This also implies that the price level P_t that satisfies both of (12) and (13) is uniquely determined.

B2. Proof of Proposition 2

Given ω^L,

$$\frac{\partial}{\partial M_t} \left(\frac{M_t + (1-\rho)\sigma^2}{\xi(M_t, \omega^L)} \right) \sim \rho(\omega^L + (1-\rho)\sigma^2)) > 0,$$

and

$$\frac{\partial}{\partial M_t} \left(\frac{1 - B(M_t, \omega^L)}{\gamma(M_t, \omega_r)} \right) = \frac{\partial}{\partial M_t} \left(\frac{(M_t + (1-\rho)\sigma^2)/\xi(M_t, \omega^L))}{(M_t + (1-\rho^2)\sigma^2)/\xi(M_t, \omega_r)} \right)$$

(after tedious calculations) which has the same sign as

$$\begin{aligned}
&\sigma^2(M_t + (1-\rho^2)\sigma^2)^2(\omega^L + (1-\rho)\sigma^2) + \omega_r\sigma^2(1-\rho)(M_t + (1-\rho^2)\sigma^2)(M_t + \sigma^2) \\
&\quad -\rho\sigma^2\omega_r(M_t + (1-\rho^2)\sigma^2)(M_t + (1-\rho)\sigma^2) + \omega_r\omega^L(M_t + \sigma^2)(1-\rho) \\
&\geq \omega_r(1-\rho)(M_t + \sigma^2)[\sigma^2(M_t + (1-\rho^2)\sigma^2) + \omega^L] \\
&> 0,
\end{aligned}$$

where the first inequality comes from $\omega^L \geq \underline{\omega} > \omega_r$.

As a result, we get $\partial H/\partial M_t > 0$ and Proof B1. shows $\partial H/\partial \omega_t^L > 0$. Therefore, the implicit function theorem implies

$$\frac{d\omega_t^L}{dM_t} = \phi_A^{'}(M_t) < 0$$

in the equilibrium.

B3. Proof of Proposition 3

By denoting $P_t = P(\omega_t^L, M_t) = P(\phi_A(M_t), M_t)$, we have

$$\begin{aligned}
\frac{dP(\phi_A(M_t), M_t)}{dM_t} &= \frac{d\phi_A(M_t)}{dM_t} \frac{\partial P_t}{\partial \omega_t^L} + \frac{\partial P_t}{\partial M_t} \\
&= \frac{\partial P_t}{\partial M_t}.
\end{aligned}$$

Note that $\frac{\partial P_t}{\partial \omega_t^L} = 0$ from the discussion in (15). Furthermore,

$$\frac{\partial P_t}{\partial M_t} = \frac{\int^{\omega_t^L} \left(\frac{\partial}{\partial M_t} \left(\frac{R_t^L(\omega)}{\gamma(\omega)} \right) - P_t \frac{\partial}{\partial M_t} \left(\frac{1}{\gamma(\omega)} \right) \right) dF(\omega) + \lambda \left(\frac{\partial}{\partial M_t} \left(\frac{R_t(\omega_r)}{\gamma(\omega_r)} \right) - P_t \frac{\partial}{\partial M_t} \left(\frac{1}{\gamma(\omega_r)} \right) \right)}{\int^{\omega_t^L} \frac{dF(\omega)}{\gamma(\omega)} + \frac{\lambda}{\gamma(\omega_r)}}.$$

Since we have

$$\frac{\partial}{\partial M_t} \left(\frac{R_t^L(\omega)}{\gamma(\omega)} \right) - P_t \frac{\partial}{\partial M_t} \left(\frac{1}{\gamma(\omega)} \right) = \frac{1}{\gamma^2(\omega)} \left[\gamma(\omega) \frac{\partial R^L(\omega)}{\partial M_t} - (R^L(\omega) - P_t) \frac{\partial \gamma(\omega)}{\partial M_t} \right] < 0$$

for $\omega < \omega_t^L$, this proves the proposition.

B4. Proof of Proposition 4

First, it has to be shown that $H(M, \underline{\omega})$ and $H(M, \bar{\omega})$ are bounded as in Fig. 1. This can be confirmed by

$$\begin{aligned} \max_M H(M, \bar{\omega}) &= \lim_{M \to +\infty} \frac{M + (1-\rho)\sigma^2}{\xi(M, \omega_r)(M + (1-\rho^2)\sigma^2)} \left[\sigma^2 E[\frac{\bar{\omega} - \omega}{\omega}](M + (1-\rho^2)\sigma^2) + \lambda \frac{\bar{\omega}}{\omega_r} \xi(M, \omega_r) \right] \\ &= \frac{\sigma^2}{\omega_r + \sigma^2} E[\frac{\bar{\omega} - \omega}{\omega}] + \lambda \frac{\bar{\omega}}{\omega_r} \\ &\equiv A_3 < +\infty \end{aligned}$$

and

$$\begin{aligned} \min_M H(M, \underline{\omega}) &= \lim_{M \to 0} \frac{M + (1-\rho)\sigma^2}{M + (1-\rho^2)\sigma^2} \frac{\xi(M, \omega_r)/\omega_r}{\xi(M, \underline{\omega})/\underline{\omega}} \lambda \\ &= \frac{\lambda}{1+\rho} \frac{1 + (1-\rho^2)\omega_r^{-1}}{1 + (1-\rho^2)\underline{\omega}^{-1}} \\ &\equiv A_0 > -\infty \end{aligned}$$

Therefore, H can be depicted as Fig. 1.

If $A(\delta) < A_0$ (in the region (i) of Fig. 1), then $A(\delta)$ never intersects with H and all traders are inactive no matter what the level of M_t is. Second, suppose that

$$A_0 \leq A(\delta) < \min_M \max_{\omega^L} H(M, \omega^L) \equiv A_1$$

where

$$\min_M \max_{\omega^L} H(M, \omega^L) = \frac{\sigma^2(1-\rho^2)E[(\bar{\omega} - \omega)/\omega] + \lambda \frac{\bar{\omega}}{\omega_r}(\omega_r + (1-\rho^2)\sigma^2)}{(1+\rho)(\bar{\omega} + (1-\rho^2)\sigma^2)}.$$

This case corresponds to the region (ii) in Fig. 1 and the no-traders equilibrium exists when M_t is relatively high. Also, the mixed equilibrium can exist as well when M_t is relatively low.

The third case is given by

$$A_1 \leq A(\delta) < \max_M \min_{\omega^L} H(M, \omega^L) \equiv A_2$$

where

$$\max_M \min_{\omega^L} H(M, \omega^L) = \frac{1 + \sigma^2 \omega_r^{-1}}{1 + \sigma^2 \underline{\omega}^{-1}} \lambda.$$

In this case, we can see from the region (iii) in Fig. 1 that the no-traders equilibrium can exist when M_t is high, the full-participation equilibrium can exists when M_t is low, and there also can be the mixed equilibrium when M_t is intermediate value.

Fourth, the region (iv) in Fig. 1 is denoted by

$$A_2 \leq A(\delta) < \max_{M, \omega^L} H(M, \omega^L) = A_3.$$

Under this condition, the mixed equilibrium materializes when M_t is relatively high and the full-participation equilibrium also can exist when M_t is low.

Finally, when $A_3 \leq A(\delta)$, then all traders participate in the market no matter what the level of M_t would be.

B5. Proof of Proposition 5

$\underline{M}$ is the (positive) solution of the following equation with respect to M.

$$\begin{aligned}\frac{\tau\sigma^2 x}{\delta\mu} &= \sigma^2 \frac{M+(1-\rho)\sigma^2}{\xi(M,\bar{\omega})} \int^{\bar{\omega}} \frac{\bar{\omega}-\omega}{\omega} dF(\omega) + \lambda \frac{1-B_t(\bar{\omega})}{\gamma_t(\bar{\omega})}. \\ &= \frac{M+(1-\rho)\sigma^2}{\xi(M_t,\bar{\omega})} \left[\sigma^2 \int^{\bar{\omega}} \frac{\bar{\omega}-\omega}{\omega} dF(\omega) + \lambda \frac{\bar{\omega}}{\omega_r}\sigma^2 + \lambda \frac{M+\sigma^2}{M+(1-\rho^2)\sigma^2}\bar{\omega}\right] \\ &\equiv H(M,\bar{\omega}).\end{aligned} \tag{32}$$

Let

$$D(y) \equiv \int^y \frac{y-\omega}{\omega} dF(\omega) + \lambda \frac{y}{\omega_r}.$$

Then, (32) can be rewritten as

$$0 = \sum_{k=0}^{2} a_k M^k$$

with

$$\begin{aligned} a_0 &= \sigma^4(1-\rho)[A(\delta)(1+\rho)(\bar{\omega}+(1-\rho^2)\sigma^2) - \sigma^2 D(\bar{\omega})(1-\rho^2) - \lambda\bar{\omega}], \\ a_1 &= \sigma^2 \left[A(\delta)(2\sigma^2(1-\rho^2) - \rho^2\bar{\omega}) - \sigma^2 D(\bar{\omega})(2-\rho-\rho^2) - \lambda\bar{\omega}(2-\rho)\right], \\ a_2 &= A(\delta)(\bar{\omega}+\sigma^2) - \sigma^2 D(\bar{\omega}) - \lambda\bar{\omega}. \end{aligned}$$

Now, under the region (iii) and (iv), we have $A(\delta) \leq A_3$. Thus,

$$\begin{aligned} a_2 &= A(\delta)(\bar{\omega}+\sigma^2) - \left[\sigma^2 \int^{\bar{\omega}} \frac{\bar{\omega}-\omega}{\omega} dF + \lambda \frac{\bar{\omega}}{\omega_r}\sigma^2 + \lambda\bar{\omega}\right] \\ &\leq \sigma^2 E[\frac{\bar{\omega}-\omega}{\omega}] + \lambda(\omega_r+\sigma^2)\frac{\bar{\omega}}{\omega_r} - \left[\sigma^2 \int^{\bar{\omega}} \frac{\bar{\omega}-\omega}{\omega} dF + \lambda \frac{\bar{\omega}}{\omega_r}\sigma^2 + \lambda\bar{\omega}\right] \\ &= 0 \end{aligned}$$

Also, we have $A(\delta) \geq A_1$, so that

$$\begin{aligned} \frac{a_0}{\sigma^4} &= A(\delta)(1+\rho)(\bar{\omega}+(1-\rho^2)\sigma^2) - \sigma^2 D(\bar{\omega})(1-\rho^2) - \lambda\bar{\omega} \\ &\geq \sigma^2(1-\rho^2)E[\frac{\bar{\omega}-\omega}{\omega}] + \lambda\frac{\bar{\omega}}{\omega_r}(\omega_r+(1-\rho^2)\sigma^2) - \sigma^2 D(\bar{\omega})(1-\rho^2) - \lambda\bar{\omega} \\ &= 0. \end{aligned}$$

In both equations, the equality holds only if $A(\delta) = A_j$. Therefore, there is a unique positive M_t that satisfies $\omega_t^L = \phi_A(M_t) = \bar{\omega}$.

B6. Proof of Lemma 4 and Proposition 8

The market clearing condition with explicitly keeping the information content is written as

$$\begin{aligned} \sigma^2\tau x &= \int^{\omega_t^L} \frac{\mu_L + \beta^s(M_t,\omega)(s_t(\omega)-\mu_L) + \beta^q(M_t,\omega)(q_{t-1}(M_t) - E[q_{t-1}(M_t)]) - P_t}{\gamma(M_t,\omega)} dF(\omega) \\ &+ \lambda \frac{\mu + \beta^s(M_t,\omega_r)(s_t(\omega_r)-\mu) + \beta^q(M_t,\omega_r)(q_{t-1}(M_t) - E[q_{t-1}(M_t)]) - P_t}{\gamma(M_t,\omega_r)}. \end{aligned}$$

Solving this for P_t (with normalizing the coefficient of v_{t+1}) yields the information content of the price as

$$q_t = v_{t+1} + \overbrace{\int^{\omega_t^L} \pi(\omega_t^L, \omega) e_{t+1}(\omega) d\omega + \lambda \pi_r(\omega_t^L) e_{t+1}(\omega_r)}^{\equiv m_{t+1}}.$$

When $e = \sqrt{\omega/2}(\hat{e} + e^{\omega})$, the results in Lemma 4 and 8 are straightforward.

Consider the general covariance structure given by $Var(e(\omega)) = \omega$ and $Cov(e(\omega), e(\omega')) = l(\omega, \omega')$. We assume that l is not diminishing, i.e., a condition such as $\lim_{|\omega-\omega'|\to\infty} l(\omega, \omega') = 0$ does not hold. This makes the LLN for the noise e in m does not hold as well.

The noise component of the price information has the following variance (where I suppress the element ω^L in π):

$$Var\left(\int^{\omega_t^L} \pi(\omega) e_{t+1}(\omega) d\omega + \pi_r e_{t+1}(\omega_r)\right) = Var\left(\int^{\omega_t^L} \pi_t(\omega) e_t(\omega) d\omega\right) + \pi_r^2 \omega_r.$$

The first term specifies the covariance effect as follows.

$$\begin{aligned}
Var\left(\int^{\omega_t^L} \pi_t(\omega) e_t(\omega) d\omega\right) &= Cov\left(\int^{\omega_t^L} \pi_t(\omega) e_t(\omega) d\omega, \int^{\omega_t^L} \pi_t(\omega') e_t(\omega') d\omega'\right) \\
&= \int^{\omega_t^L} Cov\left(\pi_t(\omega) e_t(\omega), \int^{\omega_t^L} \pi_t(\omega') e_t(\omega') d\omega'\right) d\omega \\
&= \int^{\omega_t^L} \left[\int^{\omega_t^L} Cov\left(\pi_t(\omega) e_t(\omega), \pi_t(\omega') e_t(\omega')\right) d\omega'\right] d\omega \\
&= \int^{\omega_t^L} \int^{\omega_t^L} \pi_t(\omega) \pi_t(\omega') l(\omega, \omega') d\omega' d\omega \\
&= \frac{1}{(G(\omega_t^L) + \lambda \omega_r^{-1})^2} \int^{\omega_t^L} \int^{\omega_t^L} l(\omega, \omega') g(\omega) g(\omega') d\omega' d\omega \qquad (33)
\end{aligned}$$

The second and third equalities come from the linearity of the covariance operator. Therefore, by denoting (33) by $C(\omega_t^L)$, we have (25).

B7. Proof of Proposition 10

(i): The right hand side of (22) is monotonically increasing in ω^L. Thus, there is a unique ω^L such that (23) holds. By setting $\omega^L = \underline{\omega}$, we find $\phi_N'(\underline{\omega}) > 0$ for all $\lambda\omega_r^{-1}$, which implies $\omega^N > \underline{\omega}$.
(ii): We also have $\phi_N'(\bar{\omega}) = -(\lambda\omega_r^{-1})^2 + \lambda\omega_r^{-1}\left(\sqrt{\frac{\bar{\omega}}{2}} \int^{\bar{\omega}} g(\omega)\sqrt{\frac{\omega}{2}} d\omega\right) + k$ with a positive constant k. Thus, $\phi_N'(\bar{\omega}) = 0$ has a unique positive solution with respect to $\lambda\omega_r^{-1}$, which defines $\tilde{\lambda}$ in the proposition.

B8. Covariance Structure and Approximation of A Continuum of Random Variables

In (33), define a normalized covariance, $\hat{l}$, by

$$l(\omega, \omega') = \frac{\hat{l}(\omega, \omega')}{g(\omega) g(\omega')},$$

so that

$$C(\omega_t^L) \equiv Var\left(\int^{\omega_t^L} \pi(\omega_t^L, \omega) e_{t+1}(\omega) d\omega\right) = \frac{1}{(G(\omega^L) + \lambda\omega_r^{-1})^2} \int^{\omega_t^L} \int^{\omega_t^L} \hat{l}(\omega, \omega') d\omega' d\omega.$$

In the numerical examples, I approximate the continuum of random variables $\boldsymbol{e} = \{e(\omega) : \omega \in \Omega\}$ as a limit of discrete random variables. Formally, consider a N-partition of Ω such that $\Omega_N \equiv \cap_{j=0}^{N} [\omega_j, \omega_{j+1})$ with $\underline{\omega} = \omega_0 <$

$\cdots < \omega_N = \bar{\omega}$ and some integer N. A set of discrete random variables is given by $\boldsymbol{e}_N \equiv \{e(\omega_j) : j = 0, 1, \cdots, N\}$. Note that $\boldsymbol{e}_N$ defines a set of simple functions over Ω_N.

To see the validity of this approximation, the probability space and index set are formally defined. Let $(\Theta, \mathcal{F}, \mathcal{P})$ be a probability space and define the inner product of random variables by

$$\langle f, g \rangle = \int_{\Theta} f(u) g(u) du.$$

L_2-norm is accordingly defined by $\langle f, f \rangle^{1/2}$. Then $e(\omega)$ can be seen as a function $e : \Omega \to L_2$. The set $\boldsymbol{e}$ is weakly measurable if each member of the family $\{\omega \mapsto \langle g, e(\omega) \rangle : g \in L_2\}$ of real-valued function is measurable. Then, Main Theorem of Al-Najjar (1995) shows that $\boldsymbol{e}$ can be obtained by the limit $N \to \infty$ of the set $\boldsymbol{e}_N$.

Accordingly, we consider a covariance matrix S in the discrete case where each element is $S_{i,j} = l(\omega_i, \omega_j)$. To obtain the multiplicity, I use the formulation such that

$$\hat{l}(\omega_i, \omega_j) = \begin{cases} L_0(\omega_i + \omega_j) + L_1 & \text{if } \omega_i, \omega_j \leq \tilde{\omega} \\ L_2(\omega_i + \omega_j) + L_3 & \text{if } \omega_i, \omega_j \geq \tilde{\omega} \\ L_4(\omega_i + \omega_j) + L_5 & \text{otherwise,} \end{cases} \tag{34}$$

with some $\tilde{\omega} \in (\underline{\omega}, \omega^L)$, $L_0, L_2 \geq 0$, and $L_4 < 0$. Note that this covariance structure is symmetric, and I check that the parameters make S positive semi-definite. Intuitively, the set Ω is separated into two groups by $\tilde{\omega}$. Within these two groups, the noise components of traders are positively correlated ($L_0$and L_1), while the correlation is negative (L_3) across two groups. This highlights the effect (i) and (ii) in Subsection 5.. Note that this is just one of the examples that provide the multiplicity, and other formulations are also possible as long as they satisfy the condition (i) and (ii), l is symmetric, and S is p.s.d.

B9. Proof of Proposition 11

Let

$$\hat{H}(\omega^L, \varepsilon) = \sigma^2 \frac{M_t + (1-\rho^2)\sigma^2}{\xi(M_t, \omega^L)} \int_{\Omega(\omega^L, \varepsilon)} \frac{\omega^L - \omega}{\omega} dF + \lambda \frac{1 - B(M_t, \omega^L)}{\gamma(M_t, \omega_r)}.$$

Then (see also Appendix B1.),

$$\frac{\partial \hat{H}(\omega^L, \varepsilon)}{\partial \omega^L} = \sigma^2 \frac{M_t + (1-\rho^2)\sigma^2}{\xi^2(M_t, \omega^L)} \int_{\Omega(\omega^L, \varepsilon)} \frac{\xi(M_t, \omega)}{\omega} dF + \lambda \frac{\partial}{\partial \omega^L} \left(\frac{1 - B(M_t, \omega^L)}{\gamma(M_t, \omega_r)} \right) > 0$$

$$\frac{\partial \hat{H}(\omega^L, \varepsilon)}{\partial \varepsilon} = -\sigma^2 \frac{M_t + (1-\rho^2)\sigma^2}{\xi(M_t, \omega^L)} \left[\frac{\omega^L - (\tilde{\omega} - \varepsilon)}{\tilde{\omega} - \varepsilon} f(\tilde{\omega} - \varepsilon) + \frac{\omega^L - (\tilde{\omega} + \varepsilon)}{\tilde{\omega} + \varepsilon} f(\tilde{\omega} + \varepsilon) \right] < 0.$$

By the implicit function theorem, we have $d\omega^L / d\varepsilon > 0$.

C Current Price Reference

Traders still have $s_t = v_{t+1} + e_{t+1}(\omega)$ where $e_{t+1}(\omega)$ becomes public at the *end* of date $t+1$. Then, guess that traders use the current price signal q_t and the past price signal q_{t-1} as well. This is relevant because the past price signal is not nested by the current price signal as shown below. Suppose that $\lambda = 0$ for simplicity.

In this case, with a relevant set of variables h_t,

$$R_t^L(h_t, \omega) = \mu_L + \beta^s(s_t(\omega) - \mu_L) + \beta^c(q_t - \mu_L) + \beta^p(q_{t-1} - \mu_L)$$

with filtering coefficients β^k. Note that we need to have noise traders to avoid the perfectly revealing price and the "no-trading theorem" of Milgrom and Stokey (1982). Let the noise traders make the supply x stochastic such that

$X_t \sim N(x, \sigma_x^2)$. Then, if the cutoff equilibrium is still alive, the market clearing condition is

$$\tau\sigma^2 X_t = \int^{\omega^L} \frac{R^L(h_t,\omega) - P_t}{\gamma(h_t,\omega)} dF.$$

Our guess for the form of the price signal is again linear in the fundamental part of return,

$$q_t = v_{t+1} + m_{t+1}. \tag{35}$$

Given this guess, the price would be such that

$$P_t \propto \int^{\omega^L} \frac{(1 - \sum_{k\in\{s,c,p\}} \beta^k(h_t,\omega))\mu_L + \beta^s(h_t,\omega)s_t + \beta^c(h_t,\omega)q_t + \beta^p(h_t,\omega)q_{t-1}}{\gamma(h_t,\omega)} dF - \tau\sigma^2 X_t.$$

Since, $q_{t-1} = v_t + m_t = \rho^{-1}v_{t+1} - \rho^{-1}u_{t+1} + m_t +$ constant and m_t has not been public at the timing of the investment by trader at t,[38] the current price signal is denoted by

$$q_t = \int^{\omega^L} \frac{\beta^s(h_t,\omega)(v_{t+1} + e_{t+1}(\omega)) + \rho^{-1}\beta^p(h_t,\omega)(v_{t+1} - u_{t+1} + \rho m_t)}{\gamma(h_t,\omega)} dF - \tau\sigma^2 X_t.$$

By normalizing it, we get

$$\begin{aligned} &q_t = v_{t+1} + m_{t+1}, \\ &m_{t+1} = \left[\int^{\omega^L} \frac{\beta^s(h_t,\omega) + \rho^{-1}\beta^p(h_t,\omega)}{\gamma(h_t,\omega)} dF\right]^{-1} \left(\int^{\omega^L} \frac{\beta^s(h_t,\omega)e_{t+1}(\omega) - \rho^{-1}\beta^p(h_t,\omega)(u_{t+1} - \rho m_t))}{\gamma(h_t,\omega)} dF - \tau\sigma^2 X_t\right). \end{aligned} \tag{36}$$

Thus, given the initial condition for the price signal m_0, we have shown that the guess (35) is true and has a recursive form as (36). Note that m_{t+1} involves m_t. This works as a noise when the signal q_t is used by traders at date t but it does not matter any more when the traders in the next trading round at date $t+1$ use q_t as a past price reference. Therefore, the price information is formally described as

$$q_t = v_{t+1} + \begin{cases} m^c_{t+1} & \text{for the current use} \\ m^p_{t+1} & \text{for the usage at } t+1, \end{cases}$$

and each evolves as

$$\begin{aligned} m^c_{t+1} &= \left[\int^{\omega^L} \frac{\beta^s(h_t,\omega) + \rho^{-1}\beta^p(h_t,\omega)}{\gamma(h_t,\omega)} dF\right]^{-1} \left(\int^{\omega^L} \frac{\beta^s(h_t,\omega)e_{t+1}(\omega) - \rho^{-1}\beta^p(h_t,\omega)(u_{t+1} - \rho m^p_t))}{\gamma(h_t,\omega)} dF - \tau\sigma^2 X_t\right), \\ m^p_{t+1} &= \left[\int^{\omega^L} \frac{\beta^s(h_t,\omega) + \rho^{-1}\beta^p(h_t,\omega)}{\gamma(h_t,\omega)} dF\right]^{-1} \left(\int^{\omega^L} \frac{\beta^s(h_t,\omega)e_{t+1}(\omega) - \rho^{-1}\beta^p(h_t,\omega)u_{t+1}}{\gamma(h_t,\omega)} dF\right) \end{aligned}$$

since $x = X_t$ is also public in the beginning of date $t+1$.

Thus, we can define the precision component h_t. By the filtering problem, what matters to construct conditional expectation R is the variance of each signal. Then, h_t is the vector of $Var(y_t)$ with $y_t = (s_t(\omega), q_t, q_{t-1})$. Since m^j_{t+1} is determined by the cutoff at date t,

$$M^j_{t+1} = Var(m^j_{t+1}) = \phi^j(\omega^L_t),\ j \in \{c, p\},$$

$$\therefore h_t = h(\omega, M^c_{t+1}, M^p_t) = h(\omega, \phi^c(\omega^L_t), \phi^p(\omega^L_{t-1})).$$

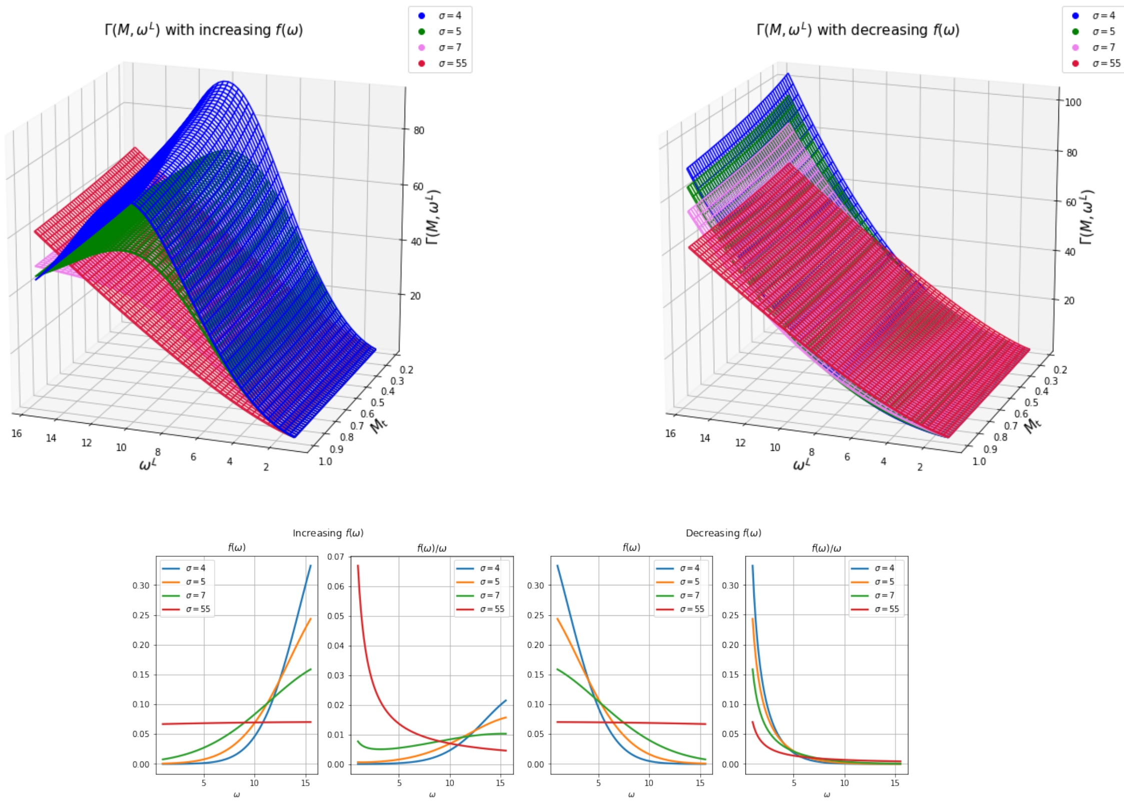
Hence, we can see that, if the cutoff equilibrium exists, that has persistent dynamics as the current behavior depends on the past price signal which is determined by the past participants in the market. However, the current price reference makes the participation decision of individual trader affects the other traders decision not only

[38] This is shown to be true since m_t is an aggregation of e_t.

through the price level but also through the price information (see M^c_{t+1} depends on ω^L_t). As a result, this is extremely complicated to solve for the cutoff ω^L and the existence of cutoff equilibria cannot be guaranteed with a general f.

D Numerical Analysis for Adverse Selection

Figure 19: Adverse Selection: $\Gamma(M_t, \omega^L)$



The sign of the indirect effect of M_t in (20) depends on the distribution f. When the shock hits, the cutoff increases as (19). This implies that the intensive margin of each participant declines due to the shock because the increment in R^L is smaller than that of P_t. Thus, the increment of the cutoff covers the decline in the intensive margin in each trader by increasing the extensive margin.

However, when f increases steeply as ω^L increases, the required expansion of the extensive margin is small and easily achieved by a smaller increase in the cutoff. Therefore, when f is steeply increasing around the cutoff, the effect of θ-shock on the cutoff will be smaller, diminishing the adverse selection effect. In contrast, the adverse selection is more likely when the heterogeneous signals are moderately distributed, i.e., f is moderately increasing or f is decreasing. For the effect through the cutoff, Fig. 19 shows examples of Γ with various distributions, and we can confirm that the arguments above are correct. In the figure, f is approximated by Gaussian distribution with different values for variance σ. The case with $\sigma = 55.0$ is the approximation of the benchmark, i.e., the uniform F.

E Idiosyncratic Shock as a Source of SOMC

Suppose that one of $\omega \leq \omega^L_l = \phi_A(M_h)$ traders accidentally observes a private signal whose covariance structure is slightly different from l by ε:

$$Cov(e(\omega), e(\omega')) = l(\omega, \omega') + \varepsilon(\omega, \omega')$$

for $\omega \leq \omega_l^L$, $\omega^{'} \neq \omega$ and $\omega^{'} \in [\omega_0, \omega_1]$. Let the idiosyncratic shock hits the trader with $\omega = \omega_i$. Then, the noise-aggregation function yields,

$$M_{t+1} = \frac{\int^{\omega^L} \int^{\omega^L} l(\omega, \omega^{'}) g(\omega) g(\omega^{'}) d\omega d\omega^{'} + \lambda^2 \omega_r^{-1} + 2 \int_{\omega_0}^{\omega_1} \varepsilon(\omega_i, \omega) g(\omega_i) g(\omega) d\omega_i}{(G(\omega_t^L) + \lambda \omega_r^{-1})^2}.$$

Thus, by rewriting $\varepsilon \equiv \int_{\omega_0}^{\omega_1} \varepsilon(\omega_i, \omega) g(\omega_i) g(\omega) d\omega_i / (G(\omega_t^L) + \lambda \omega_r^{-1})^2$ gives the ε-shock on the state variable.

F Parameter Values for Numerical Examples

Table 1: Example Parameter Values for Benchmark

parameter	parameter values			
	$\phi_N^{'} > 0$	$\phi_N^{'} < 0$	Multi	
μ	2.00	2.00	2.00	mean return
σ^2	2.70	2.70	2.70	return variance
τ	1.00	1.00	1.00	risk aversion
x	0.30	0.30	0.53	supply
ρ	0.90	0.90	0.90	persistence of v
λ	0.40	0.09	0.62	measure of sophisticated
$\bar{\omega}$	6.00	6.00	17.0	upper bound ω
$\underline{\omega}$	3.20	3.20	3.20	lower bound ω
ω_r	2.09	2.09	2.09	type of sophisticated

As for the case with multiple steady states, denoted by "Multi" in the table1, the general covariance structure is derived by the formula in (34) with $\tilde{\omega} = \bar{\omega}/2.5$, $L_0 = L_1 = 0.001$, and $L_2 = -0.008$. This parameter setting separate the entire group of irrational traders by $\tilde{\omega}$ into two groups. Within each group, the noise components are positively correlated ($L_0, L_1 > 0$), while the cross-group correlation is negative ($L_2 < 0$).