Securitization and Heterogeneous-Belief Bubbles with Collateral Constraints

Jun Maekawa

Abstract

Earlier studies have showed that the asset price is higher in a heterogeneous model than in a common prior model . These studies have assumed that there is neither budget constraint nor limitation of financial market. Recent studies have also explored the role of financial technology in heterogeneous belief model. In this paper, I show that some financial technology including securitization, particularly loan backed security, increases the asset price as reported by previous studies.

1 Background

There is a close relation between financial innovations like leverage, securitization or MBS, and bubbles. (Brunnermeier and Oehmke(2013) [2]). In fact, the size of financial innovation grew in recent U.S. crisis. The market of the securitization grew during the housing boom, which in their peak amounted to trillions of dollars a year. After the peak of the price in early 2006, the price declined.

Fostel and Geanakoplos (2012) [5] provide models in which different financial innovations affect the economic situation. Financial innovation like tranching, one of a securitization technology, raises asset prices by increasing the ability of the optimist. By tranching, optimists can take riskier positions in the asset. At the same time, an unexpected introduction of credit default swaps (CDS) can lead to drastic reductions in asset prices, since it allows pessimists to more effectively bet against the bubble asset.

In the present study, I have showed that financial innovations improve the budget constraints of traders. The proposed model explain a theoretical framework between bubbles and financial innovations. By the innovation, it is possible for the traders to have enough cash for buying the asset. I have also showed that innovations increase the market instability, such as bubbles. I also explained the relation between asset prices and financial technologies. Harrison and Kreps (1978) [9] described heterogeneous belief bubbles . They reported that in their model, the traders ' beliefs change over time. The price of the asset can be higher than the valuation of the most optimistic agent.

Geanakoplos (2010) [6] develops a model in which agents with heterogeneous beliefs have limited wealth, such that agents with optimistic views about an asset borrow funds from more pessimistic agents, with loan contracts with collateral. Because pessimistic agents have lower evaluation about the asset return, their finances to optimistic agents are not enough for the optimistic belief. In this model, optimists tend to make risky investment. When low asset return states are realized, optimists lose their wealth and more of the asset has to be held by pessimists, The asset price must be affected not only by the optimistic belief but also by the pessimistic belief and it is lower than that of Harrison and Kreps.

Simsek (2013) [15] shows a similar result in Geanakoplos in their heterogeneous model. He focused on the belief disagreement between optimists and pessimists. The extent to which pessimists are willing to finance asset purchases by optimists depends on a specific form of the belief disagreement. Intuitively speaking, when disagreement is mostly about the upside, pessimists are more willing to provide credit than when disagreement is about the downside. Hence, it is not just the amount of disagreement, but also the nature of disagreement among agents that matters for asset prices.

In this paper, I introduced the collateral loan model like Simsek . The asset price gets higher again and heterogeneous belief bubbles re-arise.

There are two original factors, loan securitizations and new optimists.

Lenders can sell loan payments to other traders by securitizations. Loan lenders can shift default risks of loan contracts to other traders. As a result, speculative lending occur in equilibrium. In my proposed model, new optimists come daily to buy the security issued by the loan lenders. The asset price is raised by their optimistic trades.

Though new optimists have little cash, their cash is indirectly used for buying the asset by the securitization. The loan contract can be securitized and new optimists buy the security. For the securitization, the lender of the loan has speculative incentive for lending cash. The borrower can have large cash

by the loan and securitization.
These two factors improve the budget constraints of traders. Each optimistic traders have little cash. They need to make loan contracts. In this economy, the loan borrower need to have a collateral. Under this constraint, traders' asset demand is put down. These constraints are drastically improved by the loan securitization.
If the asset return is high, the optimists can return cash to the pessimists. If the asset return is low, the asset will be held by pessimists because the asset is used as the collateral. The pessimistic beliefs also affect the price through collateral. Then, the asset price is lower than optimistic expectation.
Importantly, each optimist cannot cooperate with each other. If the securitization is not allowed, Only small cash is used for buying the asset. This price is equal to that of Simsek's study. Optimists must rely on pessimists cash and the asset price is lower.
Recent U.S. housing bubble is closely connected with financial innovation like securitization(Brunnermeier(2009) [1]). The securitized loan is traded in repo market in recent crisis(Gorton and Metrick(2011) [7], Krishnamurthy, Negel and Orlov(2011) [11]).
In this paper, by introducing a new financial technology, i.e., loan backed security, to heterogeneous belief model with collateral, the asset price is as high as that of Miller(1977) [12] or Harrison and Kreps(1978).
This security make the loan contract itself risk-less. Since the lender of the loan contract can sell the security to some more optimistic one, the lender need not to hesitate to lend cash to optimistic traders. For pessimists, the security also raises their payoffs, and they can finance a new investment by the securitization.
Optimistic traders behave like "noise trader"(DeLong, Shleifer, Summers and Waldmann(1990) [4]). Although there are many pessimistic traders, the influence of optimistic traders is very large. Pessimists can get high return by exploiting optimists.
My research is a part of theory that concerns borrowing constraints on asset prices as mentioned in Shleifer and Vishny(1992, 1997) [13] [14], Kiyotaki and Moore(1997) [10],Gromb and Vayanos (2002) [8], Brunnermeier and Pedersen (2009) [3]. Financial technologies can eliminate the borrowing constraint by speculative incentives of traders. In heterogeneous belief models, borrowing constraints are important role for preventing bubble economies.
First, I will explain the model settings in section II. In section III, two types of equilibrium are explained. One type is the small generation case in which

the asset price is lower than the optimistic expectation . The other type of equilibrium is the large generation case. The asset price is raised up to optimistic expectation. In section IV, the model has been analyzed. We can connect Simsek (2013) and Miller (1977) in the model. Section V is an extension. Until section IV, there is only one type of optimists in the model. I have introduced many types of optimist in section V. If there are various types of optimists, the asset price can get higher than any traders ' expectation as in Harrison and Kreps (1978).

2 Settings

In this paper, we consider a symmetric equilibrium. Same actions are chosen by same type traders on each date. The model is dynamic finite date model($t = 0, 1, ...T$). On date 0, one unit of risky asset is supplied by a monopolistic firm. There is a continuum of states on date T, denoted by $s \in S = [0, s^{max}]$. The asset pays s dollars at state s. There are two types of traders, optimists and pessimists. Both traders have risk neutral utility functions. However, they have different beliefs about the asset return. Type j traders have a belief about the asset return , distribution function $F_j(s)(j = o, p)$. Optimists and pessimists agree to disagree about their beliefs.

Optimists are optimistic about the asset return. They believe that a high state will be realized more frequently. It is assumed the optimistic belief is stochastically dominate the pessimistic belief according to the hazard rate.

Assumption 1 $\frac{f_o}{1-F_o} < \frac{f_p}{1-F_p}$ *for each* $s \in [0, s^{max})$

This assumption implies that First Order Stochastic Dominance(FOSD).

$$F_o < F_p \; for \; each \; s \in [0, s^{max}) \tag{1}$$

On date 0, the continuum of optimists (total population is one) and countably many pessimists come to the market. The number of pessimists is larger than T. At each date t, new optimists (total population is one) come to the market. All traders live up to the final date T.

Each optimist has n units of cash. Pessimists have no budget constraint.

The assumption implies optimists are natural buyers of the asset. Since n is small, optimists must borrow cash from pessimists.

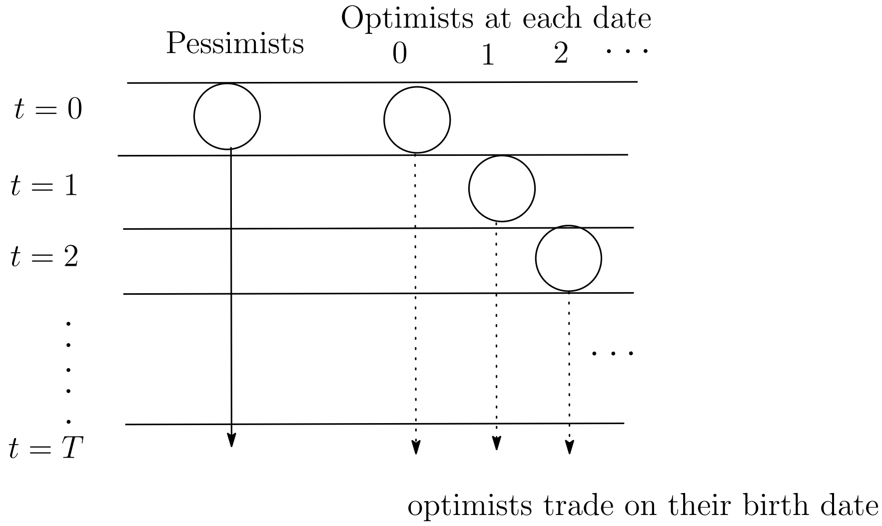

Figure 1: Generation

Optimists who come to the market on date t have a chance to buy the asset or the security only on date t. Optimists buy the security or the asset, and they wait until s is realized (date T). This optimists settings is similar to over-lapping generation models, in which traders can participate in markets only in two date. In this model, the date t optimists and the date $t+1$ optimists is linked through pessimists. This assumption implies that pessimists are professional traders (like managers in hedge funds) and optimists are not. Optimists have limited cash, an optimistic belief and little financial network advantages.

Another interpretation of this settings is that professional traders can trader very faster than optimists. Although the model is dynamic, the date T may mean one day or one hour later from the date 0. Only pessimists can use a computer and can trade each "date" t. The equilibrium of this model is very simple one and these securitization can be easily generated by hedge funds.

Pessimists have a plenty of cash and have no budget constraints. They are natural lenders of cash. Optimists don't have enough cash for buying the asset. So, they must borrow some cash from pessimists. All borrowing contract in this economy is subject to a collateral constraint. Promises made by borrowers must be collateralized by assets or securities. These constraints are the same as that of Simsek(2013) which is originally in Geanakoplos(2003, 2010). In this paper, consider a simple debt contract. The borrowers promise does not depend on the state s on date T. These contract are very simple,

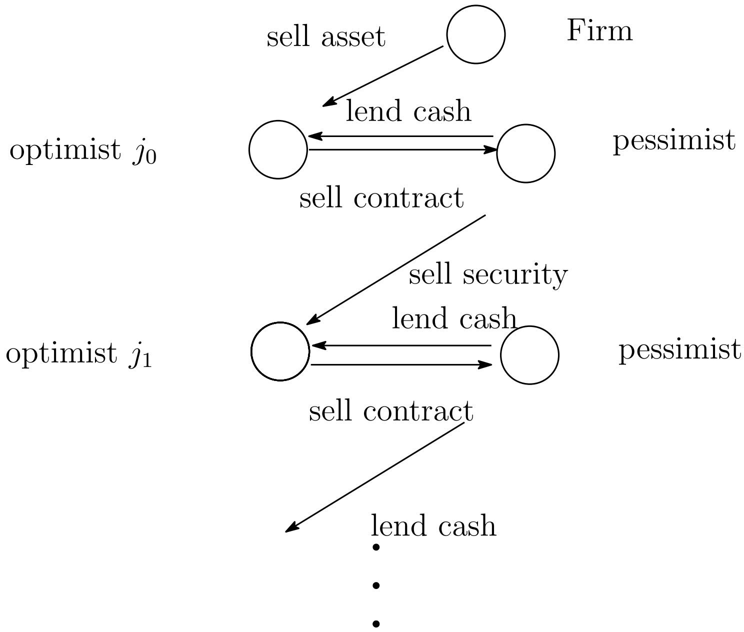

Figure 2: The sequence of the trade

but the equilibrium of the model can explain one struture of common collateralized loan model. Moreover, we can introduce the securitzation to the model very easily.

A collateraled loan contract on date 0 is defined as (γ_0, ϕ_0) for one unit of the asset collateral. ϕ_0 is the amount of cash borrowing and γ_0 is the borrowers promise payment after the revealment of s(date T). If the borrowers cannot pay γ_0 on date T, they must give one unit of the asset as a collateral to the lenders. The cash borrowers make a take-it-or-leave-it offer (γ_0, ϕ_0) to lenders.

Since only one type of the asset exists on date 0, the loan contract must be collateralized by the asset.

- If the asset return s is high $(s > \gamma_0)$, the optimist can pay the promise γ_0 to the pessimist.
- If the asset return s is low $(s \leq \gamma_0)$, the optimist gives the asset return s to the pessimist.

In other words, borrowers give $min(s, \gamma_0)$ to lenders in a loan contract (γ_0, ϕ_0).
Then, the loan payoff is $min(s, \gamma_0)$. Pessimists evaluate the payoff as $E_p[min(s, \gamma)]$ and optimists evaluate it as $E_o[min(s, \gamma_0)]$. The evaluation of optimists is higher than the pessimists' evaluation for the assumption of FOSD.

$$E_o[min(s, \gamma_0)] > E_p[min(s, \gamma_0)]$$

So, there is an incentive for trading the loan contract.
The loan lender can securitize the loan contract and sell it to the other traders at next date 1. He can issue loan backed securities whose payoff is $min(s, \gamma_0)$ with price q_1. For the sake of simplicity, suppose only one pessimist lends cash to all optimist at each date. (Pessimists have no budget constraints.) This pessimist can issue the security with monopolistic power On next date. On date 1, traders can make loan contract with collateralized the security. A loan contract collateralized by the security is defined as (γ, ϕ) for one unit of the security. Consider a security which has a return $min(s, \gamma')$ on date T is used as collateral. ϕ is the amount of cash borrowing and γ is the borrowers' promise payment after the revealment of $min(s, \gamma')$, i.e., $(\gamma' > \gamma)$. If the borrowers cannot pay γ, they must give the securities as collateral to the lenders. The cash borrowers make a take-it-or-leave-it offer (γ, ϕ) to lenders. The security payoff is $min(s, \gamma')$. If s is low on date T, the loan default may occur. In this case, the borrowers must give the security to the lenders.

- If the security return is high $(min(s, \gamma') > \gamma)$, the optimist can pay the promise γ to the pessimist.
- If the security return is low $(min(s, \gamma') \leq \gamma)$, the optimist gives the security return $min(s, \gamma)$ to the pessimist.

That is, borrowers give $min[min(s, \gamma'), \gamma] = min(s, \gamma)$ to lenders in a loan contract (γ, ϕ).
On each date $t \geq 1$, traders who lent cash at previous date $t-1$ with loan contract (γ, ϕ) can issue loan backed securities which securitize loan contracts and the security payoff is $min(s, \gamma)$. ϕ can be interpreted as a cost of issuing the security for lenders.

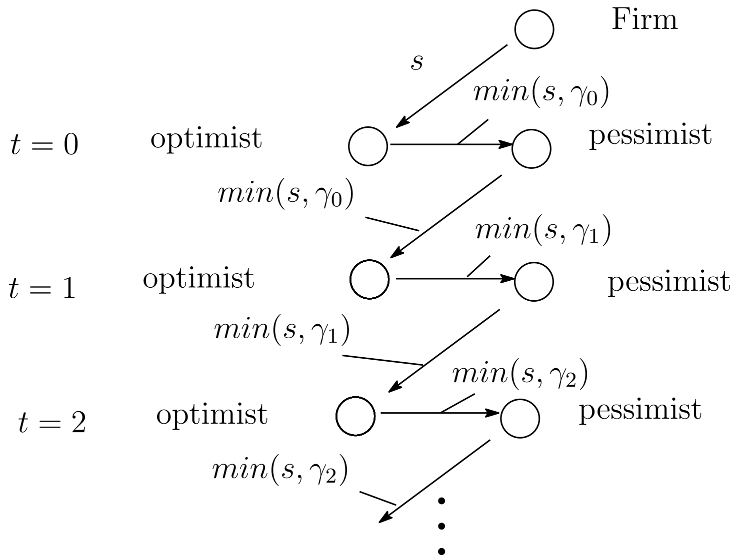

Figure 3: The stream of loans and securities

3 Equilibrium

The general equilibrium is difficult to solve. In this paper, let us consider a simple equilibrium:

- On date 0, the asset is bought by all optimists
- On each date $t > 0$, each security is bought by all optimists who come to the market on date t

On date $t(t < T - 1)$, the date $t - 1$ loan lender can sell the security. In this model, there is no profit to sell the security at later date. (On each date, optimists' total cash is equal to n and they can trade only at their coming date.) Then, the pessimist, who lent cash on date t, sells the security on date $t + 1$. (This limitation is loosed in section 5.)

- On each date optimists make loan contract with one pessimist
- Pessimists who make loan contract on date t securitize the loan on date $t + 1$

If no loan contract occur on date t, there is no security markets on date $t + 1$.

3.1 Pessimists' Loan Problem

Optimists make a take-it-or-leave-it offer (γ_t, ϕ_t) to the pessimist on date $t \leq T-2$. If the pessimist accept the offer, he lends ϕ_t cash and sells the security with price q_{t+1} on date $t+1$. Then, pessimist accept the offer if $\phi_t \leq q_{t+1}$. Since the security price q_{t+1} depends on the payment γ_t(the security return is $min(s, \gamma_t)$), let $q_{t+1}(\gamma_t)$ be the price of the security collateralized by the loan payment γ_t.
For pessimists, the loan contract is only way to earn cash. Since there are many pessimists, their lending competition implies no-arbitrage condition:

$$\phi_t = q_{t+1}(\gamma_t) \tag{2}$$

on date $T-1$, the pessimist cannot sell the security at next date T. The loan payoff is $min(s, \gamma_{T-1})$. Then, pessimist competition imply:

$$\phi_{T-1} = E_p[min(s, \gamma_{T-1})] \tag{3}$$

On date $t+1$, the date t lender can issue the security with monopolistic power. Optimists have their own cash n and can borrow cash ϕ_{t+1} per unit of the security from pessimists with loan contract. Then, let a_{t+1} be the demand of the security, the optimists' budget constraint is:

$$a_{t+1}q_{t+1} \leq n + a_{t+1}\phi_{t+1} \tag{4}$$

However, the security seller (and the date t lender) can choose the secuiry price q_{t+1}. The seller's profit is $a_{t+1}q_{t+1}$. So, he can raise the price until optimists' budget constratin is equally satisfied. That is:

$$a_{t+1}q_{t+1} = n + a_{t+1}\phi_{t+1} \tag{5}$$

on date 0, monopolistic firm choose the asset price by the same manner. The date 0 optimists' budget constraint is:

$$a_0 p = n + a_0\phi_0 \tag{6}$$

a_0 is optimists' asset demand.

3.2 Optimists' Maximization Problem

If the $T-1$ security exists, the date $T-1$ optimists send offer the loan contract $(\gamma_{T-1}, \phi_{T-1})$ to pessimists and the offers that satisfy pessimists' arbitrage condition.
As noted in the pessimists' problem, the date $T-1$ optimists' budget constraint is:

$$a_{T-1}q_{T-1} = n + a_{T-1}\phi_{T-1} \tag{7}$$

For the date $T-1$ optimists, the security return $min(s, \gamma_{T-2})$ is given from the previous date $(T-2)$.
Optimists know the lenders' arbitrage condition, $\phi_{T-1} = E_p[min(s, \gamma_{T-1})]$.
Then, the date $T-1$ optimists problem is:

$$\begin{aligned} \max_{a_{T-1},\gamma_{T-1}} \; & a_{T-1}(E_o[min(s,\gamma_{T-2})] - q_{T-1}) \\ & + a_{T-1}(\phi_{T-1} - E_o[min(s,\gamma_{T-1})]) \\ & s.t. \; a_{T-1}q_{T-1} = n + a_{T-1}\phi_{T-1}, \\ & \phi_{T-1} = E_p[min(s,\gamma_{T-1})] \end{aligned} \tag{8}$$

a_{T-1} is the security demand.
The first line of the utility is the profit of the security. The security has return $min(s, \gamma_{T-2})$ and it is sold with price q_{T-1}. The second line is the profit of the loan contract. Optimists borrow cash ϕ_{T-1} and they promise repayment $min(s, \gamma_{T-1})$ for each security a_{T-1}.
The problem can be rewritten as follows:

$$\begin{aligned} \max_{a_{T-1},\gamma_{T-1}} \; & a_{T-1}[E_o[min(s,\gamma_{T-2})] - E_o[min(s,\gamma_{T-1})]] \\ & s.t. a_{T-1}q_{T-1} = n + a_{T-1}E_p[min(s,\gamma_{T-1})] \end{aligned} \tag{9}$$

If no security market on date $T-1$, optimists' strategy is simply $a_{T-1} = \gamma_{T-1} = \phi_{T-1} = 0$
By the same way, the date t optimists problem can be define. Let q_t be the security price (if it exists) given by the seller. Given the security price q_t and the security return $min(s, \gamma_{t-1})$, optimists maximize their utilities. The date t optimists' optimization problem is:

$$
\begin{aligned}
\max_{a_t,\gamma_t}\ & a_t(E_o[min(s,\gamma_{t-1})]-q_t) \\
& + a_t(\phi_t - E_o[min(s,\gamma_t)]) \\
& s.t.\ a_t q_t = n + a_t\phi_t, \\
& \phi_t = q_{t+1}(\gamma_t)
\end{aligned}
\tag{10}
$$

a_t is the security demand. $q_{t+1}(\gamma_t)$ is the price function of the date $t+1$ security. The date $t+1$ security depends on the date t loan contract (γ_t). This function will be calculated by backward induction.
If no security exists, $a_t = \gamma_t = \phi_t = 0$
on date 0, the asset is supplied by a monopolistic firm with price p. Similarly, the date 0 optimists' problem:

$$
\begin{aligned}
\max_{a_0,\gamma_0}\ & a_0(E_o[s]-p) \\
& + a_0(\phi_0 - E_o[min(s,\gamma_0)]) \\
& s.t.\ a_0 p = n + a_0\phi_0, \\
& \phi_0 = q_1(\gamma_0)
\end{aligned}
\tag{11}
$$

a_0 is the asset demand. $q_1(\gamma_0)$ is the price function of the date 1 security. The date 1 security depends on the date 0 loan contract (γ_0).

3.3 Equilibrium Definition

Definition 1 *An equilibrium consists of the asset price p, the each date security price $\{q_t\}_{t=1,2,..,T-1}$, loan contract $\{(\gamma_t,\phi_t)\}_{t=0,1,...,T-1}$, the asset demand a_0 and the security demand $\{a_t\}_{t=1,2,..,T-1}$. They satisfy the following conditions:*

- *Given γ_{t-1} and q_t, a_t and γ_t solves the date t optimists' problem*
- *At each date t, the loan contract (γ_t,ϕ_t) satisfy the pessimists' arbitrage condition*
- *Market clearing condition*

$$a_0 = 1 \;\; \textit{(asset market clearing on date 0)}$$
$$a_t = \begin{cases} 1 & \textit{if the date } t \textit{ security exists at } t \geq 1 \\ 0 & \textit{otherwise} \end{cases}$$

3.4 Small T Case

The equilibrium is solved by backward induction from $T-1$. The date $T-1$ optimists are price takers. The security price q_{T-1} is given by the seller. For the date $T-1$ optimists, the security return $min(s, \gamma_{T-2})$ is given by the previous date $(T-2)$. Optimists know the arbitrage condition of pessimists $\phi_{T-1} = E_p[min(s, \gamma_{T-1})]$.
Then, the date $T-1$ optimists problem is:

$$\begin{aligned} \max_{a_{T-1}, \gamma_{T-1}} \; & a_{T-1}[E_o[min(s, \gamma_{T-2})] - E_o[min(s, \gamma_{T-1})]] \\ & s.t. a_{T-1} q_{T-1} = n + a_{T-1} E_p[min(s, \gamma_{T-1})] \end{aligned} \tag{12}$$

By solving the problem, q_{T-1} and γ_{T-1} are determined.

Lemma 1 *Given γ_{T-2}, the equilibrium security price (q_{T-1} and the loan contract γ_{T-1}) is determinded by the following two equations:*

$$q_{T-1} = \int_0^{\gamma_{T-1}} s dF_p + \frac{1 - F_p(\gamma_{T-1})}{1 - F_o(\gamma_T)} \int_{\gamma_{T-1}}^{s^{max}} min(s, \gamma_{T-1}) dF_o$$

$$q_{T-1} = n + E_p[min(s, \gamma_{T-1})]$$

Proof 1 *In the equilibrium, all security is bought by optimists. Then, $a_{T-1} > 0$. From the budget constraint:*

$$a_{T-1} = \frac{n}{q_{T-1} - E_p[min(s, \gamma_{T-1})]}$$

Substitute this by the objective function:

$$\frac{n/\alpha}{q_{T-1} - E_p[min(s, \gamma_{T-1})]} [E_o[min(s, \gamma_{T-2})] - E_o[min(s, \gamma_{T-1})]] \tag{13}$$

FOC imply:

$$\frac{1-F_o(\gamma_{T-1})}{q_{T-1}-E_p[min(s,\gamma_{T-1})]}+(1-F_p(\gamma_{T-1}))\frac{E_o[min(s,\gamma_{T-2})]-E_o[min(s,\gamma_{T-1})]}{(q_{T-1}-E_p[min(s,\gamma_{T-1})])^2}=0$$

Let q^1_{T-1} be the security price that satisfies FOC:

$$q^1_{T-1}=\frac{1-F_p(\gamma_{T-1})}{1-F_o(\gamma_{T-1})}\{E_o[min(s,\gamma_{T-2})]-E_o[min(s,\gamma_{T-1})]\}+E_p[min(s,\gamma_{T-1})]$$

By differentiating in γ_{T-1}:

$$\begin{aligned}\frac{dq^1_{T-1}}{d\gamma_{T-1}}&=(\frac{1-F_p(\gamma_{T-1})}{1-F_o(\gamma_{T-1})})'\{E_o[min(s,\gamma_{T-2})]-E_o[min(s,\gamma_{T-1})]\}\\&=(\frac{f_p(\gamma_{T-1})}{1-F_o(\gamma_{T-1})}+\frac{f_o(\gamma_{T-1})(1-F_p(\gamma_{T-1}))}{(1-F_o(\gamma_{T-1}))^2})\{E_o[min(s,\gamma_{T-2})]-E_o[min(s,\gamma_{T-1})]\}\end{aligned}$$

From the assumption 1 ($\frac{f_o}{1-F_o}<\frac{f_p}{1-F_p}$), this q^1_{T-1} is decreasing in γ_{T-1}. Because $0\leq\gamma_{T-1}\leq\gamma_{T-2}$, the range of q^1_{T-1} is determined:

$$E_p[min(s,\gamma_{T-2})]\leq q^1_{T-1}\leq E_o[min(s,\gamma_{T-2})]$$

In equilibrium, all securities are bought by optimists. The market clearing implies $a_{T-1}=1$. From the budget constraint:

$$q_{T-1}=n+E_p[min(s,\gamma_{T-1})]$$

Let q^2_{T-1} be the price q_{T-1} satisfies this condition:

$$q^2_{T-1}=n+E_p[min(s,\gamma_{T-1})]$$

Because $0\leq\gamma_{T-1}\leq\gamma_{T-2}$, the range of q^2_{T-1} is determined:

$$n\leq q^2_{T-1}\leq n+E_p[min(s,\gamma_{T-2})]$$

q^2_{T-1} is increasing in γ_{T-1}.

Because both q^1_{T-1} and q^2_{T-1} are continuous, there is only one γ_{T-1} which satisfies two equations. (see Appendix)
Given γ_{T-2}, γ_{T-1} is uniquely determined .
Then q_{T-1} is also uniquely determined.
//

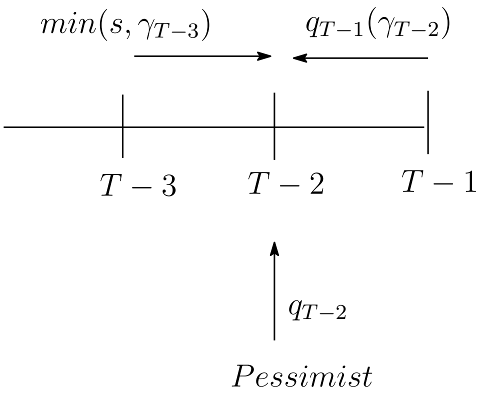

Figure 4: The date $T-2$ optimists' problem

Given γ_{T-2}, the date $T-1$ optimists determine γ_{T-1}. Let a price function $q_{T-1}(\gamma_{T-2})$ denote the equilibrium price:

$$q_{T-1}(\gamma_{T-2}) = \frac{1-F_p(\gamma_{T-1})}{1-F_o(\gamma_{T-1})}\{E_o[min(s,\gamma_{T-2})]-E_o[min(s,\gamma_{T-1})]\}+E_p[min(s,\gamma_{T-1})]$$

$$q_{T-1}(\gamma_{T-2}) = n + E_p[min(s,\gamma_{T-1})]$$

For the date $T-2$ optimists, security return $min(s,\gamma_{T-3})$ and the price function $q_{T-1}(\gamma_{T-2})$ are given. From pessimists' lending arbitrage condition, $\phi_{T-2} = q_{T-1}(\gamma_{T-2})$.
Then, the date $T-2$ optimist problem can be define:

$$\max_{a_{T-2},\gamma_{T-2}} a_{T-2}[E_o[min(s,\gamma_{T-3})] - E_o[min(s,\gamma_{T-2})]]$$
$$s.t. a_{T-2}q_{T-2} = n + a_{T-2}q_{T-1}(\gamma_{T-2})$$

By solving the optimists problem and the market clearing condition, γ_{T-2} is determined.
Similarly, the equilibrium on each date is determined by the backward induction.

Proposition 1 *γ_t and the security price q_t are determined by two equations in the equilibrium:*

- $$q_t = \frac{q'_{t+1}(\gamma_t)}{1 - F_o(\gamma_t)}\{E_o[min(s, \gamma_{t-1})] - E_o[min(s, \gamma_t)]\} + q_{t+1}(\gamma_t)$$

- $$q_t = n + q_{t+1}(\gamma_t)$$

Proof 2 *The date $T-2$ optimists' problem:*

$$\max_{a_{T-2}, \gamma_{T-2}} a_{T-2}[E_o[min(s, \gamma_{T-3})] - E_o[min(s, \gamma_{T-2})]]$$
$$s.t. a_{T-2} q_{T-2} = n + a_{T-2} q_{T-1}(\gamma_{T-2})$$

By FOC imply:

$$q_{T-2} = \frac{q'_{T-1}(\gamma_{T-2})}{1 - F_o(\gamma_{T-2})}\{E_o[min(s, \gamma_{T-3})] - E_o[min(s, \gamma_{T-2})]\} + q_{T-1}(\gamma_{T-2})$$

In the equilibrium, all securities are bought by optimists. The market clearing implies $a_{T-t} = 1$.
From the budget constraint:

$$q_{T-2} = n + q_{T-1}(\gamma_{T-2})$$

γ_{T-2} is uniquely determined by these two equations. Given γ_{T-3}, q_{T-2} is uniquely determined. Let $q_{T-2}(\gamma_{T-3})$ denote the equilibrium security price given γ_{T-3}.

By using $q_{T-2}(\gamma_{T-3})$, the date $T-3$ optimists' problem is written.

$$\max_{a_{T-3}, \gamma_{T-3}} a_{T-3}[E_o[min(s, \gamma_{T-4})] - E_o[min(s, \gamma_{T-3})]]$$
$$s.t. a_{T-3} q_{T-3} = n + a_{T-3} q_{T-2}(\gamma_{T-3})$$

By continuing the backward induction, the date t problem is written by using a security price function $q_{t+1}(\gamma_t)$.
Given γ_{t-1} and q_t, the date t optimists problem:

$$\max_{a_t,\gamma_t} a_t(E_o[min(s,\gamma_{t-1})] - E_o[min(s,\gamma_t)])$$
$$s.t.\ a_t q_t = n + a_t q_{t+1}(\gamma_t) \tag{14}$$

FOC and the market clearing imply:

- $$q_t = \frac{q'_{t+1}(\gamma_t)}{1-F_o(\gamma_t)}\{E_o[min(s,\gamma_{t-1})] - E_o[min(s,\gamma_t)]\} + q_{t+1}(\gamma_t)$$

- $$q_t = n + q_{t+1}(\gamma_t)$$

Appendix A.2 shows the equilibrium γ_t uniquely exists.
//

Similarly, the asset price p is written by using $q_1(\gamma_0)$. The date 0 optimists problem is written by using $q_1(\gamma_0)$.

$$\max_{a_0,\gamma_0} a_0\{E_o[s] - E_o[min(s,\gamma_0)]\}$$
$$s.t. a_0 p = n + a_0 q_1(\gamma_0)$$

Proposition 2 γ_0 *and the asset price* p *are determined by these two equations in the equilibrium:*

- $$p = \frac{q'_1(\gamma_0)}{1-F_o(\gamma_0)}\{E_o[s] - E_o[min(s,\gamma_0)]\} + q_1(\gamma_0)$$

- $$p = n + q_1(\gamma_0)$$

Proof 3 *By FOC of the date* 0 *problem:*

$$\frac{1-F_o(\gamma_0)}{p-q_1(\gamma_0)} + q'_1(\gamma_0)\frac{E_o[s] - E_o[min(s,\gamma_0)]}{p-q_1(\gamma_0)} = 0$$

FOC imply:

- $$p = \frac{q'_1(\gamma_0)}{1-F_o(\gamma_0)}\{E_o[s] - E_o[min(s,\gamma_0)]\} + q_1(\gamma_0)$$

In equilibrium, $a_0 = 1$ by the market clearing.

$$p = n + q_1(\gamma_0) \tag{15}$$

//

From the asset price and the security price:

$$\begin{aligned} p &= n + q_1(\gamma_0) \\ &= 2n + q_2(\gamma_2) = ... \\ &= Tn + E_p[min(s, \gamma_{T-1})] \end{aligned} \tag{16}$$

Optimists' total cash Tn raise the asset price. In this market, the highest asset price is optimistic expectation of asset return $E_o[s]$. At next subsection, this price is achieved in the large generation case.

3.5 Large T Case

If T is large enough, at some date t', the security is bought by the optimists own cash n. on date t', the security payoff is $min(s, \gamma_{t'-1})$. Since the seller of the security have monopolistic power, the security price is equal to the optimistic expectation:

$$q_{t'} = E_o[min(s, \gamma_{t'-1})] \tag{17}$$

After date t', there is no security market. ($q_t = 0$ for $t > t'$)
Given $\gamma_{t'-2}$, the date $t' - 1$ optimist problem is:

$$\begin{aligned} \max_{a_{t'-1}, \gamma_{t'-1}} \quad & a_{t'-1}\{E_o[min(s, \gamma_{t'-2})] - E_o[min(s, \gamma_{t'-1})]\} \\ & s.t. \ a_{t'-1}q_{t'-1} = n + a_{t'-1}E_o[min(s, \gamma_{t'-1})] \end{aligned} \tag{18}$$

By solving the problem, the security price $q_{t'-1}$ is:

$$\begin{aligned} q_{t'-1} &= \int_0^{\gamma_{t'-1}} s dF_o + \frac{1 - F_o(\gamma_{t'-1})}{1 - F_o(\gamma_{t'-1})} \int_{\gamma_{t'-1}}^{s^{max}} min(s, \gamma_{t'-2}) dF_o \\ &= E_o[min(s, \gamma_{t'-2})] \end{aligned}$$

This equation and $q_{t'-1} = n + E_o[min(s, \gamma_{t'-1})]$(From $a_{t'-1} = 1$) determine $\gamma_{t'-1}$.
By the backward induction, the date t security price is calculated.

proposition 1 *If T is large enough, the security price at each date t given γ_{t-1} is:*

$$q_t = E_o[min(s, \gamma_{t-1})]$$

The asset price is:

$$p = E_o[s]$$

Proof 4 *Assume the security price function $q_{t+1}(\gamma_t) = E_o[min(s, \gamma_t)]$ Given γ_{t-1} and q_t, the date t optimist problem is:*

$$\begin{aligned} &\max_{a_t, \gamma_t} a_t \{E_o[min(s, \gamma_{t-1})] - E_o[min(s, \gamma_t)]\} \\ &s.t.\ a_t q_t = n + a_t E_o[min(s, \gamma_t)] \end{aligned} \tag{19}$$

This is almost same form as the date $t' - 1$ problem.
FOC implies the security price q_t is $E_o[min(s, \gamma_{t-1})]$.
Since the security price on date t' is $q_{t'} = E_o[min(s, \gamma_{t'-1})]$, the security price q_t is $E_o[min(s, \gamma_{t-1})]$ by the backward induction. γ_t is determined by the market clearing condition $a_t = 1$:

$$E_o[min(s, \gamma_{t-1})] = n + E_o[min(s, \gamma_t)]$$

The date 0 problem is:

$$\begin{aligned} &\max_{a_0, \gamma_0} a_0 \{E_o[s] - E_o[min(s, \gamma_0)]\} \\ &s.t.\ a_0 p = n + a_0 E_o[min(s, \gamma_0)] \end{aligned} \tag{20}$$

This is also the same form on date $t' - 1$. FOC implies that the asset price is:

$$p = E_o[min(s, s^{max})] = E_o[s] \tag{21}$$

γ_0 is determined by the market clearing condition $a_t = 1$:

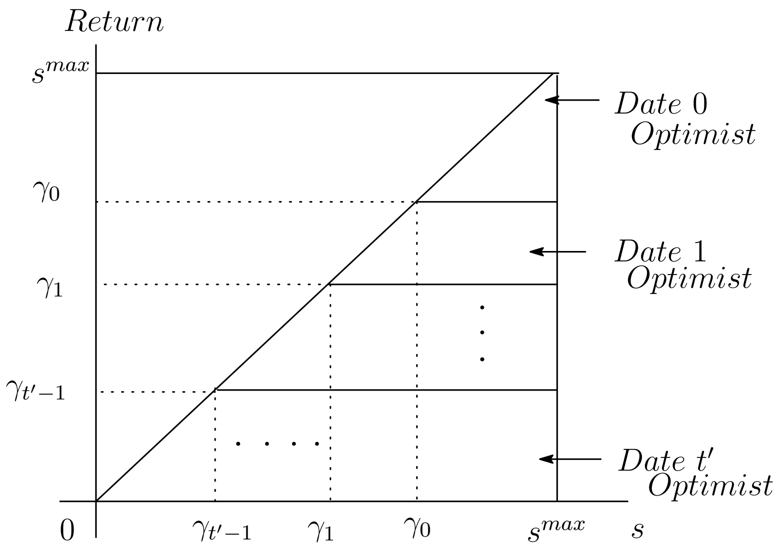

Figure 5: Return distribution

$$E_o[s] = n + E_o[min(s, \gamma_0)]$$

//

T is large enough, the asset is bought by optimists cash.$(T - 1 \geq t')$ The necessary number of generationo (t') is determined by the value of $E_o[s]$. The budget constraint of the date 0 optimists implies asset price is sum of optimist cash n.

$$p = n + n + n + ... + n + E_o[min(s, \gamma_{t'})] = t'n + E_o[min(s, \gamma_{t'})]$$

In equilibrium, the asset price is $E_o[s]$. Then, the necessary number of optimists' generation is:

$$t' = \frac{E_o[s] - E_o[min(s, \gamma_{t'})]}{n}$$

on date T, the optimists pay the loan promise to pessimists. The pessimists give the loan return to optimists who buy the security. The date t optimists receive $min(s, \gamma_{t-1}) - min(s, \gamma_t)$.

4 Analysis

4.1 The Structure of The Asset Price

The equilibrium security price is determined by two equations:

- $$q_t = \frac{q'_{t+1}(\gamma_t)}{1-F_o(\gamma_t)}\{E_o[min(s,\gamma_{t-1})] - E_o[min(s,\gamma_t)]\} + q_{t+1}(\gamma_t)$$

- $$q_t = n + q_{t+1}(\gamma_t)$$

The equilibrium asset price is also determined by two equations:

- $$p = \frac{q'_1(\gamma_0)}{1-F_o(\gamma_0)}\{E_o[s] - E_o[min(s,\gamma_0)]\} + q_1(\gamma_0)$$

- $$p = n + q_1(\gamma_0)$$

These price equations imply:

$$n = \frac{q'_{t+1}(\gamma_t)}{1-F_o(\gamma_t)}\{E_o[min(s,\gamma_{t-1})] - E_o[min(s,\gamma_t)]\} \tag{22}$$

$$n = \frac{q'_1(\gamma_0)}{1-F_o(\gamma_0)}\{E_o[s] - E_o[min(s,\gamma_0)]\} \tag{23}$$

n is the optimists' cash on each date.
On date t, optimists use their cash to receive the divided return:

$$E_o[min(s,\gamma_{t-1})] - E_o[min(s,\gamma_t)] \tag{24}$$

On date 0:

$$E_o[s] - E_o[min(s,\gamma_0)] \tag{25}$$

The optimists buy the right to receive one part of the asset return. The asset price structures show the return distribution.
Each return division can be interpreted as one of the tranche.

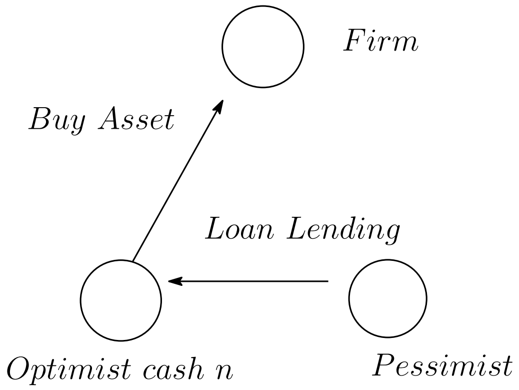

Figure 6: Cash flow in one generation case

The early optimists (like $t = 0$) is very risky because the optimists can get positive payoff only in the high return state, like a "junior" tranche. Similarly, the middle return state is like "mezzanine" tranche and the lower return state is like "senior" tranche.
By a collateral loan contract and a simple securitization, the complex financing technologies can be constructed in the model.

4.2 The Linkage: From Simsek to Miller

If the securitization is not allowed, the asset price is equal that given by Simsek(2013), which is equal to that of the $T = 1$ model.

Only one generation of the traders participate in the market. Optimists' loan borrowing is $\phi_0 = E_p[min(s, \gamma_0)]$. Their budgets are $n + E_p[min(s, \gamma_0)]$ In $T = 1$ case, the equilibrium asset price is deduced by two equations.

$$p = \int_0^{\gamma_0} sdF_p + \frac{1 - F_p(\gamma_0)}{1 - F_o(\gamma_0)} \int_{\gamma_0}^{s^{max}} sdF_o$$

and

$$p = n + E_p[min(s, \gamma_0)]$$

These two equations are exactly the same as that of Simsek(2013).
In his model, the asset price is heavily affected by the collateral economy.
If high return states occur, optimists repay the loan and still have the asset.
If low return states occur, optimists must give the asset to the pessimists as

collateral.
The first equation pictures this situation. High return states ($s > \gamma_0$) are evaluated by optimistic belief(F_o) and low return states($s < \gamma_0$) are evaluated by pessimistic belief(F_p).
In $T = 1$ case, the economy is the same as Simsek(2013). However, if T gets larger, the economy go to the model of the heterogeneous belief bubbles, like Miller(1977).
In $T = 2$ case, the securitization is allowed. Optimists loan borrowing on date 0 is $\phi_0 = q_1(\gamma_0)$. Their budget is $n + q_1(\gamma_0)$. Because $q_1(\gamma_0) > E_p[min(s, \gamma_0)]$, optimists can use bigger budget. As a result, the asset gets higher.
In $T = 2$ case, the equilibrium asset price:

$$p = n + q_1(\gamma_0) = 2n + E_p[min(s, \gamma_1)]$$

The high price is sustained by new optimists' cash.
Assume pessimists' evaluation about the asset return is 0 ($F_p(0) = 1$). In this case, pessimists evaluation about the loan lending $E_p[min(s, \gamma)] = 0$ for any γ. So, the $T = 2$ security price on date 1 is equal to $q_1(\gamma_0) = n$. The asset price of $T - 2$ case is equal to $n + q_1(\gamma_0) = 2n$.
In two generation case, the date 0 optimists can use cash from the date 1 optimists. The existence of the date 1 optimists raise the asset price.
By making loan contracts, pessimists can sell securities on next date. The pessimist has a speculative incentive for the loan contract.

By the same logic, the asset price gets higher if T gets larger. Large T implies that many optimists come to the market. Their total cash raise the asset price.

From small T case, the asset price:

$$p = Tn + E_p[min(s, \gamma_{T-1})]$$

If T is large enough, the asset price is equal to $E_o[s]$.
In Miller(1977), the asset price is equal to the optimistic expectation. In my model, Miller price is achieved by the coordination through a financial technology, securitization.
Optimists have incentive to buy the asset, but they does not have enough cash. Without the security market, they have no way to cooperate with each other. The security and the loan contract play a role in enabling their cooperation.
The loan contract causes the security, which in turn causes the loan contract.

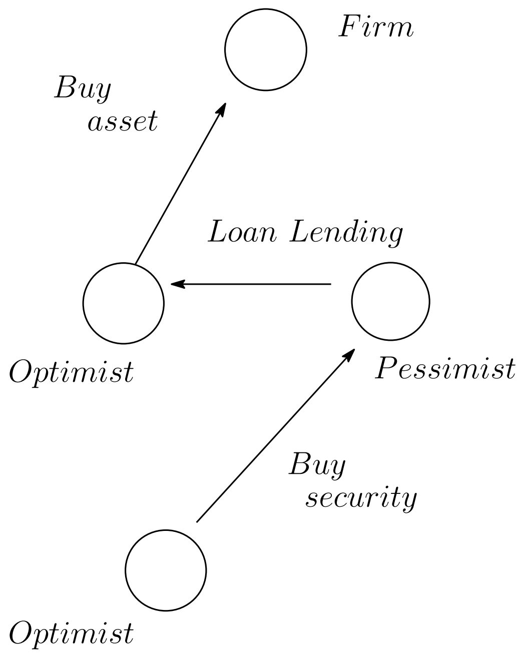

Figure 7: Cash flow in two generation case

The scheme allows each optimist to participate in the market.
In Simsek(2013), optimists does not have enough cash, he must borrow cash from pessimist. Since the loan contract is limited, he must collateralize the asset themselves. The asset price must be influenced by pessimistic belief(F_p). If T is large enough, pessimistic belief vanishes in the asset price. However, if there are no pessimists, the date 0 optimist have no way to bring enough cash to buy the asset.
The pessimists know that the new optimists will come to the market. So, they have strong incentive to lend their cash to the optimists. Lending cash is speculative action for the pessimist. They can sell the risk of the default of optimists to the other optimists by securitization. This is one of risk-shifting problem.(Shleifer and Vishny(1992))
As a result, the asset return is shared by the optimists and the asset price rises. This securities raise the utilities for both optimists and pessimists. The security technology compensate for incomplete financial market.
If there are heterogeneous beliefs exists, completeness of security market make economy riskier. The advanced security market allow heterogeneous investor to act freely.

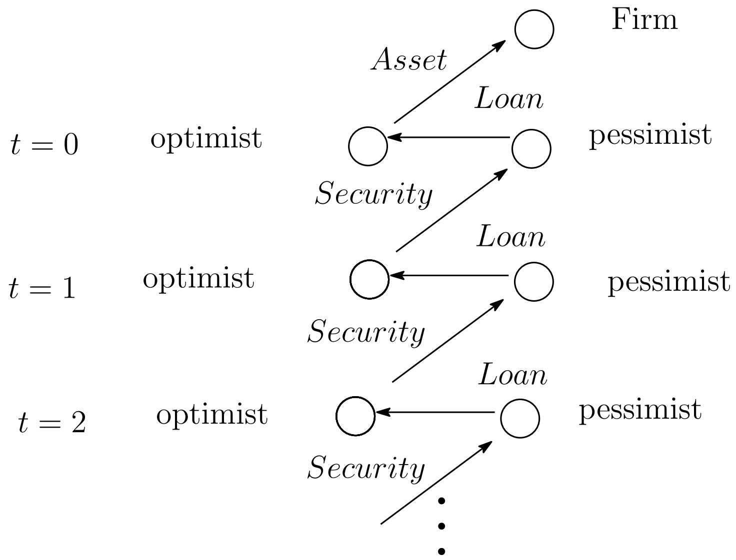

Figure 8: Cash Flow in large generation case

5 Multi Generation with Various Optimists

In this section, I will show that the asset prices exceed any traders' expectations in various optimistic cases of optimistic types.

In this section, multi generation with various optimists model will be analyzed. The basic idea is the same as the single optimist type case in section 3.

There are $J + 1$ type of traders in this economy. J types of the optimists and one type of the pessimist. The optimists' type j have optimistic belief $F_{O_j}(s)$. All types of optimists have the same expectation regarding the return of the asset ($E_{O_j}[s] = E_o[s]\ \forall j \in J$).
Each optimistic belief F_j first order stochastically dominates pessimistic belief F_p.

Assumption 2 $F_{O_j}[s] \geq F_p[s]\ \forall s$ *and* $\forall j \in J$

The asset, the securities and the loan contracts are the same in section 3. On date 0, one type of the continuum of optimists (total population is one) and countably many pessimists come to the market. It is assumed the number of pessimists is larger than T. On each date t, new optimists (total population is one) come to the market. Only one type of optimists come

on each date t. Let the date t optimists' type be j_t. Assume the order of $\{j_t\}_{t-0,1,2,..T-1}$ is exogenously given. All traders live up to the final date T. All optimists are initially endowed with small amount of cash $n > 0$ dollars and zero unit of the asset. Pessimists have no budget constraint.
Optimists who come to the market have a chance to buy the asset or the security only on date t. They buy the asset or the security, and wait until s is realized (date T).
Pessimists have a plenty of cash and they have no budget constraints. They are natural lenders of cash.
These settings are almost the same as single optimist models. However, the existence of various optimists change the security sellers' strategy. The date t seller can get higher return by selling the securities to high type optimists after date t. Then, there is an incentive for the seller to hold the security and to postpone the trade.
I have focused on the highest asset price equilibriium. If T is large enough and all types j comes frequently, all sellers can trade with the highest type of optimist.
The focused equilibrium is, the same as the single type case, the single lender and the monopolistic seller on each date. Note p^t be the asset price when the trade occur on date t.

$$argmax_t \ \ p^t = t_0$$

The asset holder sells it on t_0. On date t_0, traders make loan contract. The lender (a pessimist) can sell the security after t_0.
Note q_1^t be the security price when the trade occur on date t.

$$argmax_{t>t_0} \ \ q_1^t = t_1$$

By continuing this method, we get the stream t_τ ($\tau = 0, 1, 2..$). By using τ instead of t, the asset price is calculated like the single type optimists model.

Let p^* and q_τ^* be the highest price.

$$p^* = p^{t_0}$$
$$q_\tau^* = q_\tau^{t_\tau}$$

Let j_τ^* be the type of optimists on date t_τ.

$$j_\tau^* = j_{t_\tau}$$

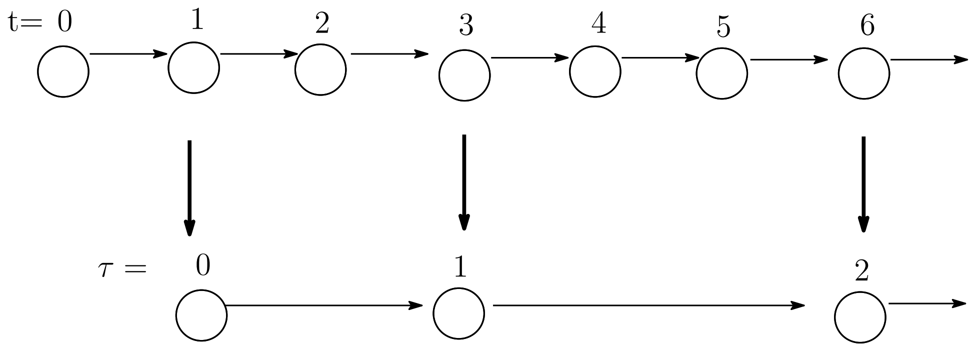

Figure 9: Convert t to τ

On date t_τ, the optimist buys the security with the loan contract. In this section, for simplicity, T is assumed to be large enough. Some optimist can buy the security at t'_s by his own cash.
For analyzing the asset price or the security price, the type j_{t_τ} problem on each t_τ is solved. From each generation τ, ϕ_{t_τ} is solved by pessimists' arbitrage condition.

$$\phi_{t_\tau} = q^*_{\tau+1}$$

$q^*_{\tau'}$ is solved by pessimists' arbitrage condition.

$$q^*_{\tau'} = E_{O_{j^*_{\tau'}}}[min(s, \gamma_{\tau'-1})]$$

So, optimist type j^*_0 problem at generation 0:

$$\max_{a_0,\gamma_0} a_0\{E_{O_{j^*_0}}[s] - E_{O_{j^*_0}}[min(s, \gamma_0)]\}$$
$$s.t. a_0 p = n + a_0 q^*_1(\gamma_0)$$

The budget constraint is satisfied by monopolistic power of the security seller. Because type j^*_0 is the highest optimists' type, they evaluate $E_{O_{j^*_0}}[s] - E_{O_{j^*_0}}[min(s, \gamma_0)]$ higher than any other types.
The type j^*_τ optimist problem at generation t_τ:

$$\max_{a_\tau,\gamma_\tau} a_\tau\{E_{O_{j^*_\tau}}[min(s, \gamma_{\tau-1})] - E_{O_{j^*_\tau}}[min(s, \gamma_\tau)]\}$$
$$s.t. a_\tau q^*_\tau = n + a_\tau q^*_{\tau+1}(\gamma_\tau)$$

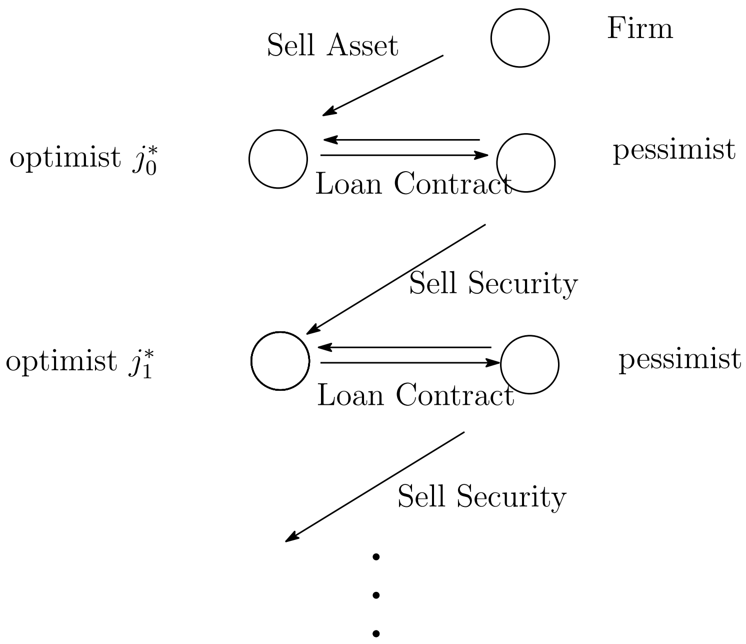

Figure 10: The stream of trade with various optimists

At generation τ', the security price is simply $q_{\tau'} = E_{O_{j^*_{\tau'}}}[min(s, \gamma_{\tau'-1})]$. $j^*_{t'}$ is the type with the highest security price in equilibrium.

By solving the problem on each trading date τ, p^* and q^*_τ are calculated.

Definition 2 *Given trading date $\tau = 0, 1, 2, , ..\tau'$, an equilibrium of various types of optimists model consists of the asset price p^*, the each date security price $\{q^*_\tau\}_{\tau=1,2,..,\tau'}$, loan contract $\{(\gamma_\tau, \phi_\tau)\}_{\tau=0,1,...,\tau'}$, the asset demand a_0 and the security demand $\{a_\tau\}_{\tau=1,2,..,T-1}$. They satisfy the following conditions:*

- *Given $\gamma_{\tau-1}$ and q^*_τ, a_τ and γ_τ solves type j^*_τ optimists' problem on date τ*
- *At each date τ, the loan contract (γ_τ, ϕ_τ) satisfy pessimists arbitrage condition*
- *market clearing condition $a_\tau = 1$ for all τ*

In the single optimist case, the asset price is equal to the optimistic expected return: $E_o[s]$. In this model, the asset price is solved by the backward induction like the single optimist model. However, in each generation, the different types of the optimists buy the asset and the security. In this case, the asset price exceeds any trader's expectation of the asset return.

proposition 2 *In multi-generation model with the various types of the optimists, the asset price exceeds optimistic traders' expectation of the asset return:*

$$p^* \geq E_o[s]$$

Proof 5 *As noted above, the security price on date $t_{\tau'}$ is $E_{o_{j^*_{\tau'}}}[min(s, \gamma_{\tau'-1})]$.*

Since $j^*_{\tau'}$ is the highest type of optimists, the security price $q_{\tau'} = E_{o_{j^*_{\tau'}}}[min(s, \gamma_{\tau'-1})]$ is higher than the other types' expected return of the security.

$$q^*_{\tau'} = E_{o_{j^*_{\tau'}}}[min(s, \gamma_{\tau'-1})] \geq E_{O_j}[min(s, \gamma_{\tau'-1})] \quad \forall j$$

The optimist type $j^*_{\tau'-1}$ buys the date $t_{\tau'-1}$ security. The optimist problem on date $t_{\tau'-1}$ is:

$$\max_{a_{\tau'-1}, \gamma_{\tau'-1}} a_{\tau'-1}\{E_{O_{j^*_{\tau'-1}}}[min(s, \gamma_{\tau'-2})] - E_{o_{j^*_{\tau'-1}}}[min(s, \gamma_{\tau'-1})]\}$$
$$s.t. a_{\tau'-1}q^*_{\tau'-1} = n + a_{\tau'-1}E_{o_{j^*_{\tau'}}}[min(s, \bar{\gamma_{\tau'-1}})]$$

By solving this problem,

$$q^*_{\tau'-1} = \int_0^{\gamma_{\tau'-1}} sdF_{O_{j^*_{\tau'}}} + (1 - F_{O_{j^*_{t'}}}(\gamma_{\tau'-1})) \int_{\gamma_{\tau'-1}}^{s^{max}} min(s, \gamma_{\tau'-2}) \frac{dF_{O_{j^*_{\tau'-1}}}}{1 - F_{O_{j^*_{\tau'-1}}}(\gamma_{\tau'-1})}$$

The security demand is 1 in the equilibrium:

$$q^*_{\tau'-1} = n + E_{o_{j^*_{\tau'}}}[min(s, \gamma_{\tau'-1})]$$

The equilibrium is determined by these two equations. (See appendix A.3 for the existence and uniqueness).
If $j^*_{\tau'-1} = j^*_{\tau'}$, i.e., the same type optimists buy the securities on both dates, the date $\tau' - 1$ security price is simply $E_{O_{j^*_{\tau'-1}}}[min(s, \gamma_{\tau'-1})]$ (it is the same as the single optimist model in large T case). In the various optimists model,

the optimist type who buys the security can be different on each date. As noted above, the security is bought by the optimists who can pay the highest security price. Then, the security price is higher than the single optimist model.

$$q^*_{\tau'-1} \geq E_{O_j}[min(s, \gamma_{\tau'-2})] \quad \forall j$$

The same calculation imply the security price exceeds the single optimist type model on each date. If pessimists sell the security to the type j optimists, the security price on date t_τ is equal to $E_{O_j}[min(s, \gamma_\tau)]$. However, the security seller can choose the highest type optimists. So, the security price is higher than any types of optimists' expected return.

$$q^*_\tau \geq E_{o_j}[min(s, \gamma_{\tau-1})] \quad \forall j \ \ \forall \tau$$

By the same logic, the asset price is higher than the single type optimist model.

$$p^* \geq E_o[s] = E_{o_j}[s] \quad \forall j$$

//

The existence of various optimist leads to the high asset price. This price is one of the heterogeneous-belief bubbles like Harrison and Kreps(1978). In their model, there are various types of traders. They have heterogeneous beliefs about asset returns. At some point, trader x is optimist and trader y is pessimist. However, at different point, x is pessimist and y is optimist. The holder of an asset changes on each date, and the most optimistic trader buys it. The asset holder knows that he can sell it to some other optimist at some future state. The asset price is affected by the optimistic belief on each state. Then, the asset price can be higher than the asset holders' expectations.
In the model with various types of optimists, the asset return is divided and distributed to various optimists. Each optimists have different belief about the divided return. At some point of the asset return, trader type x is the most optimistic trader. However, at different point, y is so. The division of the asset allow the partially optimistic trader to buy the right to receive the return. The asset price is also affected by the optimistic belief on each state like the model of Harrison and Kreps. This is exactly a heterogeneous-belief bubbles.
The asset price is calculated by the same manner as single optimist model.

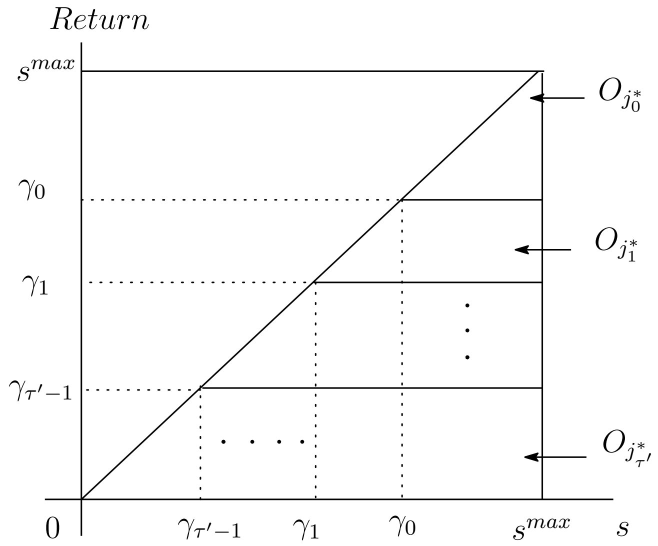

Figure 11: Return distribution with various optimists

In the single type case, each securities are bought by single optimist. Since a single type optimist evaluate the security return, the asset price is equal to the optimistic expected return. On the other hand, each return area are evaluated by the highest type optimist in various types case. As a result, the asset price exceeds the optimistic expectation.
The optimistic expectation about asset return is same:$E_o[s]$. The securitization technology divides the asset return and each optimist evaluate the each partition.

In Simsek(2013), asset return is evaluated by the optimist and the pessimist. In equilibrium, as noted section IV, the asset price:

$$p = \int_0^{\gamma_0} s dF_p + \frac{1 - F_p(\gamma_0)}{1 - F_o(\gamma_0)} \int_{\gamma_0}^{s^{max}} min(s, \gamma_0) dF_o$$

This price equation imply that the optimist evaluate upper return area, and the pessimist evaluate lower area. Only high state part of optimistic belief is used for determining the asset price in Simsek(2013). This is the reason why asset price is lower than Miller(1977).
In this model, the most optimistic trader evaluates each area. Only his optimistic part of belief is used for evaluation of asset return. Since the

asset is priced by different beliefs, it is much more optimistic than anyone's expectation.

6 Conclusions

In this paper, by introducing the securitization and the simple dynamic setting to a collateral model, heterogeneous belief bubbles rises.
In general, financial technologies, like securitizations, improve market efficiency. However, in heterogeneous belief model, securitizations allow many optimists to participate in markets and asset prices will be raised.
Financial frictions can make the asset price lower. Many studies have reported that optimists have a heavy influence on pricing. Financial frictions like borrowing constraints can prevent these traders to participate in market. In these situations, financial technologies loose this friction, and the asset price is raised by optimistic traders who have little cash.
By the dynamic securitization scheme, the asset return is distributed to many partitions. Each partition is very small, but this smallness allow various optimist to receive payoff. Most optimistic traders evaluate their partitions and the total asset price is equal to (or higher than) optimistic expected return of asset in the case of single type of optimists (in the case of various types of optimists). Heterogeneous-belief bubbles occur.
Before the asset return is realized, there are little incentive to correct their beliefs. If there are heterogeneity in markets, financial technologies can amplify it.
Financial technologies, like loan securitizations, improve market efficiency. They make the budget constrained traders to participate in the trade. Simultaneously, they may make market instability through traders' heterogeneity.

References

[1] Brunnermeier, M. "Deciphering the Liquidity and Credit Crunch 2007-2008," Journal of Economic Perspectives, 23(1), pp. 77-100. 2009.

[2] Brunnermeier, Markus K. and Martin Oehmke. "The Maturity Rat Race", Journal of Finance 68(2), 483-521.

[3] Brunnermeier, M. K., and L. H. Pedersen (2009). " Market Liquidity and Funding Liquidity, " Review of Financial Studies, 22(6), 22012238.

[4] DeLong J. B., A. Shleifer, L. H.Summers and R. J. Waldmann(1990) "Noise Tradr Risk in Financial Markets" Jouranal of Political Economy, 98(4)703-738.

[5] Fostel, A., and Geanakoplos, J. "Tranching, CDS, and asset prices: How financial innovation can cause bubbles and crashes," AEJ:Macroeconomics, 4(1)190-225. 2012.

[6] Genakoplos, J. "The Leverage Cycle" in NBER Macroeconomics Annual 2009, vol. 24 University of Chicago Press. 2010.

[7] Gorton, G., and Metrick, A (2011): "Securitized banking and the run on repo," Journal of Financial Economics, 104(3), 425-451.

[8] Gromb, D. and D. Vayanos . "Equilibrium and Welfare in Markets with Financially Constrained Arbitrageurs," Journal of Financial Economics, (66)361-407. 2002.

[9] Harrison.J.M., and D. Kreps. "Speculative Investor Behavior in a Stock Market with Heterogeneous Expectations" Quarterly Journal of Economics, (89) 323-326 1978.

[10] Kiyotaki, N. and J. Moore (1997), "Credit Cycles," Journal of Political Economy,Vol.105 (2), p.211-248.

[11] Krishnamurthy, A., S. Nagel, and D. Orlov (2011): "Sizing Up Repo," Working Paper, Stanford University.

[12] Miller, E. M.. "Risk, Uncertainty, and Divergence of Opinion," Journal of finance, 32(4), 1151-1168. 1977.

[13] Shleifer, A. and R. Vishny (1992), "Liquidation Values and Debt Capacity: A Market Equilibrium Approach,"Journal of Finance, 47, p.1343-1366.

[14] Shleifer, A. and R. Vishny(1997), "The Limits of Arbitrage",Journal of Finance, 52, p.35-55.

[15] Simsek, A.. "Belief Disagreements and Collateral Constraints" Econometrica, 81(1), 1-53.2013.

7 Mathematical Appendix

7.1 Uniqueness of γ_{T-1}

Let q^1_{T-1} be the price q_{T-1} satisfies FOC:

$$q^1_{T-1} = \frac{1-F_p(\gamma_{T-1})}{1-F_o(\gamma_{T-1})}\{E_o[min(s,\gamma_{T-2})]-E_o[min(s,\gamma_{T-1})]\}+E_p[min(s,\gamma_{T-1})]$$

By differentiating in γ_{T-1}:

$$\begin{aligned}\frac{dq^1_{T-1}}{d\gamma_{T-1}} &= (\frac{1-F_p(\gamma_{T-1})}{1-F_o(\gamma_{T-1})})'\{E_o[min(s,\gamma_{T-2})]-E_o[min(s,\gamma_{T-1})]\}\\ &= (\frac{f_p(\gamma_{T-1})}{1-F_o(\gamma_{T-1})}+\frac{f_o(\gamma_{T-1})(1-F_p(\gamma_{T-1}))}{(1-F_o(\gamma_{T-1}))^2})\{E_o[min(s,\gamma_{T-2})]-E_o[min(s,\gamma_{T-1})]\}\end{aligned}$$

From the assumption 1 ($\frac{f_o}{1-F_o} < \frac{f_p}{1-F_p}$), this q^1_{T-1} is decreasing in γ_{T-1}. Because $0 \leq \gamma_{T-1} \leq \gamma_{T-2}$, the range of q^1_{T-1} is determined:

$$E_p[min(s,\gamma_{T-2})] \leq q^1_{T-1} \leq E_o[min(s,\gamma_{T-2})]$$

In equilibrium, optimists buy all securities. Then $\alpha a_{T-1} = 1$. From the budget constraint:

$$q_{T-1} = n + E_p[min(s,\gamma_{T-1})]$$

Let q^2_{T-1} be the price q_{T-1} satisfies this condition:

$$q_{T-1}^2 = n + E_p[min(s, \gamma_{T-1})]$$

Because $0 \leq \gamma_{T-1} \leq \gamma_{T-2}$, the range of q_{T-1}^2 is determined:

$$n \leq q_{T-1}^2 \leq n + E_p[min(s, \gamma_{T-2})]$$

q_{T-1}^2 is increasing in γ_{T-1}.

Because both q_{T-1}^1 and q_{T-1}^2 are continuous, there is only one γ_{T-1} which satisfies two equations.

Given γ_{T-2}, γ_{T-1} is uniquely determined

7.2 Uniqueness of γ_t in Small T Case

I will show the existence of γ_{T-2}. γ_t can be shown by the similar way.

We must show for some γ_{T-2} $(0 \leq \gamma_{T-2} \leq \gamma_{T-3})$satisfies the following equation.

$$n = \frac{q'_{T-1}(\gamma_{T-2})}{1 - F_o(\gamma_{T-2})}\{E_o[min(s, \gamma_{T-3})] - E_o[min(s, \gamma_{T-2})]\}$$

Let RHS be $G(\gamma_{T-2})$. The range of γ_{T-2} is $[0, \gamma_{T-3}]$.

$$G(\gamma_{T-3}) = 0 < n$$
$$G(0) = q'_{T-1}(0)E_o[min(s, \gamma_{T-3})]$$

If $G(0) > n$, at least one γ_{T-2} satisfies the equation $n = G(\gamma_{T-2})$.

$q_{T-1}(\gamma_{T-2})$ must lie between optimistic expected return $E_o[min(s, \gamma_{T-2})]$ and pessimistic expected return $E_p[min(s, \gamma_{T-2})]$.

$$E_o[min(s, \gamma_{T-2})] \geq q_{T-1}(\gamma_{T-2}) \geq E_p[min(s, \gamma_{T-2})]$$

At $\gamma_{T-2} = 0$, the optimistic and pessimistic evaluation of return $min(s, \gamma_{T-2})$ is 0.

$$q_{T-1}(0) = E_o[min(s, 0)] = E_p[min(s, 0)] = 0$$

By differentiating at γ_{T-2}, $q_{T-1}(\gamma_{T-2})$ must satisfy the following equation

$$1 - F_o(0) = 1 \geq q'_{T-1}(\gamma_{T-2}) \geq 1 = 1 - F_p(0)$$

This implies $q'_{T-1}(0) = 1$.

$$G(0) = q'_{T-1}(0)E_o[min(s, \gamma_{T-3})] = E_o[min(s, \gamma_{T-3})] > n$$

Then, $G(0)$ is larger than n.
(If $E_o[min(s, \gamma_{T-3})] < n$, the date $T-2$ optimists can buy the security by their own cash. (Large n case)) At least one γ_{T-2} satisfies $n = G(\gamma_{T-2})$. There may be many γ_{T-2} satisfies the equation. As the security seller can sell the security at the highest price, he can choose the equilibrium γ_{T-2} that bring the highest price q_{T-2}.

$$\begin{aligned} \gamma_{T-2} =& argmax_{\gamma} \;\; n + q_{T-1}(\gamma) \\ & s.t. n = G(\gamma) \end{aligned}$$

Then, γ_{T-2} is uniquely determined. Similarly, γ_t is uniquely determined on each date.

7.3 Uniqueness of γ_τ in Various Types of Optimists

The proof is almost same as that in single type case. I will show the existence of $\gamma_{\tau'-2}$. γ_τ can be shown by the similar way.

I must show for some $\gamma_{\tau'-2}$ $(0 \leq \gamma_{\tau'-2} \leq \gamma_{\tau'-3})$satisfies the following equation.

$$n = \frac{q^{*'}_{\tau'-1}(\gamma_{\tau'-2})}{1 - F_{o_{j^*_{\tau'-2}}}(\gamma_{\tau'-2})}\{E_{o_{j^*_{\tau'-2}}}[min(s, \gamma_{\tau'-3})] - E_{o_{j^*_{\tau'-2}}}[min(s, \gamma_{\tau'-2})]\}$$

Let RHS be $G(\gamma_{\tau'-2})$. The range of $\gamma_{\tau'-2}$ is $[0, \gamma_{\tau'-3}]$.

$$\begin{aligned} & G(\gamma_{\tau'-3}) = 0 < n \\ & G(0) = q^{*'}_{\tau'-1}(0)E_{o_{j^*_{\tau'-2}}}[min(s, \gamma_{\tau'-3})] \end{aligned}$$

If $G(0) > n$, at least one $\gamma_{\tau'-2}$ satisfies the equation $n = G(\gamma_{\tau'-2})$.
Since T is large enough, in multi types of optimists case, $q^*_{t'-1}(\gamma_{\tau'-2})$ must

be larger than $E_{o_{j^*_{\tau'-2}}}[min(s, \gamma_{\tau'-2})]$. (If not so, the seller of the security can raise the security price.)

$$q^*_{\tau'-1}(\gamma_{\tau'-2}) \geq E_{o_{j^*_{\tau'-2}}}[min(s, \gamma_{\tau'-2})]$$

At $\gamma_{\tau'-2} = 0$, the optimistic evaluation of return $min(s, \gamma_{\tau'-2})$ is 0.

$$q^*_{\tau'-1}(0) = E_{o_{j^*_{\tau'-2}}}[min(s, 0)] = 0$$

By differentiating at $\gamma_{\tau'-2}$, $q_{t'-1}(\gamma_{\tau'-2})$ must satisfy the following equation

$$q^{*'}_{\tau'-1}(\gamma_{\tau'-2}) \geq 1 = 1 - F_{o_{j^*_{\tau'-2}}}(0)$$

This implies $q^{*'}_{\tau'-1}(0) \geq 1$.

$$G(0) = q^{*'}_{\tau'-1}(0)E_{o_{j^*_{\tau'-2}}}[min(s, \gamma_{\tau'-3})] \geq E_{o_{j^*_{\tau'-2}}}[min(s, \gamma_{\tau'-3})] > n$$

Then, $G(0)$ is larger than n, and At least one $\gamma_{\tau'-2}$ satisfies $n = G(\gamma_{\tau'-2})$. There may be many $\gamma_{\tau'-2}$ satisfies the equation. As the security seller can sell the security at the highest price, he can choose the equilibrium $\gamma_{\tau'-2}$ that bring the highest price $q^*_{\tau'-2}$.

$$\begin{aligned} \gamma_{\tau'-2} = & argmax_\gamma \;\; n + q^*_{\tau'-1}(\gamma) \\ & s.t. n = G(\gamma) \end{aligned}$$

Then, $\gamma_{\tau'-2}$ is uniquely determined. Similarly, γ_τ is uniquely determined on each date.