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"ILLIQUIDITY COMPONENT OF CREDIT RISK"
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1 INTRODUCTION

Global Game: Use global game methods to solve the outcome of the coordination problem among short term creditors. Using the method, solve for the unique equilibrium and quantify the total credit risk.

Decompose the Credit Risk: Using the global game method, decompose total credit risk (conditioning on date 0 information) into two components: [1] *Asset insolvency risk*; the risk that the eventual asset value realization may be too low to pay off all debt. [2] *Illiquidity risk*; the risk that a run by the short-term creditors may precipitate the failure of the institution even though, in the absence of the run, the asset realization would have been high enough to pay all creditors.

Two Indices: The analysis by global game reveals how the composition of the balance sheet affects the two components of credit risk. Two indices are often cited to measure the credit risk; [1] *capital ratio*; the ratio of equity to total assets, [2] *illiquidity index*; depend on the ratio of total face value of short run debt to liquid assets including asset.

Two Measures: Using the comparative statistics, characterize when reinforcing the capital buffer is the most effective way to reduce total credit risk, and when it is more effective to reduce the illiquidity index by putting more cash on the balance sheet. The analysis shows that liquidity enhancing policies are more effective on the margin when [1] the current level of illiquidity index is high, and [2] when there is significant interim uncertainty about asset returns, or, equivalently when the short-term debt is ultra short-term (there remains relatively long time until the asset return finally realizes).

Trends Preceding the Crisis: [1] The cash holdings by banks has secularly declined over the last thirty or so years, until the outbreak of the recent financial crises. [2] The

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use of short-term debt such as overnight repos dramatically increased compared to the use of longer maturity term repo agreements. Thus, liquidity enhancing policies might have been more effective than solvency enhancing policies.

2 A DECOMPOSITION OF CREDIT RISK

2.1 The Balance Sheet and the Funding Game

Balance Sheet: Consider a leveraged financial institution, who holds assets in $N+1$ categories indexed by $i \in \{0, \cdots, N\}$ and liabilities in three ways—equity, short-term debt and long-term debt. Let A_i denote the face value of assets in asset class i, and E, S, L denote those of equity, short-term debt and long-term debt, respectively.

The Value of Fundamentals: There are three dates: initial (date 0), interim (date 1), and final (date 2). Let θ_t denote the value of fundamentals at date t and assume that

$$\theta_1 = \theta_0 + \sigma_1 \epsilon_1$$
$$\theta_2 = \theta_1 + \sigma_2 \epsilon_2$$

where ϵ_1 and ϵ_2 are independently distributed with 0 means and densities f_1 and f_2 respectively. The parameters σ_i measure the size of interim and final period uncertainty respectively. The relative size of σ_i plays an important role in the analysis.

The Value of Assets at Date 2: The value of asset i at date 2 per unit face value is given by

$$\alpha_i + \beta_i \theta_2$$

Let $i = 0$ be cash, which is characterized by $\alpha_0 = 1, \beta_0 = 0$[1].

Solvency: Let r_L and r_S be the promised gross return for long- and short-term debt holders[2], respectively. The bank is *solvent* if asset gross returns cover liabilities[3], i.e.,

$$\sum_{i=0}^{N} (\alpha_i + \beta_i \theta_2) A_i \geq S r_S + L r_L$$

or equivalently,

$$\theta_2 \geq \frac{S r_S + L r_L - \sum_{i=0}^{N} \alpha_i A_i}{\sum_{i=0}^{N} \beta_i A_i} \equiv \theta^*$$

Let call θ^* a solvency point. Assume that if the bank is insolvent in period 2—i.e., when $\theta_2 < \theta^*$— then the bank goes into liquidation and for simplicity assume that

[1] Definition in the paper will be a typo

[2] Maybe, adjusted rate for the same period.

[3] As is mentioned later, assuming that, if the withdrawal had not eventually result in the bankrupt in the interim date, the bank can recover the fund withdrew in the interim date by financing from the other short-term debt holders in the market.

neither short nor long term debtors receive any return.[4]

Date 1 Decision of Short-Term Debt Holders: A the interim date 1, the short run debt holders observe θ_1 and face a decision about whether to roll over their lending. If the positions of short run debt holders are not rolled over, then assets must be liquidated to cover their positions.

Liquidated Value of Assets: Assume that the liquidated value of asset i is[5]

$$\lambda_i(\alpha_i + \beta_i\theta_0)A_i$$

where $\lambda_i \in [0,1]$ represents the fire sale discount. Assume that cash is perfectly liquid, so that $\lambda_0 = 1$. Thus, the total liquidation value of the bank's assets is

$$A^* \equiv \sum_{i=0}^{N} \lambda_i(\alpha_i + \beta_i\theta_0)A_i$$

(Assume that short holder requires its face value when she decided not to roll over her lending[6].) Then, the bank suffers a run if the proportion of short term debt holders not rolling over is more than A^*/S.

Three Assumptions Regarding Run: Assume that [1] if there is a run, all value is destroyed and short term debtors get a return of 0, [2] if there is not a run, new creditors will eventually be found and the balance sheet reverts to its initial state after the failed run, and [3] short run debt holders have an alternative investment opportunity in which they can earn gross return r^*.

Assumption on Parameters: Throughout the paper, assume that

$$q \equiv \frac{r^* S}{r_S A^*} < 1 < \frac{S}{A^*} \tag{1}$$

where the second inequality means that the liquidation value of asset is smaller than the face value of short term debt (otherwise run never happens), and the first inequality means that the outside opportunity is fairly unattractive relative to the investment in this bank (the inequality implies $r^*A^* < r^*S < r_S A^* < r_S S$).

[Note] Index about Bank Run q and Index about Illiquidity A^*: In this paper, an event such that "*the required repayment value by short-term debt holders at the interim date exceeds the liquidated value of the total asset of the bank*" defines the event "*bank run*". The value of q determines the threshold for such an event, as is shown later. On the

[4] Restrictive?

[5] In the paper, β_i is set to β. This would be a typo. Setting θ_0 instead of θ_1 seems problematic since the fundamental is revealed as θ_1 at this time. Setting $\lambda_i(\alpha_i + \beta_i\theta_1)A_i$ as the liquidated value would be more appropriate. However, this makes the liquidated asset value random at period 0. Then, we need additional integration with respect to ϵ_1 in the calculation of $\mathcal{ILL}_0(\theta)$, which yield too complicated result.

[6] Added this assumption. This implies the short-term debt holders require no interest rate between date 0 and date 1. Together with the fact that the "annualized" short-term interest rate is set to r_s, this means that the interest rate between date 1 and date 2 is set to r_S. This difference between interest rate must have relation with the difference in the riskiness, σ_1, σ_2, though which is not taken into account in this model

other hand, "*illiquidity*" is defined by "*the liquidated value of the total asset A^* per its face value*". As the fire sale discount for the asset is high, the asset is said to be more illiquid.

Thus, calling q as illiquidity index is, in principle, not correct. However, in this model, the face value of total asset, the face value of short-term debt S, and r^* and r_S are all exogenously determined, so that there is an one-to-one relation between q and the illiquidity of the asset. This one-to-one relation due to the model assumption only allows us to call q an illiquidity index.

2.2 *Solving the Coordination Game and Interim Credit Risk*

The Interim Solvency Risk: The probability that the bank will fail *if there is no run* at date 1 is

(2) $$\mathcal{INS}_1(\theta_1) = F_2\left(\frac{\theta^* - \theta_1}{\sigma_2}\right)$$

The Total Expected Return at Date 1: The total expected return to rolling over, conditional on there not being a run, is

$$r_S\left(1 - F_2\left(\frac{\theta^* - \theta_1}{\sigma_2}\right)\right)$$

while the return to not rolling over is r^*.

Noise in Observing θ_1: Suppose that instead of perfectly observing θ_1, each short run creditor observed it with a small amount of noise.

The Strategy at the Equilibrium Selected by Global Game Method

> The global game method suggest that *the short-term debt holders follows a threshold strategy in which if one receives a signal below a switching point θ^{**}, she decides not to roll over, and otherwise, decides to roll over.*

The Belief over the Ratio of "Not Roll Over"

> Let π be the proportion of short-term creditors who do not roll over their debt. The standard result for global games suggests that, *in the limiting case as noise goes to zero, and conditional on being at the switching point between rolling over and not rolling over short-term debt, the belief over π is the uniform belief over* $[0, 1]$. $\cdots (*)$

The Indifference Condition: Consider characterizing the threshold point θ^{**}. It is indifferent for short run debt holders to rolling over and not rolling over when θ_1 is at the *run point* θ^{**} satisfying

$$\frac{A^*}{S}\left(1 - F_2\left(\frac{\theta^* - \theta^{**}}{\sigma_2}\right)\right) r_S = r^*$$

where $A^*/S = P[NotRun] = U[\pi \leq A^*/S]$, which is a belief of short-term debt holders who received a signal at the threshold point θ^{**}. Rewriting the condition gives us

(3) $$\theta^{**} = \theta^* - \sigma_2 F_2^{-1}(1 - q)$$

The condition states that a run will occur whenever the interim probability of solvency (provided there is no run) is below q. Let call q the *illiquidity index*.

The Interim Illiquidity Risk: The interim illiquidity risk, i.e., the probability that the bank will fail because of a run, when it would not have been insolvent in the absence of a run, is given by

$$(4) \quad \mathcal{ILL}_1(\theta_1) = \begin{cases} 1 - F_2\left(\dfrac{\theta^* - \theta_1}{\sigma_2}\right) & \text{if } \theta_1 \leq \theta^{**} \\ 0 & \text{if } \theta_1 > \theta^{**} \end{cases}$$

As is mentioned, as one receives a signal more than θ^{**}, she rolls over with probability one. However, otherwise, she does not roll over. There, subtracting $F_2((\theta^* - \theta_1)/\sigma_2)$ since we do not incorporate the event "the bank runs in the interim date and the bank would be insolvent in the date 2 even if the bank did not run in the interim date" into a bankrupt event due to illiquidity problem.

The Total Interim Credit Risk: Total interim credit risk is $\mathcal{C}_1(\theta_1) = \mathcal{INS}_1(\theta_1) + \mathcal{ILL}_1(\theta_1)$, so that

$$(5) \quad \mathcal{C}_1(\theta_1) = \begin{cases} 1 & \text{if } \theta_1 \leq \theta^{**} \\ F_2\left(\dfrac{\theta^* - \theta}{\sigma_2}\right) & \text{if } \theta_1 > \theta^{**} \end{cases}$$

2.3 Three Measures of Ex Ante Credit Risk

Ex Ante Insolvency Risk: The ex ante probability of insolvency, that is, the probability that the bank is insolvent in period 2, conditional on there not being a run in period 1 is given by

$$(6) \quad \mathcal{INS}_0(\theta_0) = \int_{-\infty}^{\infty} F_2\left(\frac{\theta^* - \theta_1}{\sigma_2}\right)\frac{1}{\sigma_1}f_1\left(\frac{\theta_1 - \theta_0}{\sigma_1}\right)d\theta_1$$

Ex Ante Illiquidity Risk: The ex ante probability of illiquidity, that is, the probability that the bank fails due to run whereas it would be solvent without a run, is given by

$$(7) \quad \mathcal{ILL}_0(\theta_0) = \int_{-\infty}^{\theta^* - \sigma_2 F_2^{-1}(1-q)} \left(1 - F_2\left(\frac{\theta^* - \theta_1}{\sigma_2}\right)\right)\frac{1}{\sigma_1}f_1\left(\frac{\theta_1 - \theta_0}{\sigma_1}\right)d\theta_1$$

Ex Ante Total Credit Risk: The total ex ante total credit risk is given by

$$(8) \quad \begin{aligned} \mathcal{C}_0(\theta_0) &= \mathcal{INS}_0(\theta_0) + \mathcal{ILL}_0(\theta_0) \\ &= F_1\left(\frac{\theta^* - \theta_0 - \sigma_2 F_2^{-1}(1-q)}{\sigma_1}\right) \\ &\quad + \int_{\theta^* - \sigma_2 F_2^{-1}(1-q)}^{\infty} F_2\left(\frac{\theta^* - \theta_1}{\sigma_2}\right)\frac{1}{\sigma_1}f_1\left(\frac{\theta_1 - \theta_0}{\sigma_1}\right)d\theta_1 \end{aligned}$$

3 BALANCE SHEET COMPOSITION AND CREDIT RISK

Assumptions for Analysis: Make assumptions that ensure that we can get closed form solutions to expressions for ex ante credit risk described in equations (6) to (8) above. Assume that ϵ_1 and ϵ_2 are both independently uniformly distributed, so that

$$F_1(x) = F_2(x) = \begin{cases} 0 & \text{if } x \leq -\dfrac{1}{2} \\ x + \dfrac{1}{2} & \text{if } -\dfrac{1}{2} \leq x < \dfrac{1}{2} \\ 1 & \text{if } \dfrac{1}{2} \leq x \end{cases}$$

Closed Forms of Interim Variables: The formal expressions of main interim variables under this assumption are given by

$$\theta^{**} = \theta^* + \sigma_2 \left(q - \frac{1}{2} \right)$$

$$\mathcal{INS}_1(\theta_1) = \begin{cases} 1 & \text{if } \theta_1 \leq \theta^* - \dfrac{1}{2}\sigma_2 \\ \dfrac{1}{2} + \dfrac{1}{\sigma_2}(\theta^* - \theta_1) & \text{if } \theta^* - \dfrac{1}{2}\sigma_2 < \theta_1 \leq \theta^* + \dfrac{1}{2}\sigma_2 \\ 0 & \text{if } \theta^* + \dfrac{1}{2}\sigma_2 < \theta_1 \end{cases}$$

$$\mathcal{ILL}_1(\theta_1) = \begin{cases} 0 & \text{if } \theta_1 \leq \theta^* - \dfrac{1}{2}\sigma_2 \\ \dfrac{1}{2} - \dfrac{1}{\sigma_2}(\theta^* - \theta_1) & \text{if } \theta^* - \dfrac{1}{2}\sigma_2 < \theta_1 \leq \theta^* + \sigma_2 \left(q - \dfrac{1}{2} \right) \\ 0 & \text{if } \theta^* + \sigma_2 \left(q - \dfrac{1}{2} \right) < \theta_1 \end{cases}$$

$$\mathcal{C}_1(\theta_1) = \begin{cases} 1 & \text{if } \theta_1 \leq \theta^* + \sigma_2 \left(q - \dfrac{1}{2} \right) \\ \dfrac{1}{2} + \dfrac{1}{\sigma_2}(\theta^* - \theta_1) & \text{if } \theta^* + \sigma_2 \left(q - \dfrac{1}{2} \right) < \theta_1 \leq \theta^* + \dfrac{1}{2}\sigma_2 \\ 0 & \text{if } \theta^* + \dfrac{1}{2} < \theta_1 \end{cases}$$

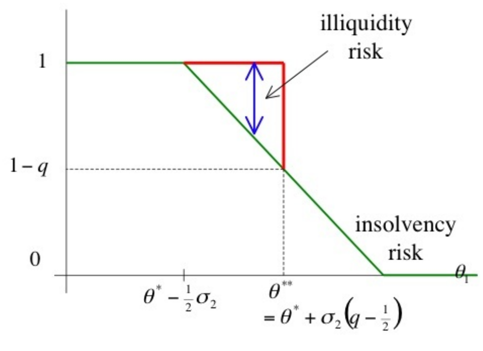

Figure 4: Total Credit Risk

Assumptions for Simplicity: Assume that all values of θ_1 where the insolvency risk is strictly between 0 and 1 are included in the support of the prior distribution: [1] the ex ante noise is sufficiently high ($\sigma_1 > \sigma_2$) and that [2] θ_0 is not too extreme, i.e.,

$$\theta_0 \in \left[\theta^* - \frac{1}{2}(\sigma_1 - \sigma_2), \theta^* + \frac{1}{2}(\sigma_1 - \sigma_2)\right]$$

Closed Forms of Ex Ante Variables:

$$\begin{aligned}
\mathcal{INS}_0(\theta_0) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathcal{INS}_1(\theta_0 + \sigma_1\epsilon) f_1(\epsilon) d\epsilon \\
&= \int_{\theta^*-\frac{1}{2}\sigma_2}^{\theta^*+\frac{1}{2}\sigma_2} \left[\frac{1}{2} + \frac{1}{\sigma_2}(\theta^* - \theta_1)\right] \frac{1}{\sigma_1} d\theta_1 + \int_{\theta_0-\frac{1}{2}\sigma_1}^{\theta^*+\frac{1}{2}\sigma_2} \frac{1}{\sigma_1} d\theta_1 \\
&= \frac{1}{\sigma_1}\left[\left(\frac{1}{2} + \frac{\theta^*}{\sigma_2}\right)\sigma_2 - \theta^* + \theta^* - \theta_0 + \frac{1}{2}\sigma_1\right] \\
&= \frac{1}{2} + \frac{\theta^* - \theta_0}{\sigma_1}
\end{aligned}$$

$$\begin{aligned}
\mathcal{ILL}_0(\theta_0) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathcal{ILL}_1(\theta_0 + \sigma_1\epsilon) f_1(\epsilon) d\epsilon \\
&= \int_{\theta^*-\frac{1}{2}\sigma_2}^{\theta^*+\sigma_2(q-\frac{1}{2})} \left[\frac{1}{2} - \frac{1}{\sigma_2}(\theta^* - \theta_1)\right] \frac{1}{\sigma_1} d\theta_1 \\
&= \frac{1}{\sigma_1}\left(\frac{1}{2} - \frac{\theta^*}{\sigma_2}\right)\sigma_2 q + \frac{1}{2\sigma_1\sigma_2}\left[\left(\theta^* + \sigma_2\left(q - \frac{1}{2}\right)\right)^2 - \left(\theta^* - \frac{1}{2}\sigma_2\right)^2\right] \\
&= \frac{\sigma_2}{2\sigma_1} q^2
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_0(\theta_0) &= \mathcal{INS}_0(\theta_0) + \mathcal{ILL}_0(\theta_0) \\
&= \frac{1}{2} + \frac{\theta^* - \theta_0}{\sigma_1} + \frac{\sigma_2}{2\sigma_1} q^2
\end{aligned} \tag{9}$$

Intuition under the Formula: The illiquidity risk rises as [1] the illiquidity index q gets higher, and as [2] the relative size of σ_2 to σ_1 gets larger. Note that q is larger when the face value of the short run debt is higher, the outside opportunity is more attractive, and the liquidation value of the asset is smaller. The relative size of σ_2 to σ_1 can be interpreted as the measure of maturity of the debt.

3.1 Balance Sheet Comparative Statistics

The Effect of Solvency and Liquidity on the Credit Risk: In the component of $\mathcal{C}_0(\theta_0)$, θ^* represents the difficulty of being solvent and q does the illiquidity. Differentiating $\mathcal{C}_0$ with respect to these variables gives us

$$\sigma_1 \frac{\partial \mathcal{C}_0}{\partial \theta^*} = 1$$

$$\sigma_1 \frac{\partial \mathcal{C}_0}{\partial q} = \sigma_2 q$$

Thus, improving the liquidity of the bank's balance sheet may be more effective in mitigating overall credit risk when compared to increasing solvency (by capital buffer?[7]) when σ_2 is large (there is more interim uncertainty about the bank's assets, and short-term debt is ultra short-term) and when q is large (the current liquidity of the balance sheet is low).
[8]

The Market Value of Equity at Date 0: The market value of equity at time 0, denoted by E_0, is given by

$$E_0 = \sum_{i=0}^{N} A_i(\alpha_i + \beta_i\theta_0) - Sr_S - Lr_L$$

Let write

$$V = \sum_{i=1}^{N} A_i\beta_i$$

which can be interpreted as the measure of variability of the firm's portfolio.

Effect of Face Value of Asset on Solvency Point: The solvency point can be rewritten as

$$\theta^* = \frac{Sr_S + Ls_L - \sum_{i=0}^{N} \alpha_i A_i}{\sum_{i=1}^{N} \beta_i A_i} = \theta_0 - \frac{E_0}{V}$$

Writing $\bar{r}_i = \alpha_i + \beta_i\theta_0$, we obtain

$$\frac{\partial\theta^*}{\partial A_i} = -\frac{\bar{r}_i}{V} + \frac{E_0\beta_i}{V^2}$$

[7] Why the capital buffer can reinforce solvency?

[8] We wonder why they take derivatives with respect to q instead of θ^{**}, while the latter seems more appropriate to compare the effectiveness of solvency-enhancing and bank run-immunization policies. Notice that from (3), we obtain

$$\frac{\partial q}{\partial\theta^{**}} = \frac{1}{\sigma_2}$$

under uniform fluctuation assumption. Then, the chain rule gives us

$$\sigma_1\frac{\partial\mathcal{C}_0}{\partial\theta^{**}} = q$$

and by assumption, we have $q < 1$. This implies that "*policy levering down the bank run point is always less effective than policy levering down the solvency point*", which is a converse result with that the authors intended.

In the first place, comparing the elasticity of $\mathcal{C}_0$ on θ^* and either of θ^{**} or q is nonsense without introducing the cost to execute the policies. Otherwise, we have no common unit of measure to compare the magnitude of the effects between two policies. We have no measure to evaluate the difficulty in changing θ^* and θ^{**} by one percent.

Effect of the Face Value of Asset on Illiquidity Index: Recall that

$$q \equiv \frac{r^* S}{r_S \sum_{i=0}^{N} \lambda_i \bar{r}_i A_i}$$

Thus

$$\frac{\partial q}{\partial A_i} = -\frac{q \lambda_i \bar{r}_i}{A^*}$$

Effect of the Face Value of Asset on Total Credit Risk: The effect of change in asset i holding is given by

$$\sigma_1 \frac{\partial \mathcal{C}_0}{\partial A_i} = -\frac{\bar{r}_i}{V} + \frac{E_0 \beta_i}{V^2} - \frac{\sigma_2 q^2 \lambda_i \bar{r}_i}{A^*}$$

The effect of change in cash holding, which is characterized by $\bar{r}_0 = 1, \beta_0 = 0$ and $\lambda_0 = 1$, can be written as

$$\sigma_1 \frac{\partial \mathcal{C}_0}{\partial A_0} = -\frac{1}{V} - \frac{\sigma_2 q^2}{A^*}$$

The effect of change in holding of riskless illiquid asset k with gross return R, which is characterized by $\bar{r}_k = R, \beta_k = 0$, and $\lambda_k = 0$, is given by

$$\sigma_1 \frac{\partial \mathcal{C}_0}{\partial A_k} = -\frac{R}{V}$$

Then, the multiplier of cash is

$$\frac{A^* + \sigma_2 V q^2}{A^* R} \tag{10}$$

Thus, when $\sigma_2 V q^2$ is large, the *liquid asset* is more important than *safe asset*.

4 APPENDIX

4.1 *Derivation of Laplacian Rule in* $(*)$

Information Structure: Let player i's private signal be given by

$$x_i = \theta + u_i$$

where θ is a Gaussian random variable with mean y and variance $1/v_\theta$, and u_i is Gaussian with mean zero and variable $1/v_u$. The random variables $\{u_i\}$ are mutually independent, and independent of θ. Denote by $\pi(x)$ be the proportion of players whose signal is x or less. Then, $\pi(x)$ is a random variable with realizations in the unit interval, and which is a function of the random variables $\{\theta, \epsilon_i\}_{i \in [0,1]}$ with threshold x.

Cumulative Distribution of $\pi(x)$: Let $G(z|x_i)$ be the cumulative distribution of $\pi(x_i)$ conditional on x_i. In other word,

$$G(z|x_i) = P[\pi_i(x_i) \leq z | x_i] \tag{12}$$

$\pi(x_i)$ **Conditional on θ:** Given θ, the proportion of players who have signal below x_i is

(13) $\Phi(\sqrt{v_u}(x_i - \theta))$

where $\Phi(\cdot)$ is a cumulative distribution function for standard normal. Let define the realization of θ at which this proportion is z, say $\hat{\theta}$, is defined by

(14) $\hat{\theta} = x_i - \dfrac{\Phi^{-1}(z)}{\sqrt{v_u}}$

Hence, we obtain

$$G(z|x_i) = P[\theta \geq \hat{\theta}|x_i]$$

Distribution of θ Conditional on x_i: Conditional on x_i, the density of θ is given by

$$\begin{aligned}
P[\theta = z|x] &= \frac{P[x|\theta = z]P[\theta = z]}{P[\theta - \epsilon = x]} \\
&= \frac{\sqrt{\frac{v_u}{2\pi}} \exp\left[-\frac{v_u}{2}(x-z)^2\right] \sqrt{\frac{v_\theta}{2\pi}} \exp\left[-\frac{v_\theta}{2}(z-y)^2\right]}{\int \sqrt{\frac{v_\theta}{2\pi}} \exp\left[-\frac{v_\theta}{2}(x+\epsilon-y)^2 \sqrt{\frac{v_u}{2\pi}} \exp\left[-\frac{v_u}{2}\epsilon^2\right]\right] d\epsilon} \\
&= \sqrt{\frac{v_\theta + v_u}{2\pi}} \exp\left[-\frac{v_\theta + v_u}{2}\left(z - \frac{v_\theta y + v_u x}{v_\theta + v_u}\right)\right]
\end{aligned}$$

Hence, we obatain

(15) $\theta|x_i \sim N\left(\dfrac{v_\theta y + v_u x_i}{v_\theta + v_u}, \dfrac{1}{v_\theta + v_u}\right)$

Resulting $G(z|x_i)$: Then, we obtain

(16) $G(z|x_i) = P[\theta \geq \hat{\theta}|x_i] = 1 - \Phi\left(\sqrt{v_u + v_\theta}\left(\hat{\theta} - \dfrac{v_\theta y + v_u x_i}{v_\theta + v_u}\right)\right)$

Substituting $\hat{\theta}$ into this gives

(17) $G(z|x_i) = \Phi\left(\dfrac{v_\theta}{\sqrt{v_\theta + v_u}}(y - x_i) + \sqrt{\dfrac{v_\theta + v_u}{v_u}}\Phi^{-1}(z)\right)$

Diminish the Noise: Taking $v_u \to \infty$ gives us

$$G(z|x_i) \to \Phi(\Phi^{-1}(z)) = z$$

This means that "*a player who received signal x believes that the proportion of players who received less than or equal to x should follow uniform distribution over $[0,1]$ in the limit where the noise goes to zero*".

4.2 Derivation of Threshold Strategy

Payoff Matrix: Consider a payoff matrix such that

		Player 2	
		A	B
Player 1	A	t_1, t_2	$t_1 - 1, 0$
	B	$0, t_2 - 1$	$0, 0$

where t_1, t_2 are written as

$$\begin{aligned} t_1 &= \theta + \epsilon_1 \\ t_2 &= \theta + \epsilon_2 \end{aligned} \tag{18}$$

and $\theta, \epsilon_1, \epsilon_2$ distribute independently as

$$\begin{aligned} &\theta \sim U[L, U] \\ &\epsilon_i \sim U[-D, D] \end{aligned}$$

Strategy: Let $\sigma(t_i) \in \{A, B\}$ denote the strategy of player i given an observation t_i. Then, the expected payoff of choosing A and B are respectively,

$$\begin{aligned} &A : t_i P[\sigma_j = A|t_i] + (t_i - 1)[1 - P[\sigma_j = A|t_i]] = t_i - 1 + P[\sigma_j = A|t_i] \\ &B : 0 \end{aligned}$$

Hence, the conditions for every equilibrium are given by

$$\begin{aligned} &(A, A) : t_1 - 1 + P[\sigma_2 = A|t_1] \geq 0, t_2 - 1 + P[\sigma_1 = A|t_2] \geq 0 \\ &(A, B) : t_1 - 1 + P[\sigma_2 = A|t_1] \geq 0, t_2 - 1 + P[\sigma_1 = A|t_2] < 0 \\ &(B, A) : t_1 - 1 + P[\sigma_2 = A|t_1] < 0, t_2 - 1 + P[\sigma_1 = A|t_2] \geq 0 \\ &(B, B) : t_1 - 1 + P[\sigma_2 = A|t_1] < 0, t_2 - 1 + P[\sigma_1 = A|t_2] < 0 \end{aligned}$$

Conjecture: Notice that $|t_1 - t_2| < |\epsilon_1| + |\epsilon_2| \leq 2D$. Suppose $D = 0$. Then, $t_1 = t_2$, and the game reduces to the complete information case. In this case, there are two equilibria (A, A) and (B, B). However, there is a support of ϵ in which there exists a unique equilibrium:

Proposition 1. *Let* $0 < D \leq \min\{1 - L, U - 1\}$. *Then, we have the unique equilibrium*

$$\sigma_i(t_i) = \begin{cases} A & \text{if } t_i \geq 0.5 \\ B & \text{if } t_i < 0.5 \end{cases}$$

Definition 1. *In general, for $k \in R$, a strategy in which one chooses A if the revealed private information is equal to or more than k and chooses B otherwise, is said to be a k-threshold strategy, and denoted by $s[k]_i$.*

Proof. If $t_i < 0$, then $\sigma_i(t_i) = B$ is the dominant strategy, and $\sigma_i(t_i) = A$ is dominant if $t_i > 1$. Hence, it suffices to examine the case $t_i \in [0, 1]$. Here,

$$P[t_j = c|t_i] = P[t_i + \epsilon_j - \epsilon_i = c|t_i] = P[\epsilon_j - \epsilon_i = c - t_i|t_i] = f(c - t_i)$$

where $f(\cdot)$ can be calculated by the transformation of variables as

$$f(y) = \begin{cases} \frac{1}{8D^2}(4D - 2y) \\ \frac{1}{8D^2}(4D + 2y) \end{cases}$$

Hence, $P[t_i > t_j] = P[t_i < t_j] = 0.5$. Now, to verify that the pair of 0.5-threshold strategy $s[0.5]_i$ is the equilibrium, derive the optimal response of player i against $s[0.5]_j$. Given $s[0.5]_j$, we have

$$P[s[k]_j = A|t_i] = P[t_j \geq k|t_i] = P[\epsilon_j - \epsilon_i \geq k - t_i|t_i] = \int_{k-t_i}^{\infty} f(x)dx$$

Hence, the expected payoff of player i when he choose A observing t_i is

$$(19) \quad \pi(t_i, k) = t_i - 1 + P[s[k]_j = A|t_i] = t_i - 1 + \int_{k-t_i}^{\infty} f(x)dx$$

Moreover, since $P[t_i > t_j] = P[t_i < t_j] = 0.5$, we have

$$(20) \quad \pi(k, k) = k + 0.5 - 1 = k - 0.5$$

From (19) and (20), $\pi(t_i, k)$ have the following properties:

[1] $\pi(t_i, k)$ is continuous in $(t_i, k) \in [0, 1] \times [0, 1]$, and strictly increasing in $t_i \in [0, 1]$ and nonincreasing in $k \in [0, 1]$. Moreover, $\pi(0, k) \leq 0$ and $\pi(1, k) \geq 0$ for all k.

[2] $\pi(k, k)$ is strictly increasing in $k \in [0, 1]$, $\pi(0, k) = -0.5 < 0$, and $\pi(1, k) = 0.5 > 0$ and hence $\pi(k, k) = 0$ has a unique solution $k = 0.5$.

Remember that player i chooses A if $\pi(t_i, k) \geq 0$ and B otherwise. By property one, for any k, there exists unique $b(k)$ such that $\pi(b(k), k) = 0$, and $\pi(t_i, k) \geq 0$ if $t_i \geq b(k)$ and $\pi(t_i, k) < 0$ if $t_i < b(k)$. Hence, the optimal response against $s[k]_j$ is given by $s[b(k)]_i$. Especially, if $k = b(k)$, then the pair $(s[k]_1, s[k]_2)$ is an equilibrium. Now, we have $\pi(k, k) = 0$, and $k = 0.5$ is the unique solution by property 2. Thus, the pair of 0.5-threshold strategies is an equilibrium.

Show that this is the unique equilibrium. First, we show that when (σ_1, σ_2) is an equilibrium, for all $n \in N$, we have

$$(21) \quad \sigma_i(t_i) = \begin{cases} A & \text{if } t_i > b^{n-1}(1) \\ B & \text{if } t_i < b^{n-1}(0) \end{cases}$$

where

$$b^n(k) = \begin{cases} k & \text{if } n = 0 \\ b(b^{n-1}(k)) & \text{if } n \geq 1 \end{cases}$$

Show this by induction. When $n = 1$, (21) implies that the player follows his dominant strategy, and hence the claim holds. Suppose the claim holds when $n = k \geq 1$. Since if $t_j > b^{k-1}(1)$, then $\sigma_j(t_j) = A$, for $i \neq j$, we have

$$P[\sigma_j = A|t_i] \geq P[s[b^{k-1}(1)]_j = A|t_i]$$

Hence, the expected payoff of taking A satisfies

$$t_i + P[\sigma_j = A|t_i] - 1 \geq \pi(t_i, b^{k-1}(1))$$

Here, if $t_i > b(b^{k-1}(1))$, we have $\pi(t_i, b^{k-1}(1)) > 0$ by the definition of $b(\cdot)$. Then, $\sigma_i(t_i) = A$.

On the other hand, since if $t_j < b^{k-1}(0)$, then $\sigma_j(t_j) = B$, for $i \neq j$, we have

$$P[\sigma_j = A|t_i] \leq P[s[b^{k-1}(0)]_j = A|t_i]$$

Hence, the expected payoff of taking A satisfies

$$t_i + P[\sigma_j = A|t_i] - 1 \leq \pi(t_i, b^{k-1}(0))$$

Here, if $t_i < b(b^{k-1}(0))$, then we have $\pi(t_i, b^{k-1}(0)) < 0$ by the definition of $b(\cdot)$. Then, $\sigma_i(t_i) = B$. Thus, we obtained

$$\sigma_i(t_i) = \begin{cases} A & \text{if } t_i > b^k(1) \\ B & \text{if } t_i < b^k(0) \end{cases}$$

Thus, (21) holds for any n.

Finally, show $b^n(1) \to 0.5, b^n(0) \to 0.5$ as $n \to \infty$. By property 2, for $k > 0.5$, we have $\pi(k,k) > \pi(0.5, 0.5) = 0$. Then, property 1 implies $b(k) < k$. In addition, by property 1, for $k > 0.5$, we have $\pi(0.5, k) \leq 0$. Hence, we obtain $0.5 \leq b(k)$. Therefore, $0.5 \leq b(k) < k$ for $k > 0.5$. In the same way, we have $k < b(k) \leq 0.5$ for $k < 0.5$. Notice that $b(\cdot)$ is nondecreasing by definition. Therefore, $b^n(0)$ is a nondecreasing function bounded above by 0.5, and $b^n(1)$ is a nonincreasing function bounded below by 0.5. Hence, both sequences converge. By the continuity, we have $b(k^*) = k^*$, but only $k^* = 0.5$ satisfies this equation.

■

The assumption on the probability distribution is used only when calculated f. Thus, the above proposition holds for any probability distribution.

Theorem 1. *Let $\theta, \epsilon_1, \epsilon_2$ be independent random variables, and suppose that ϵ_1 and ϵ_2 follows the same distribution. Assume the following conditions:*

[1] $t_i < 0$ and $t_i > 1$ has positive support.

[2] $\pi(t_i, k)$ satisfies the properties 1 and 2.

Let k^* be the solution to $\pi(k,k) = 0$. Then, the pair of k^*-threshold strategies is the unique equilibrium.

5 INDEX

p. 2

A_i: asset, $i = \{0, \cdot, N\}$.
A_0: cash.
E, S, L: face value of equity, short- and long-term debt.

θ_t: fundamental at date t.
σ_t: measure of uncertainty between date $t-1$ and t.
ϵ_t: error component of the fundamental at date t, which distributes according to F_i.
$\alpha_i + \beta_i\theta_2$: value of asset i at date 2 per unit face value.
$\alpha_0 = 1, \beta = 0$: cash is safe.
r_S, r_L: term-adjusted rate of return of short- and long-term debt, respectively.
θ^*: solvency point of θ_2.

p. 3

$\lambda_i(\alpha_i + \beta_i\theta_0)$: liquidate value of asset i per unit of face value, $\lambda_i \in [0, 1]$.
$\lambda_0 = 1$: cash is liquid.
A^*: total liquidation value of the bank asset at date 1.
r^*: rate of return of outside option for short-term debt holders.
q: illiquidity index.

p. 4

$\mathcal{INS}_1(\theta_1)$: interim solvency risk, i.e., the probability that the bank will fail if there is no run at date 1, conditional on θ_1.
θ^{**}: the run point of θ_1.
$\mathcal{ILL}_1(\theta_1)$: interim illiquidity risk, i.e., the probability that the bank will fail because of a run, when it would not have been insolvent in the absence of a run, conditional on θ_1.

p. 5

$\mathcal{C}_1(\theta_1)$: total interim credit risk, the sum of $\mathcal{ILL}_1$ and $\mathcal{INS}_1$.
$\mathcal{ILL}_0(\theta_0)$: ex ante insolvency risk.
$\mathcal{INS}_0$: ex ante illiquidity risk.

p. 9

x_i: private signal that each short-term debt holders receive.
u_i: noise in private signal. (written as ϵ_i in the paper).
y: mean of fundamental θ.
v_θ: variance of fundamental θ (written as α in the paper).
v_u: variance of noise u_i (written as β in the paper).
$\pi(x)$: the proportion of players whose signal is x or less.
$G(z|x_i)$: cumulative distribution of $\pi(x)$ conditional on x_i.