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A multi-agent incomplete equilibrium model and its applications to reinsurance pricing and life-cycle investment

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Abstract

This paper develops an incomplete equilibrium model with multi-agents' different risk attitudes and heterogeneous income/payout profiles. Particularly, we apply its concrete and computationally tractable model to reinsurance derivatives pricing and life-cycle investment, which are important for insurance and asset management companies in practice. In numerical experiments, we explicitly obtain endogenously determined expected returns of the risky asset in equilibrium, agents' specific reinsurance prices with their stochastic discount factors (SDF) and optimal life-cycle trading strategies. Moreover, we investigate how each agent's degree of risk aversion and income/payout profile, and correlations between an insurance or economic factor and the risky asset price affect reinsurance claims pricing and optimal portfolios in life-cycle investment.

Key words:
incomplete market; multi-agent model; equilibrium model; reinsurance pricing; life-cycle investment.
JEL classification codes:
D15; G11; G12; G13; G22.

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1 Introduction

This study proposes a multi-agent's incomplete equilibrium model, which characterizes an excess return process of a risky asset, each agent's optimal portfolio and stochastic discount factor (SDF), equivalently, an agent's specific risk-neutral probability measure in equilibrium. Moreover, when state processes are of square-root types, those variables are explicitly obtained, which are effectively applied to reinsurance claim pricing and life-cycle investment in numerical experiments.

Insurance companies manage large portfolios for their insurance payments, exposing them to risks that cannot be fully hedged with market instruments. Reinsurance is a tool used by insurance companies to hedge their insurance risk, and how to price the risk is an important issue. Since insurance and reinsurance companies face risks with heterogeneous risk preferences and income or payout profiles, those companies have different risk-neutral probability measures for contingent claim pricing. Then, it is not always possible for them to agree on the reinsurance claim pricing. Therefore, it is essential to price reinsurance claims under heterogeneous risk-neutral measures to investigate conditions when insurance and reinsurance companies can agree on the price to trade. Particularly, we show reinsurance pricing for the financial stop loss contract using a marginal pricing approach, where the pricing of reinsurance claims is done with given the equilibrium expected return process as a result of the original individual portfolio optimization.

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Also, on behalf of individuals who aim to optimize their financial portfolios with considering their lifetime income and payment, pension funds plan and offer products that suit individuals' life cycles, and optimally trade by taking into account the markets' and individuals' net income movement, which is also a central topic in the insurance industry.

Since the risks insurance companies face and changes in economic factors affecting individuals' income and payment cannot be completely hedged with tradable securities, pricing a contingent claim and investigating an optimal investment with such risks and factors are quite important problems. Moreover, as the expected return of a risky asset is a key element in pricing associated financial products and determining relevant asset allocations in portfolio management, considering an equilibrium model where the excess expected return process is endogenously determined seems also useful in practice.

Hence, the motivation of this study is to develop a computationally tractable equilibrium model to evaluate contingent claims and investigate optimal portfolios with an endogenously determined expected return process in an incomplete market setting with multi-agents' different risk attitudes and heterogeneous income/payout profiles. In detail, we incorporate the heterogeneity of agents into the modeling of equilibrium in an incomplete market, namely, risk aversion parameters of exponential utilities, income/payout profiles at maturity driven by a common economic factor through functional specific to respective agents, and individual income factors, which are sources of market incompleteness. Specifically, we solve the problem through a transformation of the probability measure, which enables us to reduce the dimension of the problems substantially. As examples including numerics, we present computationally tractable cases with the common and individual-specific factors following square-root processes.

The contribution of this study is to develop a concrete multi-agent equilibrium model in an incomplete market to investigate practically important problems for insurance and asset management companies. In particular, we apply the model to reinsurance claim pricing and life-cycle investment, extracting the essence from practical situations. In numerical experiments we explicitly obtain agents' specific reinsurance prices and optimal trading strategies with endogenously determined expected returns of a risky asset in equilibrium. Moreover, we investigate how each agent's degree of risk aversion and income/payout profile, and correlations between an insurance or economic factor and the risky asset price affect reinsurance claim pricing and optimal portfolios in life-cycle investment, whose implications agree with the insights from practice.

For related literature, Choi and Larsen (2015) develop exponential-quadratic models with the motivation of approximating a general class of incomplete Radner equilibrium models. Kizaki et al. (2022) consider a multi-agent equilibrium in an incomplete market with conservative views on Brownian motions by a backward-stochastic differential equation approach. These studies focus on equilibrium in the sense of the financial market that satisfies market clearing conditions. Another approach can be found in the game-theoretic literature, which examines the multi-agent equilibrium model. Xia (2004) investigated a cooperative investment game and provided a characterization of Pareto optimal cooperative strategies in incomplete markets. Bensoussan et al. (2014) addressed a non-zero-sum stochastic differential investment and reinsurance game between two insurance companies. They solved the game problem by applying the dynamic programming principle. Han et al. (2022) developed a dynamic model to study the effect of relative performance evaluation (RPE) in delegated portfolio management. They introduced a non-zero-sum game among managers, in addition to the hierarchical Stackelberg game between the shareholders and managers.

For reinsurance pricing, Becherer (2003) deals with an indifference pricing, and for the life-cycle investment, Henderson (2005) investigates an optimal investment problem with stochastic income, where those studies consider a single agent model with an exogenously given expected return of a risky asset. Our study is different from Becherer (2003) and Henderson (2005) in that we consider a multi-agent model in an incomplete market, where the expected return process is endogenously determined in equilibrium. Namely, we introduce insurance and reinsurance parties with different risk profiles before their contract's agreement, or a representative of individuals considering life-cycle investment and the other market participants, whose different optimization behaviors determine an equilibrium expected return of a risky asset and provide agents' specific reinsurance prices or optimal life-cycle investment strategies.

The organization of the paper is as follows: Section 2 introduces the setting of a multi-agent equilibrium model in an incomplete market, Section 3 solves the problem of each agent's optimal investment and an equilibrium expected return process of the risky asset. Section 4 investigates the case where state factors' processes are of square-root types and provide a computational procedure, which enables us to concretely obtain each agent's optimal trading strategy and the equilibrium expected return of the risk asset. By applying this method, Section 5 presents numerical examples for reinsurance pricing and life-cycle investments. Finally, Section 6 concludes. Appendix A discusses another example of a multi-agent equilibrium in an incomplete market setting in a log utility case.

2 Settings

2.1 Economy and financial market

Firstly, we describe the settings of the economy and financial market in this study.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, $[0, T]$ be the time horizon, and $W = (W_Y, W_S, W_1, \ldots, W_I)^\top$ be an $I+2$-dimensional standard Brownian motion, where $I \geq 2$. Let $\{\mathcal{F}_t\}_{0\leq t\leq T}$ be the augmented filtration generated by W. We consider an economy consisting of I agents endowed with income at maturity and the financial market, where there are two tradable assets, a money market account and a risky asset whose price processes are denoted by $\{B_t\}_{0\leq t\leq T}$ and $\{S_t\}_{0\leq t\leq T}$, respectively. Particularly, we assume $B_t \equiv 1$, which indicates the risk-free interest rate is 0 for simplicity in this study.

Let $Y_i,\ i = 1, \ldots, I$ be the individual income process of agent i, and Y be a factor process that drives the final income/payment. We assume that there are two types of exogenous state variables in this economy, a common factor process $\{Y_t\}_{0\leq t\leq T}$ and individual factor processes $\{Y_{i,t}\}_{0\leq t\leq T}$ $(i = 1, \ldots, I)$, and both the individual income $Y_{i,T}$ and the final income or payment $F_i(Y_T)$ are given at maturity T, where $F_i : \mathcal{R} \to \mathcal{R}$ are continuous functions.

We assume that Y, Y_i and S satisfy the following stochastic differential equations (SDEs)

$$dY_t = \mu_{Y,t}dt + \sigma_{Y,t}dW_{Y,t},\ Y_0 = y_0, \tag{1}$$

$$dY_{i,t} = \mu_{i,t}dt + \sigma_{i,t}(\rho_{i,t}dW_{Y,t} + \hat{\rho}_{i,t}dW_{i,t}),\ Y_{i,0} = y_{i,0}\ (i = 1, \ldots, I), \tag{2}$$

$$\frac{dS_t}{S_t} = \mu_{S,t}dt + \sigma_{S,t}(\rho_{S,t}dW_{Y,t} + \hat{\rho}_{S,t}dW_{S,t}),\ S_0 = s_0 > 0, \tag{3}$$

where $\mu_Y, \mu_i, \mu_S, \sigma_Y, \sigma_i, \sigma_s, \rho_i, \hat{\rho}_i, \rho_S, \hat{\rho}_S$ are $\{\mathcal{F}_t\}$-progressively measurable processes, the volatility coefficients $\sigma_{Y,t}, \sigma_{i,t}, \sigma_{s,i} > 0$, and the correlation coefficients, ρ_i, $\hat{\rho}_i$, ρ_S, and $\hat{\rho}_S$ satisfy $|\rho_{i,t}| \leq 1$, $|\hat{\rho}_i| \leq 1$ with $\hat{\rho}_{i,t} = \sqrt{1-\rho_{i,t}^2}$, $i = 1, \ldots, I$, $|\rho_{S,t}| \leq 1$, $|\hat{\rho}_{S,t}| \leq 1$ with $\hat{\rho}_{S,t} = \sqrt{1-\rho_{S,t}^2}$ for all $t \in [0, T]$. Also, $\int_0^T |\mu_{Y,s}|ds, \int_0^T |\mu_{i,s}|ds, \int_0^T |\mu_{S,s}|ds < \infty$, $\int_0^T |\sigma_{Y,s}|^2ds, \int_0^T |\sigma_{i,s}|^2ds, \int_0^T |\sigma_{S,s}|^2ds < \infty$, $i = 1, \ldots, I$, $\mathbf{P}-a.s.$

We further suppose that $\mu_Y, \mu_i, \sigma_Y, \sigma_i, \rho_i(\hat{\rho}_i), \rho_S(\hat{\rho}_S)$, $\theta := \frac{\mu_S}{\sigma_S}$ are functions of t and Y_t, which indicates that these coefficients are driven by the common factor Y and can be stochastic.

In the following, given the volatility process of the risky asset σ_S, we aim to obtain the representation of the expected return process μ_S, or equivalently the market price of risk θ in equilibrium where a market clearing condition is satisfied, which will be defined later.

2.2 Individual optimization problem and market equilibrium

Next, we introduce individual optimization problems of the agents and market equilibrium. Firstly, let $\{X_{i,t}^{(\pi_i)}\}_{0\leq t\leq T}$ be the wealth process of the i-th agent. We consider the situation where the i-th agent invests its wealth $X_{i,t}^{(\pi_i)}$, where the initial wealth is set to be 0 for simplicity, into the money market account and the risky asset over the time horizon $[0, T]$ and receives its income $Y_{i,T}$, which is inherent to the agent, and the income/payment $F_i(Y_T)$, which is driven by the common factor Y, at maturity T.

Next, let $\{\pi_{i,t}\}_{0\leq t\leq T}$ be a portfolio process of the i-th agent for the risky asset, which is $\mathcal{R}$-valued $\{\mathcal{F}_t\}$-progressively measurable satisfying $\int_0^T \pi_{i,s}^2 ds < \infty$, $\mathbf{P}-a.s.$ Here, we express the portfolio process $\pi_{i,t}$ in terms of the value of the risky asset position at t, the number of units the i-th agent holds times the risky asset price per unit, and assume the trading strategy to be self-financing. Since the risk-free interest rate is 0, the SDE of $X_i^{(\pi_i)}$ is as follows.

$$dX_{i,t}^{(\pi_i)} = \pi_{i,t}\mu_{S,t}dt + \pi_{i,t}\sigma_{S,t}(\rho_{S,t}dW_{Y,t} + \hat{\rho}_{S,t}dW_{S,t}),\ X_{i,0}^{(\pi_i)} = x_{i,0} = 0. \tag{4}$$

Particularly, in terms of the market price of risk $\theta_t = \frac{\mu_{S,t}}{\sigma_{S,t}}$, the SDE of $X_i^{(\pi_i)}$ is expressed as

$$dX_{i,t}^{(\pi_i)} = \pi_{i,t}\sigma_{S,t}\theta_t dt + \pi_{i,t}\sigma_{S,t}(\rho_{S,t}dW_{Y,t} + \hat{\rho}_{S,t}dW_{S,t}). \tag{5}$$

We suppose that each agent has an exponential utility $U_i(x) = -e^{-\gamma_i x}$ with its absolute risk aversion (ARA) parameter $0 < \gamma_i < \infty$ and set the market ARA parameter Γ as

$$\Gamma = \frac{1}{\sum_{i=1}^I \frac{1}{\gamma_i}}. \tag{6}$$

Then, we introduce the individual optimization problem of the i-th agent, $i = 1, \ldots, I$, as

$$\sup_{\pi_i\in\mathcal{A}_i} \mathbf{E}\left[U_i(X_{i,T}^{(\pi_i)} + Y_{i,T} - F_i(Y_T))\right] = \sup_{\pi_i\in\mathcal{A}_i} \mathbf{E}\left[-e^{-\gamma_i(X_{i,T}^{(\pi_i)}+Y_{i,T}-F_i(Y_T))}\right], \tag{7}$$

where $F_i : \mathcal{R} \to \mathcal{R}$ is continuous and $\mathcal{A}_i$ is a set of admissible portfolio strategies defined below.

Definition 1 *For the i-th agent, a portfolio strategy $\pi_i \in \mathcal{R}$ is admissible if $X_i^{(\pi_i)}$ is a supermartingale under $\mathbf{Q}_i$, where $\mathbf{Q}_i$ is a probability measure equivalent to $\mathbf{P}$ such that the risky asset price process S is a local martingale under $\mathbf{Q}_i$, which will be specified in each individual optimization problem.*

We denote the set of i-th agent's all admissible strategies as $\mathcal{A}_i$.

Here, we define an admissible strategy such that the wealth process is a supermartingale, which ensures that an arbitrage opportunity is excluded. In detail, if a strategy π_i is an arbitrage strategy, the corresponding wealth process $X_i^{(\pi_i)}$ is not a supermartingale under $\mathbf{Q}_i$. Since $X_{i,0}^{(\pi_i)} = 0$ and $X_{i,T}^{(\pi_i)} \geq 0$ is strictly positive with a positive probability under $\mathbf{P}$, and $\mathbf{Q}_i$ is equivalent to $\mathbf{P}$, $X_{i,T}^{(\pi_i)} \geq 0$ is also strictly positive with a positive probability under $\mathbf{Q}_i$, leading to $\mathbf{E}^{\mathbf{Q}_i}[X_{i,T}^{(\pi_i)}] > 0$ which indicates $X_i^{(\pi_i)}$ is not a supermartingale under $\mathbf{Q}_i$.

We note that the expectation in (7) is well-defined in the sense that it is bounded by zero and may take the value $-\infty$. This individual optimization problem indicates that the i-th agent aims to maximize its expected utility on the total amount of its wealth, the income, and the final payment at maturity. Then, we aim to obtain a representation of the instantaneous Sharpe ratio θ in equilibrium, where we define the market equilibrium as follows.

Definition 2 *We call the financial market is in an equilibrium if (i)for each $i = 1, \dots, I$, $\pi_i^* \in \mathcal{A}_i$ attains the supremum in the individual optimization problems (7), and (ii) the following market clearing conditions for the risky asset and the money market are satisfied.*

$$\sum_{i=1}^{I} \pi_{i,t}^* = 0, \ \sum_{i=1}^{I} (X_{i,t}^{(\pi_i^*)} - \pi_{i,t}^*) = 0, \tag{8}$$

for all $t \in [0, T]$.

First of all, the first equation indicates that the total of the risky asset positions among the agents is zero and the second equation shows that the total of the money market position is also zero, where $\pi_{i,t}^*$ is a risky asset position of agent i and $X_{i,t}^{(\pi_i^*)} - \pi_{i,t}^*$, which is a difference between the wealth and the risky asset position, is a money market position of agent i. We remark that we assume a net zero position for the risky asset and the money market, meaning net zero supply in the pure exchange economy. This assumption is reasonable when opposing interests exist among agents. In the log utility case discussed in Appendix A, we consider a case where a nonzero dividend at maturity is provided, which complies with the positive wealth restriction in the log utility. Extending the exponential utility model to the one with a net nonzero supply of the risky asset is a future research topic.

3 Solving individual optimization problems and equilibrium by an HJB method

In this section, we obtain the optimal portfolio processes π_i^*, $i = 1, \dots, I$ and the Sharpe ratio θ in equilibrium by an HJB approach. Specifically, after providing the transformation of probability measures to solve the individual optimization problems, we show that given the Sharpe ratio in equilibrium, the optimal portfolio processes attain the supremum in each individual optimization problem, then show that the market clearing conditions are satisfied.

Hereafter, we assume the following.

Assumption 1 *The local martingale $Z^{(i,Y)}$ defined as*

$$Z_t^{(i,Y)} = \exp\left(-\frac{\gamma_i^2}{2} \int_0^t \sigma_{i,s}^2 ds - \gamma_i \int_0^t \sigma_{i,s} \hat{\rho}_{i,s} dW_{i,s} - \gamma_i \int_0^t \rho_{i,s} \sigma_{i,s} dW_{Y,s} \right), \tag{9}$$

is a martingale under $\mathbf{P}$.

Remark 1 *When σ_i is nonrandom and $\int_0^T \sigma_{i,s}^2 ds < \infty$, since $\hat{\rho}_{i,s}^2 + \rho_{i,s}^2 = 1$, Novikov's condition is satisfied and Assumption 1 holds. When σ_i is stochastic, we need to check if Assumption 1 holds, which depends on the form of σ_i. When σ_i is stochastic, Assumption 1 holds if a weak version of Novikov's condition (e.g., Corollary 3.5.14 in Karatzas and Shreve (2012)) is satisfied, for example; there exists a partition of $[0, T]$, $0 = t_0 < t_1 < \cdots < t_N = T$, such that*

$$\mathbf{E}\left[\exp\left(\frac{1}{2} \int_{t_{n-1}}^{t_n} \gamma_i^2 \sigma_{i,s}^2 ds \right) \right] < \infty,$$
$$\text{for all } 1 \leq n \leq N. \tag{10}$$

We remark that as mentioned in Section 4.1, the weak version of Novikov's condition is used to confirm that Assumption 1 holds in the square-root case, where the condition is satisfied by Theorem 3.2 of Shirakawa (2002).

Assumption 2 *For μ_i and σ_i, $i = 1, \dots, I$, there exists a nonrandom process c_i such that*

$$\gamma_i \mu_{i,t} - \frac{\gamma_i^2}{2} \sigma_{i,t}^2 = c_i(t), \ \forall t \in [0, T]. \tag{11}$$

This indicates that we assume μ_i to be of the form $\frac{1}{\gamma_i}c_i(t) + \frac{\gamma_i}{2}\sigma_{i,t}^2$, which enables us to simplify the individual optimization problem that includes Y_i, the individual income of the agent i at maturity, by transformation of a probability measure in the next section.

3.1 Transformation of a probability measure

Firstly, to simplify the individual optimization problem (7), by Assumption 1, we define a probability measure $\mathbf{P}^{(i,Y)}$ by $\frac{d\mathbf{P}^{(i,Y)}}{d\mathbf{P}} = Z_T^{(i,Y)}$, which corresponds to measure transformation based on each agent's income profile.
Noting that

$$d(\gamma_i Y_{i,t}) = \gamma_i \mu_{i,t} dt + \gamma_i \sigma_{i,t}(\rho_{i,t} dW_{Y,t} + \hat{\rho}_{i,t} dW_{i,t}), \tag{12}$$

and

$$e^{-\gamma_i(X_{i,T}^{(\pi_i)} + Y_{i,T} - F_i(Y_T))} = e^{-\gamma_i(X_{i,T}^{(\pi_i)} - F_i(Y_T))} e^{-\int_0^T \left(\gamma_i \mu_{i,s} - \frac{\gamma_i^2}{2}\sigma_{i,s}^2\right) ds} Z_T^{(i,Y)},$$

by Assumption 2, we have

$$\mathbf{E}\left[-e^{-\gamma_i(X_{i,T}^{(\pi_i)} + Y_{i,T} - F_i(Y_T))}\right] = \mathbf{E}^{(i,Y)}\left[-e^{-\gamma_i(X_{i,T}^{(\pi_i)} - F_i(Y_T))}\right] e^{-\int_0^T c_i(s)ds}. \tag{13}$$

Also, by Girsanov's theorem, $(W_{Y,t}^{(i,Y)}, W_{S,t}^{(i,Y)})$ defined by

$$dW_{Y,t}^{(i,Y)} = dW_{Y,t} + \gamma_i \rho_{i,t}\sigma_{i,t} dt, dW_{S,t}^{(i,Y)} = dW_{S,t}, \tag{14}$$

is a two-dimensional Brownian motion under $\mathbf{P}^{(i,Y)}$.
Thus, the individual optimization problem (7) becomes

$$\sup_{\pi_i \in \mathcal{A}_i} \mathbf{E}^{(i,Y)}\left[-e^{-\gamma_i(X_{i,T}^{(\pi_i)} - F_i(Y_T))}\right], \tag{15}$$

where

$$dY_t = \hat{\mu}_{Y,t}^i dt + \sigma_{Y,t} dW_{Y,t}^{(i,Y)}, \tag{16}$$

$$\frac{dS_t}{S_t} = \sigma_{S,t}\hat{\mu}_{S,t}^i dt + \sigma_{S,t}(\rho_{S,t} dW_{Y,t}^{(i,Y)} + \hat{\rho}_{S,t} dW_{S,t}^{(i,Y)}), \tag{17}$$

$$dX_{i,t}^{(\pi_i)} = \pi_{i,t}\sigma_{S,t}\hat{\mu}_{S,t}^i dt + \pi_{i,t}\sigma_{S,t}(\rho_{S,t} dW_{Y,t}^{(i,Y)} + \hat{\rho}_{S,t} dW_{S,t}^{(i,Y)}). \tag{18}$$

Here, we set

$$\hat{\mu}_{Y,t}^i = \mu_{Y,t} - \gamma_i \rho_{i,t}\sigma_{i,t}\sigma_{Y,t}, \tag{19}$$

$$\hat{\mu}_{S,t}^i = \theta_t - \gamma_i \rho_{S,t}\rho_{i,t}\sigma_{i,t}. \tag{20}$$

3.2 Candidate of optimal trading strategies by an HJB method

Next, we define the value function $V_i(x, y, t)$ for (15) as

$$V_i(x,y,t) = \sup_{\pi_i \in \mathcal{A}_i} \mathbf{E}^{(i,Y)}\left[-e^{-\gamma_i(X_{i,T}^{(\pi_i)} - F_i(Y_T))} | X_{i,t}^{(\pi_i)} = x, Y_t = y\right]. \tag{21}$$

First noting that

$$dY_t = \hat{\mu}_{Y,t}^i dt + \sigma_{Y,t} dW_{Y,t}^{(i,Y)}, \tag{22}$$

$$dX_{i,t}^{(\pi_i)} = \pi_{i,t}\sigma_{S,t}\hat{\mu}_{S,t}^i dt + \pi_{i,t}\sigma_{S,t}(\rho_{S,t} dW_{Y,t}^{(i,Y)} + \hat{\rho}_{S,t} dW_{S,t}^{(i,Y)}), \tag{23}$$

we obtain the following HJB equation:

$$\sup_{p \in \mathcal{R}}\left[\frac{p^2\sigma_S^2}{2}(\partial_{xx}V_i) + \frac{\sigma_Y^2}{2}(\partial_{yy}V_i) + \rho_S p\sigma_S\sigma_Y(\partial_{xy}V_i) + p\sigma\hat{\mu}_S^i(\partial_x V_i) + \hat{\mu}_Y^i(\partial_y V_i) + \partial_t V_i\right] = 0. \tag{24}$$

Then, a candidate of optimal portfolio is given by the following lemma.

Lemma 1 *Suppose that there exist $f_i : \mathcal{R} \times [0, T] \to \mathcal{R},\ i = 1, \ldots, I$ of class C^2 for y and class C^1 for t, satisfying a system of partial differential equations (PDEs)*

$$\frac{\sigma_Y^2}{2}(\partial_{yy}f_i) + \hat{\rho}_S^2\frac{\gamma_i\sigma_Y^2}{2}(\partial_y f_i)^2 + \left[(\mu_Y - \gamma_i\hat{\rho}_S^2\rho_i\sigma_i\sigma_Y) - \rho_S\sigma_Y\theta\right](\partial_y f_i) + \partial_t f_i - \frac{1}{2\gamma_i}\left[\theta - \gamma_i\rho_S\rho_i\sigma_i\right]^2 = 0, \tag{25}$$

with terminal conditions

$$f_i(y,T) = F_i(y),\ i = 1, \ldots, T. \tag{26}$$

Let

$$V_i(x,y,t) = -\exp\left(-\gamma_i(x - f_i(y,t))\right). \tag{27}$$

V_i in (27) is a solution of HJB equation (24). Particularly,

$$p^*_{i,t} = \frac{1}{\sigma_{S,t}}\left(\frac{\hat{\mu}^i_{S,t}}{\gamma_i} + \rho_{S,t}\sigma_{Y,t}(\partial_y f_i)\right) = \frac{1}{\sigma_{S,t}}\left(\frac{\theta(y,t) - \gamma_i\rho_{i,t}\sigma_{i,t}\rho_{S,t}}{\gamma_i} + \rho_{S,t}\sigma_{Y,t}(\partial_y f_i)\right), \tag{28}$$

attains the supremum in (24).

Proof. Noting that each partial derivatives of V_i are given by

$$\partial_x V_i = -\gamma_i V_i,\ \partial_{xx} V_i = \gamma_i^2 V_i, \tag{29}$$

$$\partial_y V_i = \gamma_i V_i(\partial_y f_i),\ \partial_{yy} V_i = \gamma_i^2 V_i(\partial_y f_i)^2 + \gamma_i V_i(\partial_{yy} f_i), \tag{30}$$

$$\partial_{xy} V_i = -\gamma_i^2 V_i(\partial_y f_i), \tag{31}$$

$$\partial_t V_i = \gamma_i V_i(\partial_t f_i), \tag{32}$$

and $-\gamma_i V_i > 0$, we substitute these partial derivatives into the left hand side of the HJB equation (24) and obtain

$$-\gamma_i V_i \times \sup_{p\in\mathcal{A}_i}\left[-\frac{\gamma_i p^2\sigma_S^2}{2} - \frac{\sigma_Y^2}{2}((\partial_{yy} f_i) + \gamma_i(\partial_y f_i)^2) + \gamma_i\rho_S p\sigma_S\sigma_Y(\partial_y f_i) + p\sigma_S\hat{\mu}^i_S - \hat{\mu}^i_Y(\partial_y f_i) - \partial_t f_i\right]. \tag{33}$$

Since this is a quadratic function of p and the first order condition with respect to p becomes

$$-\gamma_i\sigma_S^2 p + \gamma_i\rho_S\sigma_S\sigma_Y(\partial_y f_i) + \sigma_S\hat{\mu}^i_S = 0, \tag{34}$$

the supremum is attained at $p^*_{i,t} = \frac{1}{\sigma_{S,t}}\left(\frac{\hat{\mu}^i_{S,t}}{\gamma_i} + \rho_{S,t}\sigma_{Y,t}(\partial_y f_i)\right)$. Then, we calculate

$$\begin{aligned} -\frac{\gamma_i\sigma_S^2(p^*_i)^2}{2} &= -\frac{\gamma_i\sigma_S^2}{2}\frac{1}{\sigma_S^2}\left(\frac{(\hat{\mu}^i_S)^2}{\gamma_i^2} + \rho_S^2\sigma_Y^2(\partial_y f_i)^2 + 2\frac{\hat{\mu}^i_S\rho_S\sigma_Y}{\gamma_i}(\partial_y f_i)\right) \\ &= -\frac{(\hat{\mu}^i_S)^2}{2\gamma_i} - \frac{\gamma_i\rho_S^2\sigma_Y^2}{2}(\partial_y f_i)^2 - \rho_S\sigma_Y\hat{\mu}^i_S(\partial_y f_i), \end{aligned} \tag{35}$$

$$\gamma_i\rho_S p^*_i\sigma_S\sigma_Y(\partial_y f_i) = \rho_S\sigma_Y\hat{\mu}^i_S(\partial_y f_i) + \gamma_i\rho_S^2\sigma_Y^2(\partial_y f_i)^2, \tag{36}$$

$$p^*_i\sigma_S\hat{\mu}^i_S = \frac{(\hat{\mu}^i_S)^2}{\gamma_i} + \rho_S\sigma_Y\hat{\mu}^i_S(\partial_y f_i), \tag{37}$$

the sup part in (33) becomes

$$\begin{aligned} &-\frac{(\hat{\mu}^i_S)^2}{2\gamma_i} - \frac{\gamma_i\rho_S^2\sigma_Y^2}{2}(\partial_y f_i)^2 - \rho_S\sigma_Y\hat{\mu}^i_S(\partial_y f_i) - \frac{\sigma_Y^2}{2}((\partial_{yy} f_i) + \gamma_i(\partial_y f_i)^2) \\ &+ \rho_S\sigma_Y\hat{\mu}^i_S(\partial_y f_i) + \gamma_i\rho_S^2\sigma_Y^2(\partial_y f_i)^2 + \frac{(\hat{\mu}^i_S)^2}{\gamma_i} + \rho_S\sigma_Y\hat{\mu}^i_S(\partial_y f_i) - \hat{\mu}^i_Y(\partial_y f_i) - \partial_t f_i. \end{aligned} \tag{38}$$

Rearranging this, we have

$$\frac{\sigma_Y^2}{2}(\partial_{yy} f_i) + \hat{\rho}_S^2\frac{\gamma_i\sigma_Y^2}{2}(\partial_y f_i)^2 + \left[\hat{\mu}^i_Y - \rho_S\sigma_Y\hat{\mu}^i_S\right](\partial_y f_i) + \partial_t f_i - \frac{(\hat{\mu}^i_S)^2}{2\gamma_i}, \tag{39}$$

where we used $\rho_S^2 + \hat{\rho}_S^2 = 1$.

Since

$$\hat{\mu}^i_Y = \mu_Y - \gamma_i\rho_i\sigma_i\sigma_Y, \tag{40}$$

$$\hat{\mu}^i_S = \theta - \gamma_i\rho_i\sigma_i\rho_S, \tag{41}$$

(39) is rewritten as

$$\frac{\sigma_Y^2}{2}(\partial_{yy} f_i) + \hat{\rho}_S^2\frac{\gamma_i\sigma_Y^2}{2}(\partial_y f_i)^2 + \left[(\mu_Y - \gamma_i\hat{\rho}_S^2\rho_i\sigma_i\sigma_Y) - \rho_S\sigma_Y\theta\right](\partial_y f_i) + \partial_t f_i - \frac{1}{2\gamma_i}\left[\theta - \gamma_i\rho_S\rho_i\sigma_i\right]^2, \tag{42}$$

which is 0 by (25). □

3.3 Verification of the optimality

As we have observed in Section 3.2, the candidate for the optimal trading strategy is given by

$$\pi^*_{i,t} = p^*_i(Y_t, t) = \frac{1}{\sigma_{S,t}}\left(\frac{\theta_t - \gamma_i\rho_{i,t}\sigma_{i,t}\rho_{S,t}}{\gamma_i} + \rho_{S,t}\sigma_{Y,t}(\partial_y f_i)\right). \tag{43}$$

Then, as we will observe in a square-root case in Section 4, it easily follows that if $\{\eta_{i,t}\}_{0\le t\le T}$ set as

$$\eta_{i,t} = \frac{Z^{(i,Y)}_t V_i(X^{(\pi^*_i)}_{i,t}, Y_t, t)}{V_i(x_{i,0}, y_0, 0)},\ \ 0 \le t \le T, \tag{44}$$

is a martingale under $\mathbf{P}$, $X^{(\pi^*_i)}_i$ is a local martingale under $\mathbf{Q}_i$, where $\mathbf{Q}_i$ is defined by $\frac{d\mathbf{Q}_i}{d\mathbf{P}} = \eta_{i,T}$.

Moreover, if $X^{(\pi^*_i)}_i$ is a martingale under $\mathbf{Q}_i$, we confirm that π^*_i is the optimal trading strategy for a given market

price of risk $\theta_t = \theta(Y_t, t)$.
Furthermore, if the market price of risk θ is set so that the clearing conditions (8) are satisfied for $\pi_i^*,\ i = 1, \ldots, I$ in (43), we can confirm that the financial market is in equilibrium, which is summarized as follows.

Theorem 1 *Suppose that there exist $f_i : \mathcal{R} \times [0,T] \to \mathcal{R},\ i = 1, \ldots, I$ of class C^2 for y and class C^1 for t, satisfying a system of partial differential equations (PDEs)*

$$\frac{\sigma_Y^2}{2}(\partial_{yy} f_i) + \hat{\rho}_S^2 \frac{\gamma_i \sigma_Y^2}{2}(\partial_y f_i)^2 + \left[(\mu_Y - \gamma_i \hat{\rho}_S^2 \rho_i \sigma_i \sigma_Y) - \rho_S \sigma_Y \theta\right](\partial_y f_i) + \partial_t f_i - \frac{1}{2\gamma_i}\left[\theta - \gamma_i \rho_S \rho_i \sigma_i\right]^2 = 0, \tag{45}$$

with terminal conditions

$$f_i(y,T) = F_i(y),\ i = 1, \ldots, I, \tag{46}$$

where

$$\theta(y,t) = -\Gamma \cdot \rho_{S,t} \cdot \sum_{k=1}^{I}\left((\sigma_{Y,t}\partial_y f_k) - \rho_{k,t}\sigma_{k,t}\right). \tag{47}$$

Suppose that $\{\eta_{i,t}\}_{0 \le t \le T}$ in (44) with $V_i(x,y,t) = -\exp(-\gamma_i(x - f_i(y,t)))$ is a martingale under $\mathbf{P}$ and $X_i^{(\pi_i^*)}$ is a martingale under $\mathbf{Q}_i$, where $\mathbf{Q}_i$ is defined by $\frac{d\mathbf{Q}_i}{d\mathbf{P}} = \eta_{i,T}$.
Then, the Sharpe ratio and the optimal trading strategies in equilibrium are given by

$$\theta_t = \theta(Y_t, t) = -\Gamma \cdot \rho_{S,t} \cdot \sum_{k=1}^{I}\left((\sigma_{Y,t}\partial_y f_k(Y_t,t)) - \rho_{k,t}\sigma_{k,t}\right), \tag{48}$$

and

$$\pi_{i,t}^* = \frac{1}{\sigma_{S,t}}\left(\frac{\theta_t - \gamma_i \rho_{i,t}\sigma_{i,t}\rho_{S,t}}{\gamma_i} + \rho_{S,t}\sigma_{Y,t}(\partial_y f_i(Y_t,t))\right). \tag{49}$$

Proof. Firstly, we show that π_i^* attains supremum of each agent's expected utility in (15).

Lemma 2 *$\pi_i^* \in \mathcal{A}_i$ attains supremum of the expected utility for agent i in (15).*

Proof. First, we set the convex conjugate of U_i as $\tilde{U}_i$,

$$\tilde{U}_i(y) = \sup_{x \in \mathcal{R}}(U_i(x) - xy),\ \text{for } y > 0. \tag{50}$$

Then, it holds that

$$U_i(x) \le \tilde{U}_i(y) + xy, \tag{51}$$

$$U_i(x) = \tilde{U}_i(U_i'(x)) + xU_i'(x), \tag{52}$$

for any $x \in \mathcal{R}$ and $y > 0$.
For any $\pi_i \in \mathcal{A}_i$, let us set $x = X_{i,T}^{(\pi_i)} - F_i(Y_T)$ and $y = \mathbf{E}^{(i,Y)}[U_i'(X_{i,T}^{(\pi_i^*)} - F_i(Y_T))]\frac{d\mathbf{Q}_i}{d\mathbf{P}^{(i,Y)}}$ in (51).
Then, we have

$$\begin{aligned}
&\mathbf{E}^{(i,Y)}[U_i(X_{i,T}^{(\pi_i)} - F_i(Y_T))] \\
&\le \mathbf{E}^{(i,Y)}\left[\tilde{U}_i\left(\mathbf{E}^{(i,Y)}[U_i'(X_{i,T}^{(\pi_i^*)} - F_i(Y_T))]\frac{d\mathbf{Q}_i}{d\mathbf{P}^{(i,Y)}}\right)\right] + \mathbf{E}^{(i,Y)}[U_i'(X_{i,T}^{(\pi_i^*)} - F_i(Y_T))]\mathbf{E}^{(i,Y)}\left[\frac{d\mathbf{Q}_i}{d\mathbf{P}^{(i,Y)}}(X_{i,T}^{(\pi_i)} - F_i(Y_T))\right] \\
&\le \mathbf{E}^{(i,Y)}\left[\tilde{U}_i\left(\mathbf{E}^{(i,Y)}[U_i'(X_{i,T}^{(\pi_i^*)} - F_i(Y_T))]\frac{d\mathbf{Q}_i}{d\mathbf{P}^{(i,Y)}}\right)\right] + \mathbf{E}^{(i,Y)}[U_i'(X_{i,T}^{(\pi_i^*)} - F_i(Y_T))]\mathbf{E}^{(i,Y)}\left[\frac{d\mathbf{Q}_i}{d\mathbf{P}^{(i,Y)}}(x_{i,0} - F_i(Y_T))\right] \\
&= \mathbf{E}^{(i,Y)}\left[\tilde{U}_i\left(\mathbf{E}^{(i,Y)}[U_i'(X_{i,T}^{(\pi_i^*)} - F_i(Y_T))]\frac{d\mathbf{Q}_i}{d\mathbf{P}^{(i,Y)}}\right)\right] + \mathbf{E}^{(i,Y)}[U_i'(X_{i,T}^{(\pi_i^*)} - F_i(Y_T))]\mathbf{E}^{(i,Y)}\left[\frac{d\mathbf{Q}_i}{d\mathbf{P}^{(i,Y)}}(X_{i,T}^{(\pi_i^*)} - F_i(Y_T))\right] \\
&= \mathbf{E}^{(i,Y)}[U_i(X_{i,T}^{(\pi_i^*)} - F_i(Y_T))].
\end{aligned} \tag{53}$$

The second inequality and the first equality follow from the fact that $X_i^{(\pi_i)}$ is a supermartingale and $X_i^{(\pi_i^*)}$ is a martingale in $\mathbf{Q}_i$, respectively. In the last equality, we used (52) and

$$\frac{d\mathbf{Q}_i}{d\mathbf{P}^{(i,Y)}} = \frac{d\mathbf{Q}_i}{d\mathbf{P}}\left(\frac{d\mathbf{P}^{(i,Y)}}{d\mathbf{P}}\right)^{-1} = \frac{V_i(X_{i,T}^{(\pi_i^*)}, Y_T, T)}{V_i(x_{i,0}, y_0, 0)} = \frac{U_i'(X_{i,T}^{(\pi_i^*)} - F_i(Y_T))}{\mathbf{E}^{(i,Y)}[U_i'(X_{i,T}^{(\pi_i^*)} - F_i(Y_T))]}. \tag{54}$$

This shows that π_i^* attains supremum the expected utility. □

Secondly, we show that $\pi_i^*,\ i = 1, \ldots, I$ satisfy the market clearing condition with the instantaneous Sharpe ratio θ

in (48). Substituting (49) into the market clearing condition in (8), we have

$$\sum_{i=1}^{I}\pi_{i,t}^{*}=\frac{1}{\sigma_{S,t}}\left(\theta_t\sum_{i=1}^{I}\frac{1}{\gamma_i}-\rho_{S,t}\sum_{i=1}^{I}\rho_{i,t}\sigma_{i,t}+\rho_{S,t}\sigma_{Y,t}\sum_{i=1}^{I}(\partial_y f_i)\right)=0. \tag{55}$$

This completes the proof of Theorem 1. $\square$

Remark 2 *We can also confirm that $X_i^{(\pi_i)}$ and S are local martingales under $\mathbf{Q}_i$ for a general factor process Y and income processes $Y_i, i = 1\ldots, I$ satisfying Assumptions 1 and 2 as follows.*

Suppose that $\{\eta_{i,t}\}_{0\le t\le T}$ in $\eta_{i,t}=\frac{Z_t^{(i,Y)}V_i(X_{i,t}^{(\pi_i^*)},Y_t,t)}{V_i(x_{i,0},y_0,0)}$, $0\le t\le T$, with $V_i(x,y,t)=-\exp\left(-\gamma_i(x-f_i(y,t))\right)$ is a martingale under $\mathbf{P}$.
By Ito's formula, $d\eta_i$ is expressed as

$$d\eta_{i,t}=\eta_{i,t}\left(\frac{\partial_x V_i}{V_i}\pi_{i,t}^{*}\sigma_{S,t}\hat{\rho}_{S,t}dW_{S,t}+\left(\frac{\partial_x V_i}{V_i}\pi_{i,t}^{*}\sigma_{S,t}\rho_{S,t}+\frac{\partial_y V_i}{V_i}\sigma_{Y,t}-\gamma_i\sigma_{i,t}\rho_{i,t}\right)dW_{Y,t}\right). \tag{56}$$

Then, by Girsanov's theorem, $W_Y^{\mathbf{Q}_i}$ and $W_S^{\mathbf{Q}_i}$ defined by

$$\begin{aligned}dW_{Y,t}^{\mathbf{Q}_i}&=dW_{Y,t}-\left(\frac{\partial_x V_i}{V_i}\pi_{i,t}^{*}\sigma_{S,t}\rho_{S,t}+\frac{\partial_y V_i}{V_i}\sigma_{Y,t}-\gamma_i\sigma_{i,t}\rho_{i,t}\right)dt,\\ dW_{S,t}^{\mathbf{Q}_i}&=dW_{S,t}-\frac{\partial_x V_i}{V_i}\pi_{i,t}^{*}\sigma_{S,t}\hat{\rho}_{S,t}dt,\end{aligned} \tag{57}$$

are Brownian motions under $\mathbf{Q}_i$, and thus

$$\begin{aligned}\rho_{S,t}dW_{Y,t}^{\mathbf{Q}_i}+\hat{\rho}_{S,t}dW_{S,t}^{\mathbf{Q}_i}&=\rho_{S,t}dW_{Y,t}+\hat{\rho}_{S,t}dW_{S,t}-\left(\frac{\partial_x V_i}{V_i}\pi_{i,t}^{*}\sigma_{S,t}+\frac{\partial_y V_i}{V_i}\sigma_{Y,t}\rho_{S,t}-\gamma_i\sigma_{i,t}\rho_{i,t}\rho_{S,t}\right)dt\\ &=\rho_{S,t}dW_{Y,t}+\hat{\rho}_{S,t}dW_{S,t}+(\gamma_i\pi_{i,t}^{*}\sigma_{S,t}-\gamma_i\partial_y f_i\sigma_{Y,t}\rho_{S,t}+\gamma_i\sigma_{i,t}\rho_{i,t}\rho_{S,t})dt\\ &=\rho_{S,t}dW_{Y,t}+\hat{\rho}_{S,t}dW_{S,t}+\theta_t dt,\end{aligned} \tag{58}$$

where we used the relation between $\pi_{i,t}^{*}$ and θ_t in (49).
Hence, we have

$$\begin{aligned}dX_{i,t}^{(\pi_i)}&=\pi_{i,t}\sigma_{S,t}\theta_t dt+\pi_{i,t}\sigma_{S,t}(\rho_{S,t}dW_{Y,t}+\hat{\rho}_{S,t}dW_{S,t}).\\ &=\pi_{i,t}\sigma_{S,t}(\rho_{S,t}dW_{Y,t}^{\mathbf{Q}_i}+\hat{\rho}_{S,t}dW_{S,t}^{\mathbf{Q}_i}),\\ \frac{dS_t}{S_t}&=\mu_{S,t}dt+\sigma_{S,t}(\rho_{S,t}dW_{Y,t}+\hat{\rho}_{S,t}dW_{S,t})=\sigma_{S,t}(\theta_t dt+\rho_{S,t}dW_{Y,t}+\hat{\rho}_{S,t}dW_{S,t})\\ &=\sigma_{S,t}(\rho_{S,t}dW_{Y,t}^{\mathbf{Q}_i}+\hat{\rho}_{S,t}dW_{S,t}^{\mathbf{Q}_i}).\end{aligned} \tag{59}$$

Therefore, $X_i^{(\pi_i)}$ and S are local martingales under $\mathbf{Q}_i$.
We can interpret the optimal trading strategy of the i-th agent

$$\pi_{i,t}^{*}=\frac{1}{\sigma_{S,t}}\left(\frac{\theta_t-\gamma_i\rho_{i,t}\sigma_{i,t}\rho_{S,t}}{\gamma_i}+\rho_{S,t}\sigma_{Y,t}(\partial_y f_i(Y_t,t))\right), \tag{60}$$

and the Sharpe ratio in equilibrium

$$\theta_t=-\Gamma\cdot\rho_{S,t}\cdot\sum_{k=1}^{I}\left((\sigma_{Y,t}\partial_y f_k(Y_t,t))-\rho_{k,t}\sigma_{k,t}\right), \tag{61}$$

which yields the expected return $\mu_{S,t}=\theta_t\sigma_{S,t}$:

$$\mu_{S,t}=-\Gamma\cdot(\rho_{S,t}\sigma_{S,t})\cdot\sum_{k=1}^{I}\left((\sigma_{Y,t}\partial_y f_k(Y_t,t))-\rho_{k,t}\sigma_{k,t}\right), \tag{62}$$

as follows.

First, the optimal trading strategy of the i-th agent (60) consists of the mean-variance term $\theta_t/(\gamma_i\sigma_{S,t})$, proportional to the market price of risk θ, and the hedging term $\rho_{S,t}\sigma_{Y,t}(\partial_y f_i(Y_t,t))/\sigma_{S,t}-\rho_{i,t}\sigma_{i,t}\rho_{S,t}/\sigma_{S,t}$.

The hedging term describes the demand for hedging risks arising from the movement of Y and Y_i. As an interpretation, suppose $(\partial_y f_i)>0$ and $\rho_S>0$ for instance, noting that $\partial_y V_i=\gamma_i V_i(\partial_y f_i)$ and $V_i<0$, the value function V_i decreases when the economic factor Y increases. Then, since the risky asset price S is positively correlated with Y, the long position of the risky asset, $\rho_{S,t}\sigma_{Y,t}(\partial_y f_i(Y_t,t))/\sigma_{S,t}$ can have a hedging effect on the risk arising from the movement of the economic factor Y.

Moreover, since the individual income $Y_{i,T}$ is included in the agent's expected utility in (7) and the risky asset price S

is positively correlated with the income process Y_i as $\rho_i\rho_S > 0$, the short position of the risky asset, $-\rho_{i,t}\sigma_{i,t}\rho_{S,t}/\sigma_{S,t}$ has a hedging effect on the risk arising from Y_i. The other cases can be explained in the same manner.

Next, the Sharpe ratio θ in equilibrium in (61) indicates that the market price of risk is set so that it offsets the aggregate hedging demand. For example, if $\rho_{S,t}\sum_{k=1}^{I}\left((\sigma_{Y,t}\partial_y f_k(Y_t,t)) - \rho_{k,t}\sigma_{k,t}\right) > 0$, which indicates excess aggregate hedging demand, then θ is negative, which implies that the excess return process μ_S is negative and the selling demand in the mean-variance term increases.

4 Square-root case

4.1 Settings and result

In this section, we introduce a square-root model below as a specific case of the settings (1)-(3) in Section 2, which will be used in the numerical examples in Section 5. We assume that $\{\mathcal{F}_t\}$-progressively measurable processes Y, $Y_i,\ i = 1,\dots,I$, S satisfy the following SDEs,

$$dY_t = (\mu_Y^1 Y_t + \mu_Y^2)dt + \bar{\sigma}_Y\sqrt{Y_t}dW_{Y,t}, \tag{63}$$

$$dY_{i,t} = \mu_{i,t}dt + \bar{\sigma}_i\sqrt{Y_t}(\rho_i dW_{Y,t} + \hat{\rho}_i dW_{i,t}), \tag{64}$$

$$\frac{dS_t}{S_t} = \mu_{S,t}dt + \sigma_{S,t}(\rho_S dW_{Y,t} + \hat{\rho}_S dW_{S,t}), \tag{65}$$

where μ_Y^1,μ_Y^2, $\bar{\sigma}_Y$,$\bar{\sigma}_i \in \mathcal{R}$ are constant, particularly $\bar{\sigma}_Y$,$\bar{\sigma}_i > 0$, and also $\rho_i, \hat{\rho}_i,\ \rho_S, \hat{\rho}_S$ satisfying $|\rho_i|, |\rho_S| \leq 1$, $\hat{\rho}_i = \sqrt{1-\hat{\rho}_i^2}$, $\hat{\rho}_S = \sqrt{1-\hat{\rho}_S^2}$ are constant for simplicity. We assume that μ_i is given by $\mu_{i,t} = \frac{\gamma_i}{2}\bar{\sigma}_i^2 Y_t + \frac{c(t)}{\gamma_i}$ for some nonrandom function $c(t)$, which satisfies Assumption 2, μ_S and σ_S are $\{\mathcal{F}_t\}$-progressively measurable processes with $\sigma_{S,t} > 0, 0 \leq t \leq T$, and $F_i(y) = g_i^1(0)y + g_i^2(0)$ with $g_i^1(0), g_i^2(0) \in \mathcal{R},\ i = 1,\dots,I$. We can interpret this as a special case of (1)-(3) where $\mu_{Y,t} = \mu_Y^1 Y_t + \mu_Y^2$, $\mu_{i,t} = \frac{\gamma_i}{2}\bar{\sigma}_i^2 Y_t + \frac{c(t)}{\gamma_i}$, $\sigma_{Y,t} = \bar{\sigma}_Y\sqrt{Y_t}$, $\sigma_{i,t} = \bar{\sigma}_i\sqrt{Y_t}$, and $\rho_{i,t}, \hat{\rho}_{i,t}, \rho_{S,t}, \hat{\rho}_{S,t}$ are set to be constant. We also note that Assumption 1 follows from the weak version of Novikov's condition, which is satisfied by Theorem 3.2 in Shirakawa (2002). Then, the market price of risk and the optimal trading strategies of the agents are given as follows.

Theorem 2 *There exists $T_{\text{Blow-up}} > 0$ such that for $0 \leq \tau < T_{\text{Blow-up}}$, a solution $\{(g_i^1, g_i^2)\}_{i=1,\dots,I}$, for the system of ODEs*

$$\begin{aligned}
\partial_\tau g_i^1 =& \hat{\rho}_S^2\frac{\gamma_i\bar{\sigma}_Y^2}{2}(g_i^1)^2 \\
&- \left(-\rho_S\bar{\sigma}_Y\mu_Y^1 + \gamma_i\hat{\rho}_S^2\rho_i\bar{\sigma}_i\bar{\sigma}_Y\right)g_i^1 \\
&- \frac{1}{2\gamma_i}\left(-\Gamma\cdot\rho_S\cdot\sum_{j=1}^{I}\left(\bar{\sigma}_Y g_j^1 - \rho_j\bar{\sigma}_j\right) - \gamma_i\rho_S\rho_i\bar{\sigma}_i\right)^2,
\end{aligned} \tag{66}$$

$$\partial_\tau g_i^2 = \mu_Y^2 g_i^1, \tag{67}$$

with initial value conditions $g_i^1(0)$,$g_i^2(0) \in \mathcal{R}\ i = 1,\dots,I$ uniquely exists.

Moreover, if $T < T_{\text{Blow-up}}$ the instantaneous Sharpe ratio θ in equilibrium and the optimal trading strategies of the i-th agent $\pi_i^*,\ i = 1,\dots,I$ are given by

$$\theta_t = -\Gamma\cdot\rho_S\cdot\sum_{j=1}^{I}\left(\bar{\sigma}_Y g_j^1(T-t) - \rho_j\bar{\sigma}_j\right)\sqrt{Y_t}, \tag{68}$$

and

$$\pi_{i,t}^* = \frac{\sqrt{Y_t}}{\sigma_{S,t}}\left(\frac{-\Gamma\cdot\rho_S\cdot\sum_{j=1}^{I}\left(\bar{\sigma}_Y g_j^1(T-t) - \rho_j\bar{\sigma}_j\right) - \gamma_i\rho_i\bar{\sigma}_i\rho_S}{\gamma_i} + \rho_S\bar{\sigma}_Y g_i^1(T-t)\right). \tag{69}$$

Proof. For each i, the right-hand side of (66) is C^1-class in $g_1^1,\dots,g_I^1$. Thus, the right-hand side of the system of ODEs is locally Lipschitz continuous as a function of $(g_1^1,\dots,g_I^1)$. Then, by Picard-Lindelöf theorem (e.g., Theorem 2.2 in Teschl(2012)), the system of ODEs has a unique local solution. Moreover, by the extensibility of the solution (e.g., Theorem 2.13 in Teschl(2012)), there exists a constant, blow-up time, $T_{\text{Blow-up}} \in (0,\infty]$ such that the ODE system (66) has a unique solution for $0 \leq \tau < T_{\text{Blow-up}}$.

First, we show that $f_i(y,t) = g_i^1(\tau)y + g_i^2(\tau),\ i = 1,\dots,I$, where $\tau = T - t$, satisfy the system of PDEs (45).

Noting that

$$\partial_y f_i(y,t) = g_i^1(\tau),\ \partial_{yy} f_i(y,t) = 0,$$
$$\partial_t f_i(y,t) = -y\partial_\tau g_i^1(\tau) - \partial_\tau g_i^2(\tau), \tag{70}$$

and $\theta(y,t)$ in (47) becomes

$$\theta(y,t) = \sqrt{y}\phi(t), \tag{71}$$

where

$$\phi(t) = -\Gamma \cdot \rho_S \cdot \sum_{j=1}^{I} \left(\bar{\sigma}_Y g_j^1(T-t) - \rho_j\bar{\sigma}_j\right), \tag{72}$$

we calculate the left hand side of (45) as

$$y\left[\hat{\rho}_S^2 \frac{\gamma_i\bar{\sigma}_Y^2}{2}(g_i^1(\tau))^2 - \left(\rho_S\bar{\sigma}_Y\phi(T-\tau) - \mu_Y^1 + \gamma_i\hat{\rho}_S^2\rho_i\bar{\sigma}_i\bar{\sigma}_Y\right) g_i^1(\tau) - \frac{1}{2\gamma_i}\left(\phi(T-\tau) - \gamma_i\rho_S\rho_i\bar{\sigma}_i\right)^2 - \partial_\tau g_i^1(\tau)\right]$$
$$+ \left[\mu_Y^2 g_i^1(\tau) - \partial_\tau g_i^2(\tau)\right], \tag{73}$$

which is 0 due to (66) and (67), and the terminal conditions (46) hold. Thus, $f_i(y,t) = g_i^1(\tau)y + g_i^2(\tau),\ i = 1,\ldots,I$ satisfy the system of PDEs (45).

Next, we show that η_i in (44) with $V_i(x,y,t) = -\exp\left(-\gamma_i(x - f_i(y,t))\right)$ and π_i^* in (69) is a martingale under $\mathbf{P}$ for $i = 1,\ldots,I$.

Applying Ito's formula to $V_i(X_{i,t}^{(\pi_i^*)}, Y_t, t)$, since the drift term of dV_i is 0, we have

$$\begin{aligned}
dV_i &= (\partial_x V_i)(\pi_{i,t}^*\sigma_{S,t})(\rho_S dW_{Y,t}^{(i,Y)} + \hat{\rho}_S dW_{S,t}^{(i,Y)}) + (\partial_y V_i)\sigma_{Y,t} dW_{Y,t}^{(i,Y)} \\
&= ((\partial_x V_i)(\pi_{i,t}^*\sigma_{S,t})\rho_S + (\partial_y V_i)\sigma_{Y,t}) dW_{Y,t}^{(i,Y)} + (\partial_x V_i)(\pi_{i,t}^*\sigma_{S,t})\hat{\rho}_S dW_{S,t}^{(i,Y)} \\
&= ((-\gamma_i V_i)(\pi_{i,t}^*\sigma_{S,t})\rho_S + (\gamma_i V_i(\partial_y f_i))\sigma_{Y,t}) dW_{Y,t}^{(i,Y)} + (-\gamma_i V_i)(\pi_{i,t}^*\sigma_{S,t})\hat{\rho}_S dW_{S,t}^{(i,Y)} \\
&= V_i\left[-\gamma_i((\pi_{i,t}^*\sigma_{S,t})\rho_S - (\partial_y f_i)\sigma_{Y,t}) dW_{Y,t}^{(i,Y)} - \gamma_i(\pi_{i,t}^*\sigma_{S,t})\hat{\rho}_S dW_{S,t}^{(i,Y)}\right].
\end{aligned} \tag{74}$$

Moreover, by (69),

$$\pi_{i,t}^*\sigma_{S,t} = \sqrt{Y_t}\left(\frac{\phi(t) - \gamma_i\rho_i\bar{\sigma}_i\rho_S}{\gamma_i} + \rho_S\bar{\sigma}_Y g_i^1(T-t)\right). \tag{75}$$

Since $\phi(t)$ given by

$$\phi(T-\tau) = \phi(t) = -\Gamma \cdot \rho_S \cdot \sum_{j=1}^{I} \left(\bar{\sigma}_Y g_j^1(\tau) - \rho_j\bar{\sigma}_j\right), \tag{76}$$

is a linear combination of $g_i^1(T-t)$, which is bounded in $[0,T]$,

$$\left(\frac{\phi(t) - \gamma_i\rho_i\bar{\sigma}_i\rho_S}{\gamma_i} + \rho_S\bar{\sigma}_Y g_i^1(T-t)\right), \tag{77}$$

is also bounded in $[0,T]$. Then, by (74) and (75), V_i is a martingale in $\mathbf{P}^{(i,Y)}$ since the weak version of Novikov's condition holds by Theorem 3.2 in Shirakawa (2002) as Y is a square-root process satisfying an SDE

$$dY_t = (\mu_Y^1 Y_t + \mu_Y^2 - \gamma_i\rho_i\bar{\sigma}_i\bar{\sigma}_Y Y_t)dt + \bar{\sigma}_Y\sqrt{Y_t} dW_{Y,t}^{(i,Y)}. \tag{78}$$

Thus, we have

$$\begin{aligned}
\mathbf{E}[\eta_{i,T}|\mathcal{F}_t] &= \mathbf{E}\left[\frac{Z_T^{(i,Y)} V_i(X_{i,T}^{(\pi_i^*)}, Y_T, T)}{V_i(x_{i,0}, y_0, 0)}\middle|\mathcal{F}_t\right] \\
&= \frac{Z_t^{(i,Y)}\mathbf{E}^{(i,Y)}[V_i(X_{i,T}^{(\pi_i^*)}, Y_T, T)|\mathcal{F}_t]}{V_i(x_{i,0}, y_0, 0)} \\
&= \frac{Z_t^{(i,Y)} V_i(X_{i,t}^{(\pi_i^*)}, Y_t, t)}{V_i(x_{i,0}, y_0, 0)} = \eta_{i,t},
\end{aligned} \tag{79}$$

which indicates that η_i is a martingale under $\mathbf{P}$.

By (9), (74), and Girsanov's theorem, $W_{Y,t}^{\mathbf{Q}_i}, W_{S,t}^{\mathbf{Q}_i}$ defined as

$$dW_{Y,t}^{\mathbf{Q}_i} = dW_{Y,t} + \gamma_i\{(\pi_{i,t}^*\sigma_{S,t})\rho_S - \sigma_{Y,t}(\partial_y f_i) + \sigma_{i,t}\rho_i\}dt, \tag{80}$$
$$dW_{S,t}^{\mathbf{Q}_i} = dW_{S,t} + \gamma_i(\pi_{i,t}^*\sigma_{S,t})\hat{\rho}_S dt, \tag{81}$$

are Brownian motions under $\mathbf{Q}_i$.
Substituting

$$\pi^*_{i,t}\sigma_{S,t} = \left(\frac{\theta_t - \gamma_i\rho_{i,t}\sigma_{i,t}\rho_{S,t}}{\gamma_i} + \rho_{S,t}\sigma_{Y,t}(\partial_y f_i)\right), \tag{82}$$

we calculate

$$\rho_S dW^{\mathbf{Q}_i}_{Y,t} + \hat{\rho}_S dW^{\mathbf{Q}_i}_{S,t} = \rho_S dW_{Y,t} + \hat{\rho}_S dW_{S,t} + \theta_t dt. \tag{83}$$

Thus,

$$\begin{aligned} dX^{(\pi^*_i)}_{i,t} &= \pi^*_{i,t}\sigma_{S,t}\theta_t dt + \pi^*_{i,t}\sigma_{S,t}(\rho_S dW_{Y,t} + \hat{\rho}_S dW_{S,t}) \\ &= \pi^*_{i,t}\sigma_{S,t}(\rho_S dW^{\mathbf{Q}_i}_{Y,t} + \hat{\rho}_S dW^{\mathbf{Q}_i}_{S,t}). \end{aligned} \tag{84}$$

Lemma 3 *For π^*_i in (69), $X^{(\pi^*_i)}_{i,t}$ is a martingale under $\mathbf{Q}_i$.*

Proof. Since $X^{(\pi^*_i)}_i$ satisfies (84), it is sufficient to show that $\mathbf{E}^{\mathbf{Q}_i}[\int_0^T |\pi^*_{i,t}\sigma_{S,t}|^2 dt] < \infty$.
First, we note that

$$|\pi^*_{i,t}\sigma_{S,t}|^2 = Y_t\left|\frac{\phi(t) - \gamma_i\rho_i\bar{\sigma}_i\rho_S}{\gamma_i} + \rho_S\bar{\sigma}_Y g^1_i(T-t)\right|^2, \tag{85}$$

where

$$\left|\frac{\phi(t) - \gamma_i\rho_i\bar{\sigma}_i\rho_S}{\gamma_i} + \rho_S\bar{\sigma}_Y g^1_i(T-t)\right|^2, \tag{86}$$

is bounded in $[0,T]$ since $\phi(t)$ given by (76) is a linear combination of $g^1_i(T-t)$ which is bounded in $[0,T]$.
Next, under $\mathbf{Q}_i$, Y_t is also a square root process with time-varying parameters

$$dY_t = \{\mu^2_Y + (\mu^1_Y - \gamma_i\rho_i\bar{\sigma}_i\bar{\sigma}_Y\hat{\rho}^2_S - \bar{\sigma}_Y\rho_S\phi(t) + \gamma_i\bar{\sigma}^2_Y\hat{\rho}^2_S g^1_i(T-t))Y_t\}dt + \bar{\sigma}_Y\sqrt{Y_t}dW^{\mathbf{Q}_i}_{Y,t}. \tag{87}$$

Then, there exists $C > 0$ such that $\mathbf{E}^{\mathbf{Q}_i}[Y_t] < C,\ \forall t \in [0,T]$, which is proved as follows.
Setting

$$dY_t = \{\alpha(t) - \beta_i(t)Y_t\}dt + \bar{\sigma}_Y\sqrt{Y_t}dW^{\mathbf{Q}_i}_{Y,t}, \tag{88}$$

where $\alpha(t) = \mu^2_Y$ and $\beta_i(t) = -(\mu^1_Y - \gamma_i\rho_i\bar{\sigma}_i\bar{\sigma}_Y\hat{\rho}^2_S - \bar{\sigma}_Y\rho_S\phi(t) + \gamma_i\bar{\sigma}^2_Y\hat{\rho}^2_S g^1_i(T-t))$, by Ito's formula, we calculate

$$\begin{aligned} d(e^{\int_0^t \beta_i(u)du}Y_t) &= \beta_i(t)e^{\int_0^t \beta_i(u)du}Y_t dt + e^{\int_0^t \beta_i(u)du}dY_t \\ &= \alpha(t)e^{\int_0^t \beta_i(u)du}dt + \bar{\sigma}_Y e^{\int_0^t \beta_i(u)du}\sqrt{Y_t}dW^{\mathbf{Q}_i}_{Y,t}. \end{aligned} \tag{89}$$

Thus, we have

$$Y_t = e^{-\int_0^t \beta_i(u)du}y_0 + \int_0^t \alpha(s)e^{-\int_s^t \beta_i(u)du}ds + \int_0^t \bar{\sigma}_Y e^{-\int_s^t \beta_i(u)du}\sqrt{Y_s}dW^{\mathbf{Q}_i}_{Y,s}. \tag{90}$$

We define stopping times $\tau^{(k)},\ k \in \mathbf{N}$ by

$$\tau^{(k)} = \inf\{s > 0 : Y_s \geq k\} \wedge T, \text{ for } k \in \mathbf{N}. \tag{91}$$

Then, we have

$$Y_{t\wedge\tau^{(k)}} = e^{-\int_0^{t\wedge\tau^{(k)}} \beta_i(u)du}y_0 + \int_0^{t\wedge\tau^{(k)}} \alpha(s)e^{-\int_s^{t\wedge\tau^{(k)}} \beta_i(u)du}ds + \int_0^{t\wedge\tau^{(k)}} \bar{\sigma}_Y e^{-\int_s^{t\wedge\tau^{(k)}} \beta_i(u)du}\sqrt{Y_s}dW^{\mathbf{Q}_i}_{Y,s}. \tag{92}$$

Since $Y_s \leq k$ for $s \in [0, t\wedge\tau^{(k)}]$, the stochastic integral part in (92) is a martingale and its expectation under $\mathbf{Q}_i$ is 0. Hence,

$$\mathbf{E}^{\mathbf{Q}_i}[Y_{t\wedge\tau^{(k)}}] \leq C, \tag{93}$$

for some $C > 0$ and by Fatou's lemma, we obtain

$$\mathbf{E}^{\mathbf{Q}_i}[Y_t] \leq \liminf_{k\to\infty}\mathbf{E}^{\mathbf{Q}_i}[Y_{t\wedge\tau^{(k)}}] \leq C. \tag{94}$$

Thus $\mathbf{E}^{\mathbf{Q}_i}[\int_0^T |\pi^*_{i,t}\sigma_{S,t}|^2 dt] < \infty$ and $X^{(\pi^*_i)}_{i,t}$ is a martingale under $\mathbf{Q}_i$. $\square$

Therefore, by Theorem 1, θ in (68) and $\pi^*_i,\ i = 1,\dots,I$ in (69) are the Sharpe ratio and the optimal trading strategy of the i-th agent in an equilibrium. $\square$

Remark 3 *We will estimate the blow-up time $T_{Blow\text{-}up}$ numerically for each concrete problem by an arc-length transformation method proposed by Hirota and Ozawa (2006). See Section 4.2 for the computation procedure. Then, if we set the maturity $T < T_{Blow\text{-}up}$, which can be estimated by a numerical method for given parameters, the ODE system has a unique bounded solution $g^1_i(\tau)$ in $[0,T]$. We also note that the system is described as a collection of I quadratic ordinary differential equations (ODEs), where the variables interact through the common term $-\Gamma \cdot \rho_S \cdot$*

$\sum_{j=1}^{I}\left(\bar{\sigma}_Y g_j^1(\tau)-\rho_j\bar{\sigma}_j\right)$ *denoted by* $\phi(T-\tau)$ *in (76), which is not expressible in the matrix form of Riccati equations. (For the comparison principle used to establish the well-posedness of the Riccati system, see Lemma 1 of Ma et al. (2019) or Theorem 4.1.4 of Abou-Kandil et al. (2012), for example).*

4.2 Computation procedure

In the following numerical examples, we will use the square-root model, where we compute the instantaneous Sharpe ratio and the optimal portfolios by the following procedure.

(1) First, we exogenously set the common parameters $T, \mu_Y^1, \mu_Y^2, \bar{\sigma}_Y, \rho_S$ and the individual parameters $\gamma_i, \bar{\sigma}_i, \rho_i$, and $g_i^1(0), g_i^2(0)$ satisfying $F_i(y) = g_i^1(0)y + g_i^2(0)$ in (15).

(2) Second, we solve the system of ODEs (66) in $[0, T]$ by some numerical methods. In this study, we use the explicit Runge-Kutta method of order 5(4)implemented in **scipy.integrate.solve_ivp** of SciPy package (see the reference of Scipy package (2022) for details).

(3) Finally, we compute $\phi(t)$ by (72), which also determines the instantaneous Sharpe ratio $\theta_t = \phi(t)\sqrt{Y_t}$ and the optimal portfolio $\pi_{i,t}^*$ in (69).

We note that the solution may blow up at some $T_{\text{Blow-up}}, T_{\text{Blow-up}} < T$ depending on the given parameters. In this case, we estimate the blow-up time $T_{\text{Blow-up}}$ numerically by an arc-length transformation method in Hirota and Ozawa (2006). Then, we reset T so that $T < T_{\text{Blow-up}}$ and solve the system of ODEs again in the reset $[0, T]$.

5 Examples

In this section, we present numerical examples for two important topics in practice, reinsurance claim pricing and life-cycle investment, as applications of the square-root model for the multi-agent equilibrium in an incomplete market in Section 4. Specifically, we consider two agent cases and investigate the effect of an economic factor Y and the individual income factors $Y_i,\ i = 1, 2$, sources of incompleteness in the market on a reinsurance claim price and agents' trading strategies.

5.1 Reinsurance pricing

Reinsurance is a tool used by insurance companies to hedge their insurance risk. The insurance risk is generally non-tradable, and how to price the risk is an important issue for both insurance and reinsurance companies. For instance, there are some reinsurance contracts that transfer both insurance risk and investment risk from an insurer to a reinsurer (e.g., Allianz (2021)). In addition, catastrophe equity put options are also traded to transfer losses from insurance risk (e.g., Arnone et al. (2021)). Payoffs of insurance risk transferred to a reinsurer are classified into proportional (linear payoff) and non-proportional (nonlinear payoff) (see Chapter 2 in Albrecher et al. (2017)), and pricing such payoffs can be complex.

In the following sections, we present numerical examples for financial stop-loss reinsurance introduced in Møller (2003) and Becherer (2003). Financial stop-loss reinsurance is a contract where an insurance company receives a guarantee within certain limits from a reinsurance company at maturity when the insurance company is unable to compensate for payment on insurance claims (insurance risk denoted as Y) by investing in a risky asset whose price is expressed as S. Then, this contract has the following payoff,

$$\Phi(Y_T, S_T) = \min\left\{\max(Y_T + \delta(S_0 - S_T) - K_1, 0), (K_2 - K_1)\right\}, \tag{95}$$

where $\delta \geq 0$ is a constant that determines the degree of investment in the risky asset, and K_1 and K_2, $K_1 < K_2 < \infty$, describe the guarantee level the insurer requires from the reinsurer. In detail, only if the insurance risk Y_T hedged by the return from investment in the risky asset exceeds K_1, $\delta(S_0 - S_T)$, the exceeding amount capped at K_2 is paid to the insurance company.

In our model, the agents' stochastic discount factors (SDFs) in market equilibrium $\eta_{i,T},\ i = 1, \ldots, I$ are obtained and can be used to calculate the price for a reinsurance claim.

5.1.1 Settings in our model

We consider two agent cases where there are an insurance company (agent 1) and a reinsurance company (agent 2) in the economy. We suppose that agent 1 owns some dynamic insurance risk (liability) and aims to transfer some of the risks by purchasing a reinsurance contract from agent 2.

We use the square-root model in Section 4 for pricing and assume each variable describes the following. $X_{i,T}^{(\pi_i)}$ is the wealth of agent i's portfolio at maturity with the trading strategy π_i. We assume $Y_i \equiv 0,\ i = 1, 2$ and $-F_1(Y_T) = -Y_T$, $-F_2(Y_T) = -Y_T$ or 0, which indicate exogenous payoffs at maturity on the insurance risk Y of agents 1 and 2. The state variable Y expresses the insurance risk that drives the insurer's and reinsurer's exogenous payoff.

For the reinsurance contract whose payoff is given by (95), noting that the interest rate is assumed to be 0 for simplicity, under the risk-neutral measure for each agent i, $\mathbf{Q}_i$ defined by $\frac{d\mathbf{Q}_i}{d\mathbf{P}} = \eta_{i,T}$, the present value of this claim (with zero interest rate) is given by

$$p_i = \mathbf{E}\left[\eta_{i,T}\Phi(Y_T, S_T)\right] = \mathbf{E}^{\mathbf{Q}_i}[\Phi(Y_T, S_T)], \tag{96}$$

where agent i's SDF η_i satisfies

$$\frac{d\eta_{i,t}}{\eta_{i,t}} = \sqrt{Y_t}[-\gamma_i\bar{\sigma}_i\hat{\rho}_i dW_{i,t} - \gamma_i\{P_i(t)\rho_S - \bar{\sigma}_Y g_i^1(T-t) + \bar{\sigma}_i\rho_i\}dW_{Y,t} - \gamma_i P_i(t)\hat{\rho}_S dW_{S,t}],\ \eta_{i,0} = 1. \tag{97}$$

Here, $P_i(t)$ is defined as

$$P_i(t) = \pi_i^M(t) + \pi_i^H(t), \tag{98}$$

$$\pi_i^M(t) = \frac{\phi(t)}{\gamma_i}, \tag{99}$$

$$\pi_i^H(t) = \rho_S\bar{\sigma}_Y g_i^1(T-t) - \rho_i\bar{\sigma}_i\rho_S. \tag{100}$$

We note that this valuation method may be regarded as marginal indifference pricing or fair pricing (e.g., Section 33.2 in Björk (2020) or Chapter 6 in Karatzas (1997) , respectively).

Then, we can obtain the prices for both agents by simulating the payoff $\Phi(Y_T, S_T)$ and each SDF $\eta_{i,T}$ under $\mathbf{P}$ with the equations above. In fact, as $Y_i \equiv 0$ $(i = 1, 2)$ in the current example, we set $\bar{\sigma}_i = 0$ $(i = 1, 2)$ when solving the system of corresponding ODEs (66) and implementing the Monte Carlo simulations with equations (96)-(100).

Thus, a transaction price of the reinsurance claim p can be agreed between the two agents as long as their prices p_1, p_2 and the transaction price p satisfy

$$p_2 \le p \le p_1, \tag{101}$$

which implies that the insurer can buy the reinsurance claim at a lower price than p_1 and the reinsurer sells it at a higher price than p_2.

In the following, we conduct a Monte-Carlo simulation to calculate the expectation (96) for each agent's price, where we divide 1 year into 250 grids and adopt 100,000 paths. We set the volatility process σ_S of the risky asset price in (65) as a constant $\sigma_{S,t} \equiv \bar{\sigma}_S$ and the parameters in the model in accordance with Example 4.9 in Becherer (2003). In this case, the optimal trading strategies $\pi_{i,t}^*$ in (69) can be decomposed into the mean-variance part π_i^M to maximize the expectation of the terminal wealth with the insurance risk and the hedging part π_i^H to reduce the terminal insurance risk as

$$\pi_{i,t}^* = \frac{\sqrt{Y_t}}{\bar{\sigma}_S}(\pi_i^M(t) + \pi_i^H(t)). \tag{102}$$

Here, $\phi(t)$, which also determines the expected return on the risky asset by $\mu_{S,t} = \bar{\sigma}_S\phi(t)\sqrt{Y_t}$, is expressed as

$$\phi(t) = -\Gamma \cdot \sum_{i=1}^{I} \pi_i^H(t). \tag{103}$$

5.1.2 Numerical results

Table 1 shows the prices of agents 1 and 2 for sets of parameters in the square-root model in (65).

For Set 1 as a base case to set parameters, where the correlation ρ_S between the insurance risk and the risky asset price process is 0, it is confirmed that the prices of two agents are the same, since the parameters related to each agent are identical, and its price is consistent with the one for $\delta = 0.4$ in Figure 3 of Example 4.9 in Becherer (2003).

For Sets 2-7, we consider cases where the insurance company has to pay insurance risk at $(g_1^1(0) = 1)$ maturity T, while the reinsurance company has no insurance risk to pay $(g_2^1(0) = 0)$. Then, we investigate the change in their prices when the correlation between the risky asset price process S and the insurance risk Y or the level of risk aversion γ_i is shifted.

Firstly, agent 2's price p_2 in Set 3 with no payout of Y_T $(g_2^1(0) = 0)$ is lower than the one in Set 1 with the payout of Y_T $(g_2^1(0) = 1)$, since in pricing agent 2 puts a larger weight, i.e. agent 2's SDF $\eta_{2,T}$ on the payoff Φ in bad states, namely large Y_T and small S_T. In fact, Figure 1 shows the scatter plot of S_T and Y_T with the colored values of $\eta_{2,T}$, where the larger $\eta_{2,T}$ corresponds to the right bottom area in the left panel (Set 1) than in the right one (Set 3). On the contrary, we note that the payoffs in Sets 1 and 3 are the same since the expected return of the risky asset $\mu_{S,t} = \bar{\sigma}_S\phi(t)\sqrt{Y_t}$ is 0 due to $\phi(t) \equiv 0$ by (100) and (103) with $\rho_S = 0$ and $\bar{\sigma}_i = 0$.

In the following, we describe the effects of other parameters. Figure 2 shows the histogram of the payoff $\Phi(Y_T, S_T)$ for Sets 2-7 when the payoff is simulated by Monte-Carlo simulation under $\mathbf{P}$. Moreover, Figure 3 exhibits the scatter plot of Y_T and S_T with colored values of agent 1's SDF $\eta_{1,T}$ for Sets 2-7.

First, we observe that agent 1's price is higher than agent 2's in Sets 2-7. This agrees with the intuition that agent 1 with the insurance risk at maturity is willing to pay for a reinsurance claim to hedge the risk, and agent 2 with no insurance risk can sell it at a lower price.

Table 1
Parameter sets and the prices for agent 1 and agent 2. Parameters in bold show the differences from Set 1.

	Set 1	Set 2	Set 3	Set 4	Set 5	Set 6	Set 7
μ_Y^1	-1	-1	-1	-1	-1	-1	-1
μ_Y^2	0.83	0.83	0.83	0.83	0.83	0.83	0.83
$\bar{\sigma}_Y$	0.14	0.14	0.14	0.14	0.14	0.14	0.14
$\bar{\sigma}_1$	0	0	0	0	0	0	0
$\bar{\sigma}_2$	0	0	0	0	0	0	0
$\bar{\sigma}_S$	0.25	0.25	0.25	0.25	0.25	0.25	0.25
ρ_1	0	0	0	0	0	0	0
ρ_2	0	0	0	0	0	0	0
ρ_S	0	**0.5**	0	**-0.5**	**0.5**	0	**-0.5**
$\hat{\rho}_S$	1.00	**0.87**	1.00	**0.87**	**0.87**	1.00	**0.87**
γ_1	1	1	1	1	**10**	**10**	**10**
γ_2	1	1	1	1	1	1	1
$g_1^1(0)$	1	1	1	1	1	1	1
$g_2^1(0)$	1	**0**	**0**	**0**	**0**	**0**	**0**
y_0	0.83	0.83	0.83	0.83	0.83	0.83	0.83
s_0	1	1	1	1	1	1	1
K_1	0.85	0.85	0.85	0.85	0.85	0.85	0.85
K_2	1.15	1.15	1.15	1.15	1.15	1.15	1.15
δ	0.4	0.4	0.4	0.4	0.4	0.4	0.4
T	1	1	1	1	1	1	1
p_1	0.0447	0.0306	0.0447	0.0544	0.0589	0.0827	0.0802
p_2	0.0447	0.0283	0.0414	0.0520	0.0286	0.0414	0.0523
$p_1 - p_2$	0	0.0023	0.0033	0.0024	0.0303	0.0412	0.0279

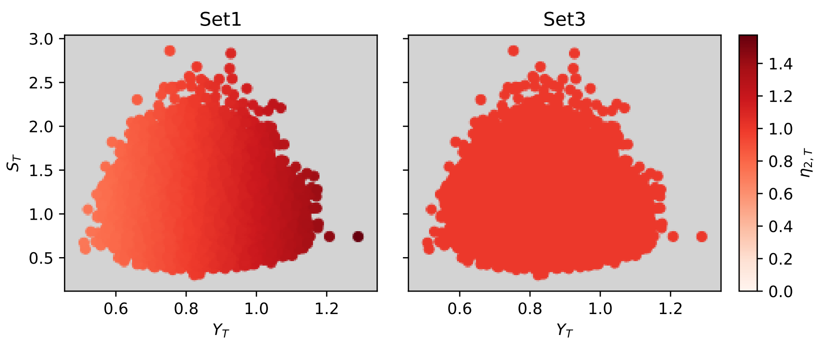

Fig. 1. Set 1 and Set 3 : Scatter plots for Y_T and S_T with colored values of $\eta_{2,T}$.

Next, we observe that each agent's price in Set 4 is the highest among Sets 2-4, which can be interpreted as follows: We note that the price p_i in (96) is the expectation of the payoff $\Phi(Y_T, S_T)$ in (95) weighted by the SDF $\eta_{i,T}$. Figure 2 shows that the payoff $\Phi(Y_T, S_T)$ with $\rho_S < 0$ in Sets 4 and 7, is distributed at higher values than in other sets. Figure 3 shows that agent 1's SDF values (darkness of the color) in Sets 2-4 are close, which also holds for agent 2. Thus, the payoff's distribution is the main determinant in each agent's price p_i $(i = 1, 2)$ for Sets 2-4, and Set 4 provides the highest prices among Sets 2-4 in Table 1.

However, we observe that agent 1's price in Set 6 is the highest among Sets 5-7 with higher risk aversion parameter $\gamma_1 = 10$. Figure 2 shows that the payoff's distributions are almost unchanged among Sets 2-7 with different γ_1. On

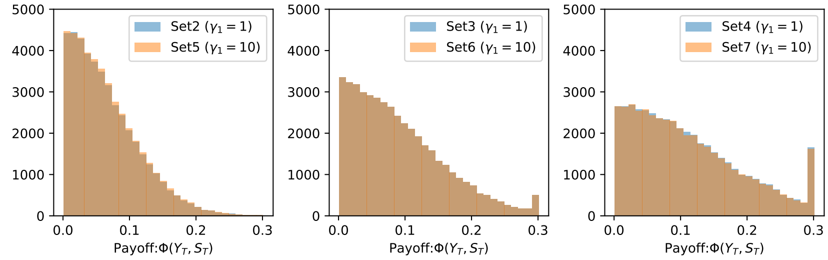

Fig. 2. Histogram for the reinsurance payoff $\Phi(Y_T, S_T)$: We show $\Phi(Y_T, S_T) > 0.001$.

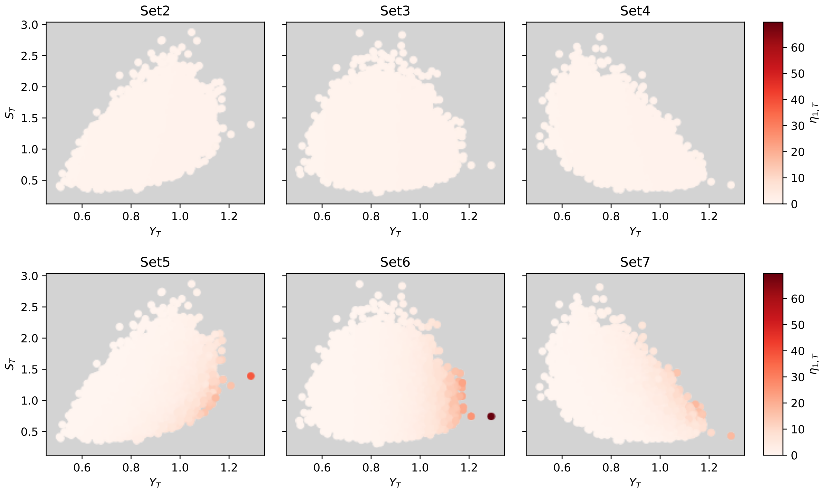

Fig. 3. Scatter plots for Y_T and S_T with colored values of $\eta_{1,T}$: Upper panels shows Set 2-4 ($\gamma_1 = 1$) and lower panels show Set 5-7 ($\gamma_1 = 10$).

the contrary, Figure 3 indicates that the agent 1's SDF value with $\gamma_1 = 10$ is higher (darker colors) at the area where Y_T is relatively large and S_T is small in Sets 5-7, particularly in Set 6 ($\rho_S = 0$), which results in the highest price p_1 shown in Set 6 among Sets 5-7 ($\gamma_1 = 10$).

Finally, we examine the price difference between agents 1 and 2, namely $p_1 - p_2$ appearing in the last row of Table 1, which is caused by the difference in each agent's SDF. We find that the price difference is much larger in Sets 5-7 with agent 1's higher risk aversion parameter ($\gamma_1 = 10$) than in Sets 2-4. This agrees with the intuition that if agent 1 has a higher risk aversion parameter, agent 1 is more willing to pay for a reinsurance claim to hedge its insurance risk. Moreover, we observe that the price difference in Set 3 (Set 6) with $\rho_S = 0$ is larger than Sets 2 and 4 (Sets 5 and 7). This is because when $\rho_S = 0$, agent 1 is not able to hedge the insurance risk at maturity by trading the risky asset, and hence, agent 1 pays a higher premium for the reinsurance claim to agent 2 with no insurance risk.

5.2 Life-cycle investment

Next, we provide numerical examples of life-cycle investment as an application of the square-root model in Section 4. When individuals consider long-term investments for their retirement, it is important to take labor income (also called human capital) into account in addition to investment in financial assets. This is called life-cycle investment.

Since labor income is volatile and not tradable though correlated with financial asset prices, it is difficult for individuals to hedge the fluctuations on their own. For this reason, asset management companies including subsidiaries of life insurance companies offer individual investors funds that are designed to substitute life-cycle investment, which is called life-cycle funds or target-year (date) funds.

For related literature, Henderson (2005) deals with an optimal portfolio problem of an individual who receives labor income and invests in a risky asset with the exogenously given constant expected return. Bruder et al. (2012) also consider an optimal portfolio problem for life-cycle funds, where the expected returns of risky assets are exogenously given a function of time.

As mentioned in Bruder et al. (2012) , the expected return on a risky asset has a significant impact on the construction of the optimal portfolio. Thus, in the following numerical examples, we suppose the equilibrium model in an incomplete market, where the expected return of the risky asset is determined endogenously by the market clearing condition of the financial assets and investigate the impact of the expected return on the optimal trading strategies, which helps asset management firms to establish life-cycle funds.

5.2.1 Setting

As in the reinsurance pricing in Section 5.1, we consider a two-agent case in the square-root model in Section 4, where agent 1 represents individual investors or asset management companies for the life-cycle fund, and agent 2 does the other market participants.

Moreover, we suppose that at retirement date T, agent 1 receives $Y_{1,T}$ that stands for his or her specific lifetime income, and $-F_1(Y_T) = -g_i^1(0)Y_T - g_i^2(0)$ with $F_i(y)$ in (1) of Section 4.2, which is agent 1's net income directly linked to Y_T representing the economic condition for Cases A and B in Section 5.2.2 and the inflation for Case C in Section 5.2.3, respectively.

As for Cases A and B, we set positive net income $-F_1(Y_T) = Y_T$ with $g_1^1(0) = -1$ and $g_1^2(0) = 0$ in Case A, and negative net income $-F_1(Y_T) = -Y_T$ with $g_1^1(0) = 1$ and $g_1^2(0) = 0$ in Case B for simplicity. On the contrary, agent 2 does not have $F_2(Y_T)$, that is, $F_2(Y_T) \equiv 0$ in both cases. However, we assume that agent 2 receives $Y_{2,T}$ with a large volatility coefficient $\bar{\sigma}_2$. As seen in Section 5.2.2, the resulting equilibrium expected return of the risky asset is positive.

As for Case C we suppose that agent 1 receives $Y_{1,T}$ and negative net income $-F_1(Y_T) = -Y_T$ with $g_1^1(0) = 1$ and $g_1^2(0) = 0$, while agent 2 does not have those income, namely $F_2(Y_T) \equiv 0$ and $Y_2 \equiv 0$.

In addition, we set $\sigma_{S,t} = \sqrt{Y_t}$ in (65), and $c(t) \equiv 0$, which implies $\mu_{i,t} = \frac{\gamma_i}{2}\bar{\sigma}_i^2 Y_t + \frac{c(t)}{\gamma_i} = \frac{\gamma_i}{2}\bar{\sigma}_i^2 Y_t$ in (64).

5.2.2 Numerical results in Case A and B

In the following, we investigate the optimal trading strategies and the excess return process of the risky asset in equilibrium for the parameter sets of Cases A and B in Table 2.

Table 2
Settings of parameters of life-cycle investment example.

	μ_Y^1	μ_Y^2	$\bar{\sigma}_Y$	$\bar{\sigma}_1$	$\bar{\sigma}_2$	ρ_1	ρ_2	ρ_S	γ_1	γ_2	$g_1^1(0)$	$g_1^2(0)$	T
Case A	-0.2	0.4	0.4	0.2	2	0.3	0.8	0.5	3	1	**-1**	0	10
Case B	-0.2	0.4	0.4	0.2	2	0.3	0.8	0.5	3	1	**1**	0	10

In both cases, we assume that the economic condition Y has the positive correlations with the risky asset return and the change in the individual's specific income, $\rho_S = 0.5$ and $\rho_1 = 0.3$, $\rho_2 = 0.8$, respectively. The difference between Case A and Case B is just the sign of agent 1's net income associated with Y_T. Particularly, in Case A, agent 1 receives a fund $(-g_1^1(0) = +1)$ thanks to a retirement allowance, for instance. In Case B, there is a payout at agent 1's retirement $(-g_1^1(0) = -1)$ due to mortgage repayment, for example.

Figures 4 and 5 exhibit ϕ, the determinant of the expected return process μ_S, and the optimal trading strategies of agents 1 and 2 in equilibrium π_1^* $(i = 1, 2)$ in (69), given as

$$\pi_{i,t}^* = \frac{\sqrt{Y_t}}{\sigma_{S,t}}(\pi_i^M(t) + \pi_i^H(t)) = \pi_i^M(t) + \pi_i^H(t), \tag{104}$$

for Cases A and B, respectively, where $\sigma_{S,t} = \sqrt{Y_t}$ and, $\pi_i^M(t)$ and $\pi_i^M(t)$ are defined in (99) and (100). In particular, we note that $\phi(t)$ determines the equilibrium expected return of the risky asset by $\mu_{S,t} = \sigma_{S,t}\theta_t = \sigma_{S,t}\sqrt{Y_t}\phi(t) = \phi(t)Y_t$, and agent i's mean-variance portfolio as $\pi_i^M(t) = \frac{\phi(t)}{\gamma_i}$.

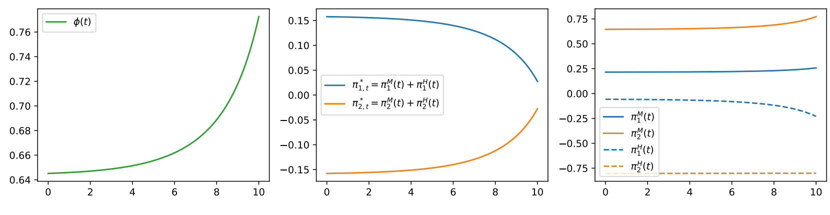

Fig. 4. Case A : ϕ, determinant of the expected return process μ_S, and optimal trading strategies π_1^* and π_2^*.

Firstly, we can interpret the positive expected return process determined by $\phi(t) > 0$ from a general equilibrium perspective. Since agent 2 representing other market participants takes a short position on the risky asset $(\pi_{2,t}^* < 0)$

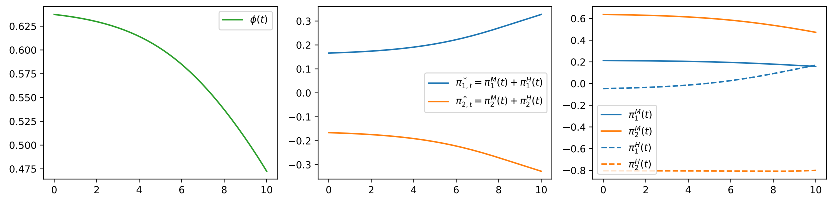

Fig. 5. Case B : ϕ, determinant of the expected return process μ_S, and optimal trading strategies π_1^* and π_2^*.

to hedge the realization of $Y_{2,T}$ due to $\rho_2\rho_S > 0$, which has a relatively large volatility $\bar{\sigma}_2$, agent 1 needs to take an opposite position, that is long, so that the market is cleared. Therefore, the expected return of the risky asset has to be positive for agent 1 to take a long position ($\pi_{1,t}^* > 0$).

Next, we examine the decreasing or increasing behavior of agent 1's long position on the risky asset towards maturity T, which can be explained in terms of the hedging portfolio for agent 1, $\pi_1^H(t)$ as follows:

Figure 4 shows that agent 1 reduces its long position towards maturity T, which corresponds to life-cycle funds where the asset management companies invest more in stocks when the customers are young, while they shift the allocation to bonds as the customers' ages become closer to retirement. In Case A, since agent 1 receives a fund Y_T, agent 1 reduces its hedging demand $\pi_1^H(t)$ towards maturity T to hedge the realization of Y_T due to $\rho_S > 0$.

On the contrary, we find that Figure 5 shows that agent 1 increases its long position towards maturity T, since agent 1 has a payout of Y_T in Case B.

5.2.3 Numerical results in Case C

In the following, we investigate the optimal trading strategies and the excess return process of the risky asset in equilibrium for the parameter set of Case C in Table 3.

Table 3
Setting of parameters of life-cycle investment example.

	μ_Y^1	μ_Y^2	$\bar{\sigma}_Y$	$\bar{\sigma}_1$	$\bar{\sigma}_2$	ρ_1	ρ_2	ρ_S	γ_1	γ_2	$g_1^1(0)$	$g_1^2(0)$	T
Case C	-0.2	0.4	0.2	0.4	0	0.8	0	-0.5	3	1	1	0	10

Particularly, Case C is where Y representing inflation has a negative correlation with the change in the stock price ($\rho_S = -0.5$), while agent 1's specific income positively correlates with Y mainly by inflation allowance, and hence has a negative correlation with the stock price movement ($\rho_1\rho_S = -0.4$) as in the last year, 2022. In addition, there is a payout at agent 1's retirement ($-g_1^1(0) = -1$) due to mortgage repayment, for example.

Figure 6 exhibits ϕ, the determinant of the expected return process μ_S, and the optimal trading strategies of agents 1 and 2 in equilibrium as in Cases A and B.

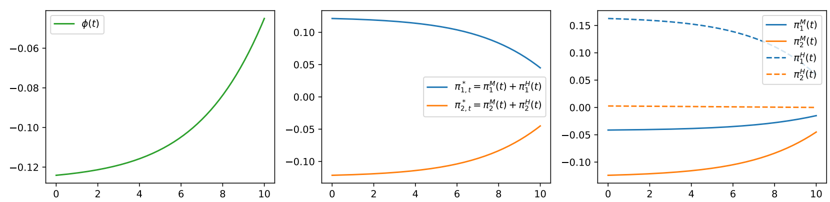

Fig. 6. Case C : ϕ, determinant of the expected return process μ_S, and optimal trading strategies π_1^* and π_2^*.

Figure 6 illustrates that agent 1 takes a long position for the risky asset ($\pi_{1,t}^* > 0$) and gradually reduces this long position as maturity T approaches. In the right panel breakdown, it can be observed that the hedging portfolio π_1^H mainly contributes to π_1^* due to the higher risk-aversion parameter γ_1 than γ_2. This observation can be understood by considering the correlation between the risky asset (S) and agent 1's total terminal labor income ($Y_{1,T} - Y_T$).

The correlation between the risky asset and the total terminal labor income is quantified and calculated as $\bar{\sigma}_1\rho_1\rho_S - \bar{\sigma}_Y\rho_S = -0.16 - (-0.1) = -0.06 < 0$. The negative sign in the first term, $\bar{\sigma}_1\rho_1\rho_S = -0.16 < 0$, indicates that agent 1 takes a long position to hedge against the realization of $Y_{1,T}$. Conversely, the positive sign in the second term, $-\bar{\sigma}_Y\rho_S = -(-0.1) > 0$, suggests that agent 1 takes a short position to hedge against the payout $-Y_T$ in Case C.

Thus, although the resulting net total hedging portfolio π_1^H gradually reduces its long position towards maturity T, it remains in a long position at T due to the overall negative correlation, $\bar{\sigma}_1\rho_1\rho_S - \bar{\sigma}_Y\rho_S = -0.06 < 0$. Consequently, the resulting net total position remains long, $\pi_{1,t}^* = \pi_1^M(t) + \pi_1^H(t) > 0$, with π_1^H accounting for the majority of π_1^*.

Next, we observe that agent 2 takes a short position for the risky asset ($\pi_2^* < 0$) under the negative expected return on the risky asset ($\phi(t) < 0$) in Case C. This is because the mean-variance portfolio part π_2^M that aims to increase agent 2's profit accounts for the majority of agent 2's portfolio π_2^*, which is due to the low risk-aversion parameter $\gamma_2 = 1$ and no terminal payoffs to hedge.

Moreover, we can interpret the negative expected return process $\phi(t) < 0$ from a general equilibrium perspective. Since agent 1 has a demand to take a long position on the risky asset to hedge the total terminal labor income in both cases, agent 2 needs to take an opposite position, i.e. short, so that the market is cleared. Therefore, the expected return of the risky asset has to be negative for agent 2 to take a short position.

Finally, as mentioned in Bruder et al. (2012) , determining the expected returns on risky assets are essential in life-cycle investment. In this study, we have used the market clearing condition to set the expected return endogenously and confirmed that the relationship among individuals' optimal portfolios, the risky asset return and the labor income is consistent with the results in Henderson (2005) , which analyzes an individual's optimal portfolio with an exogenously given expected return of the risky asset and stochastic labor income. Thus, our model provides a theoretical basis for setting the expected return of a risky asset in life-cycle investment.

6 Conclusion

This study has developed a dynamic incomplete equilibrium model with multi-agents of exponential utilities. Moreover, in numerical experiments, we have explicitly obtained agents' optimal trading strategies, their stochastic discount factors (SDFs) and expected returns of the risky asset in equilibrium.

Our research is new in that we propose a concrete equilibrium model for an incomplete market with heterogeneous income/payout profiles and different risk attitudes of agents, which is applied to two meaningful examples in practice for insurance and asset management companies, namely, reinsurance claim pricing and life-cycle investment. Particularly, our model endogenously determines an equilibrium excess return process of the risky asset, which has been exogenously given in previous works and has been considered to affect the optimal trading strategies largely.

The implications of this study are as follows: The model can be used in pricing reinsurance claims with estimations of the relevant factors and those correlations with stock prices, and in predicting how the reinsurance price changes when those correlations shift. Also, pension funds and asset management companies may utilize our model in life-cycle investment/target-year (date) funds to construct portfolios by incorporating the effects of individuals' income and payout profiles, economic factors and, their correlations with stock prices.

As for future studies, introducing intermediate consumption to determine an equilibrium stochastic interest rate in an incomplete market will be one of the important research topics. Moreover, while two examples in the current paper have extracted the essence from practical situations, developing more detailed and realistic modeling with empirical analyses will be even more useful in practice. In addition, when securities related to unhedgeable factors are newly issued and the market becomes complete, investigating how the equilibrium return process and the optimal portfolios change will be a future research topic.

Declaration of competing interest

There is no competing interest.

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A Equilibrium in an incomplete market with log utility settings

In this section, as another example of solving the equilibrium in an incomplete market, we present a case that each agent has a different state-dependent log utility, which may be interpreted as a log utility with an agent-specific subjective probability measure.

First, suppose that the zero net supply money market account with a risk-free interest rate r exists.

Suppose also that one unit of a stock is issued and that its price process is the solution to the SDE:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_{S,t} dW_t = r_t dt + \sigma_t \left[(\frac{\mu_t - r_t}{\sigma_t}) dt + (\rho_t dW_t^Y + \hat{\rho}_t dW_t^S) \right], \tag{A.1}$$

where $\sigma_{S,t} = \sigma_t(\rho_t, \hat{\rho}_t)$ with $\hat{\rho}_t = \sqrt{1-\rho_t^2}$ and $dW_t = (dW_t^Y, dW_t^S)^\top$.

Let

$$\theta_t = \sigma_{S,t}^\top (\sigma_{S,t}\sigma_{S,t}^\top)^{-1}(\mu_t - r_t) = \frac{\mu_t - r_t}{\sigma_t}(\rho_t, \hat{\rho}_t)^\top \in Range(\sigma_S^\top), \tag{A.2}$$

and note that $\sigma_{S,t}\theta_t = \mu_t - r_t$, and hence θ is a market price of risk.

We assume agent i's expected utility as $\mathbf{E}[\eta_T^i \log X_T^i]$ with $\eta_T^i = \exp\left[\int_0^T \lambda_s^i \cdot dW_s - \frac{1}{2}\int_0^T |\lambda_s^i|^2 ds\right]$, where

$\lambda_s^i = (\lambda_{1,s}^i, \lambda_{2,s}^i)^\top$, $i = 1, 2, \cdots, I$.

Particularly, we can make λ^i depend on a factor H common among all agents or/and agent i's specific factor H_i, namely, $\lambda^i(H, H_i) = (\lambda_1^i(H, H_i), \lambda_2^i(H, H_i))^\top$, $i = 1, 2, \cdots, I$.

We define the agent's wealth process as the riskless asset amount π_0^i plus (tradable) risky asset amount $\pi^i \in \mathcal{A}_i$, namely, for $t \in [0, T]$, $X_t^i = \pi_{0,t}^i + \pi_t^i$.

Here, we consider admissible strategies as the set of progressively measurable processes such that

$$\mathcal{A}_i := \left\{ \pi^i : X_t^i \geq 0,\ a.s.,\ t \in [0,T],\ \mathbf{E}\left[\int_0^T (\pi_s^i)^2 \sigma_s^2 ds\right] < \infty \right\}. \tag{A.3}$$

Then, the wealth dynamics is given as follows.

$$\begin{aligned} dX_t^i &= r_t X_t^i dt + \pi_t^i(\mu_t - r_t)dt + \pi_t^i \sigma_{S,t} dW_t \\ &= X_t^i \left[r_t dt + \tilde{\pi}_t^i \sigma_{S,t} \left\{ \theta_t dt + dW_t \right\} \right],\ X_0^i = x_0^i > 0, \end{aligned} \tag{A.4}$$

where $\pi_t^i = \tilde{\pi}_t^i X_t^i$.

Particularly, with $\hat{\theta}_s := \frac{\mu_s - r_s}{\sigma_s}$ and $B_t = e^{\int_0^t r_s ds}$,

$$X_t^i = x_0^i B_t \exp\left[\int_0^t \tilde{\pi}_s^i \sigma_s \hat{\theta}_s ds - \frac{1}{2}\int_0^t (\tilde{\pi}_s^i \sigma_s)^2 ds + \int_0^t \tilde{\pi}_s^i \sigma_{S,s} dW_s \right]. \tag{A.5}$$

Then, applying a well-known result for the log utility that is, the optimal portfolio is always given by the (instantaneous) mean-variance portfolio, with the measure change by η_T^i, we obtain agent i's optimal stock holding for the problem as follows.

(Problem)

$$\begin{aligned} &\sup_{\pi^i \in \mathcal{A}_i} \mathbf{E}[\eta_T^i \log X_T^i] = \tilde{\mathbf{E}}^i[\log X_T^i],\ \text{under the wealth dynamics} \\ &dX_t^i = r_t X_t^i dt + \pi_t^i(\mu_t - r_t)dt + \pi_t^i \sigma_{S,t}[d\tilde{W}_t^i + \lambda_t^i dt] \\ &= X_t^i \left[r_t dt + \tilde{\pi}_t^i \sigma_t \{ (\hat{\theta}_t + \rho_t \lambda_{1,t}^i + \hat{\rho}_t \lambda_{2,t}^i) dt + (\rho_t, \hat{\rho}_t) d\tilde{W}_t^i \} \right], \end{aligned} \tag{A.6}$$

where $\tilde{\mathbf{E}}^i[\cdot]$ denotes the expectation operator under the probability measure $\tilde{\mathbf{P}}^i$ induced by η_T^i, and $\tilde{W}_t^i := W_t - \int_0^t \lambda_s^i ds$ is a two-dimensional Brownian motion under $\tilde{\mathbf{P}}^i$.

We obtain agent i's optimal stock holding as $\pi_t^i = \tilde{\pi}_t^{*,i} X_t^i$ by the (instantaneous) mean-variance portfolio with the excess expected return $\{\mu_t + \sigma_t(\rho_t \lambda_{1,t}^i + \hat{\rho}_t \lambda_{2,t}^i) - r_t\}$ and variance σ_t^2, where $\tilde{\pi}_t^{*,i} = \frac{\hat{\theta}_t + (\rho_t \lambda_{1,t}^i + \hat{\rho}_t \lambda_{2,t}^i)}{\sigma_t}$, $\hat{\theta}_t = \frac{\mu_t - r_t}{\sigma_t}$, and the optimal money market account holding is $\pi_{0,t}^i = X_t^i - \pi_t^i$. Then, by (A.5), we obtain the optimal wealth X^i

as

$$\frac{X_t^i}{B_t} = x_0^i \frac{Z_t^i}{Z_t^\theta}, \text{ with} \tag{A.7}$$

$$Z_t^\theta := \exp\left[-\int_0^t \theta_s \cdot dW_s - \frac{1}{2}\int_0^t |\theta_s|^2 ds\right], \tag{A.8}$$

$$Z_t^i := \exp\left[\int_0^t \hat{\lambda}_{i,s} \cdot dW_s - \frac{1}{2}\int_0^t |\hat{\lambda}_{i,s}|^2 ds\right], \tag{A.9}$$

where

$$\theta_t = \sigma_{S,t}^\top(\sigma_{S,t}\sigma_{S,t}^\top)^{-1}(\mu_t - r_t) = \hat{\theta}_t(\rho_t, \hat{\rho}_t)^\top, \ \hat{\theta}_t = \frac{\mu_t - r_t}{\sigma_t}, \tag{A.10}$$

$$\hat{\lambda}_{i,t} = (\rho_t \lambda_{1,t}^i + \hat{\rho}_t \lambda_{2,t}^i)(\rho_t, \hat{\rho}_t)^\top. \tag{A.11}$$

Here, note that $\sigma_{S,t}\theta_t = \mu_t - r_t$, and hence $\theta \in Range(\sigma_S^\top)$ is a market price of risk. We also see $\hat{\lambda}_i$ in terms of an orthogonal decomposition of λ^i: That is, $\lambda^i = \hat{\lambda}_i \oplus \hat{\lambda}_i^\perp$, where $\hat{\lambda}_i \in Range(\sigma_S^\top)$ and $\hat{\lambda}_i^\perp \in Kernel(\sigma_S) = \{x : \sigma_S x = 0\}$. Finally, for the market clearing condition, exogenously given a terminal dividend, $\mathcal{F}_T$-measurable $\xi > 0$ which an agent holding one unit equity at T receives, the market clearing condition is given by

$$\xi = \sum_i \pi_T^i = \sum_i X_T^i - \sum_i \pi_{0,T}^i = \sum_i X_T^i, \tag{A.12}$$

since the sum of the optimal money market account holding should be zero, that is $\sum_i \pi_{0,T}^i = 0$.

Similarly, at $t \in [0, T)$ it should hold that $S_t = \sum_i X_t^i$.

Hence, we obtain

$$\xi = \sum_i x_0^i \frac{Z_T^i B_T}{Z_T^\theta}, \text{ that is, } Z_T^\theta = \sum_i x_0^i \frac{Z_T^i B_T}{\xi}, \tag{A.13}$$

and since Z_t^θ should be a martingale, we have

$$Z_t^\theta = \mathbf{E}_t\left[\frac{B_T}{\xi}\sum_i x_0^i Z_T^i\right] = \sum_i x_0^i \mathbf{E}_t\left[\frac{B_T}{\xi} Z_T^i\right]. \tag{A.14}$$

Based on this equation we will derive $\theta_t = \frac{\mu_t - r_t}{\sigma_t}(\rho_t, \hat{\rho}_t)^\top$.

Remark 1 *The following condition must be satisfied at $t = 0$:*

$$Z_0^\theta = 1 = \mathbf{E}\left[\frac{B_T}{\xi}\sum_i x_0^i Z_T^i\right] = \sum_i x_0^i \mathbf{E}\left[\frac{B_T}{\xi} Z_T^i\right]. \tag{A.15}$$

Moreover, given a constant interest rate r, this is rewritten as

$$\frac{1}{B_T} = \sum_i x_0^i \mathbf{E}\left[\frac{Z_T^i}{\xi}\right]; \ i.e., \ r = -\frac{1}{T}\log\left(\sum_i x_0^i \mathbf{E}\left[\frac{Z_T^i}{\xi}\right]\right) \tag{A.16}$$

Thus, we have the following two choices to satisfy the condition.

(i) Given $\xi > 0$, $x_0^i > 0$, $\lambda^i = (\lambda_1^i, \lambda_2^i)^\top$, $(\rho, \hat{\rho})$, (and hence Z^i), the condition determines $B_T = e^{rT}$, namely r.

(ii) Given $\xi > 0$, $\lambda^i = (\lambda_1^i, \lambda_2^i)^\top$, $(\rho, \hat{\rho})$, r, the condition provides an allocation of the initial wealth, $x_0^i > 0$ $(i = 1, 2, \ldots, I)$.

Remark 2 *The stock price dynamics in equilibrium is also obtained and thus, we can solve for the volatility σ_S along with the market price of risk θ.*

Since Z^i is a martingale, and the market clearing condition implying that $\xi = \sum_i X_T^i = \sum_i x_0^i \frac{Z_T^i B_T}{Z_T^\theta}$ and $S_t = \sum_i X_t^i = \sum_i x_0^i \frac{Z_t^i B_t}{Z_t^\theta}$, we have

$$Z_T^\theta \frac{\xi}{B_T} = \sum_i x_0^i Z_T^i, \text{ and} \tag{A.17}$$

$$Z_t^\theta \frac{S_t}{B_t} = \sum_i x_0^i Z_t^i = \sum_i x_0^i \mathbf{E}_t[Z_T^i] = \mathbf{E}_t[Z_T^\theta \frac{\xi}{B_T}]. \tag{A.18}$$

Thus, $Z^\theta S/B$ is a martingale as expected, and we have

$$\frac{S_t}{B_t} = \frac{1}{Z_t^\theta}\sum_i x_0^i Z_t^i = \frac{1}{Z_t^\theta}\mathbf{E}_t[Z_T^\theta \frac{\xi}{B_T}] = \mathbf{E}_t^*[\frac{\xi}{B_T}], \tag{A.19}$$

where $\mathbf{E}_t^*[\cdot]$ denotes the conditional expectation operator under a risk-neutral measure induced by θ. In particular, $S_0 = \sum_i x_0^i$.

Moreover, using $S_t = \frac{B_t}{Z_t^\theta}\sum_i x_0^i Z_t^i$, we obtain the stock volatility $\sigma_S = \sigma(\rho, \hat{\rho})$ and hence the expected return μ, given $\frac{\mu - r}{\sigma}$.

Concretely, we have

$$d\left(\frac{S_t}{B_t}\right) = \left(\frac{1}{Z_t^\theta}\right)\theta_t \cdot \sum_i x_0^i Z_t^i[\theta_t + \hat{\lambda}_{i,t}]dt + \left(\frac{1}{Z_t^\theta}\right)\left\{\sum_i x_0^i Z_t^i[\hat{\lambda}_{i,t} + \theta_t]\right\} \cdot dW_t, \tag{A.20}$$

$$\frac{d(S_t/B_t)}{(S_t/B_t)} = \frac{\left[\theta_t \cdot \sum_i x_0^i Z_t^i\{(\hat{\lambda}_{i,t} + \theta_t)\}dt + \left\{\sum_i x_0^i Z_t^i(\hat{\lambda}_{i,t} + \theta_t)\right\} \cdot dW_t\right]}{\sum_i x_0^i Z_t^i}. \tag{A.21}$$

Then, setting

$$\sigma_{S,t}^\top = \frac{\sum_i x_0^i Z_t^i(\hat{\lambda}_{i,t} + \theta_t)}{\sum_i x_0^i Z_t^i} = \frac{\sum_i x_0^i Z_t^i \hat{\lambda}_{i,t}}{\sum_i x_0^i Z_t^i} + \theta_t, \tag{A.22}$$

we obtain

$$\frac{dS_t}{S_t} - r_t dt = \sigma_{S,t}(\theta_t dt + dW_t). \tag{A.23}$$

Moreover, recalling $\sigma_S = \sigma(\rho, \hat{\rho})^\top$, $\hat{\lambda}_i = (\rho\lambda_1^i + \hat{\rho}\lambda_2^i)(\rho, \hat{\rho})^\top$ and $\theta = \frac{\mu - r}{\sigma}(\rho, \hat{\rho})^\top$, we define σ as

$$\sigma_t = \frac{\sum_i x_0^i Z_t^i\left(\rho_t \lambda_{1,t}^i + \hat{\rho}_t \lambda_{2,t}^i\right)}{\sum_i x_0^i Z_t^i} + \frac{\mu_t - r_t}{\sigma_t}. \tag{A.24}$$

Then, we have $\frac{dS_t}{S_t} - r_t dt = \sigma_t\left[\frac{\mu_t - r_t}{\sigma_t}dt + (\rho_t dW_t^Y + \hat{\rho}_t dW_t^S)\right]$, which is consistent with (A.1).

Next, we confirm that the market clearing condition holds. Let $X_t := \sum_i X_t^i$. Then, on one hand, by (A.4),

$$dX_t = r_t X_t dt + (\sum_i \pi_t^i)[(\mu_t - r_t)dt + \sigma_{S,t}dW_t]; \; X_0 = \sum_i x_0^i. \tag{A.25}$$

On the other hand, by (A.1),

$$dS_t = S_t[\mu_t dt + \sigma_{S,t}dW_t] = r_t S_t dt + S_t[(\mu_t - r_t)dt + \sigma_{S,t}dW_t]; \; S_0 = \sum_i x_0^i. \tag{A.26}$$

Hence, thanks to $X_t = S_t$ for all $t \in [0, T]$, it holds that $S_t = \sum_i \pi_t^i$ for all $t \in [0, T]$, particularly, $\xi_T = \sum_i \pi_T^i$, which shows that the market clearing condition is satisfied.

In the following assuming a specific form for ξ, we aim to obtain the market price of risk $\theta = \frac{\mu - r}{\sigma}(\rho, \hat{\rho})$. First, let $\xi = \xi_T$, and suppose that

$$\frac{d\xi_t}{\xi_t} = \mu_{\xi,t}dt + \sigma_{\xi,t} \cdot dW_t; \; \xi_0 > 0, \tag{A.27}$$

$$\frac{d\hat{\xi}_t}{\hat{\xi}_t} = \mu_{\hat{\xi},t}dt + \sigma_{\hat{\xi},t} \cdot dW_t, \; (\hat{\xi} := 1/\xi), \tag{A.28}$$

where

$$\sigma_{\xi,t} = (\sigma_{1,t}^\xi, \sigma_{2,t}^\xi)^\top; \; \mu_{\hat{\xi},t} = -\mu_{\xi,t} + \bar{\sigma}_{\xi,t}^2; \; \bar{\sigma}_{\xi,t} = \sqrt{(\sigma_{1,t}^\xi)^2 + (\sigma_{2,t}^\xi)^2}; \; \sigma_{\hat{\xi},t} = -\sigma_{\xi,t}. \tag{A.29}$$

We calculate $Z_t^\theta = \mathbf{E}_t\left[\frac{B_T}{\xi}\sum_i x_0^i Z_T^i\right] = \mathbf{E}_t\left[\frac{B_T}{\xi_T}\sum_i x_0^i Z_T^i\right]$ to obtain $\theta = \frac{\mu - r}{\sigma}(\rho, \hat{\rho})^\top$.

Namely, we calculate dZ_t^θ / Z_t^θ with

$$Z_t^\theta = B_t \sum_i x_0^i \mathbf{E}_t\left[\frac{B_T}{B_t \xi} Z_T^i\right] = B_t \hat{\xi}_t \sum_i x_0^i Z_t^i \mathbf{E}_t\left[\frac{B_T}{B_t}\frac{\hat{\xi}_T}{\hat{\xi}_t}\frac{Z_T^i}{Z_t^i}\right], \tag{A.30}$$

where

$$\mathbf{E}_t\left[\frac{B_T}{B_t}\frac{\hat{\xi}_T}{\hat{\xi}_t}\frac{Z_T^i}{Z_t^i}\right] = \mathbf{E}_t\left[\exp\left\{\int_t^T (r_s + \mu_{\hat{\xi},s} - \frac{1}{2}|\sigma_{\hat{\xi},s}|^2 - \frac{1}{2}|\hat{\lambda}_{i,s}|^2)ds + \int_t^T (\sigma_{\hat{\xi},s} + \hat{\lambda}_{i,s}) \cdot dW_s\right\}\right], \tag{A.31}$$

and compare the volatility term of (A.30) with the one in $dZ_t^\theta = -Z_t^\theta \theta_t \cdot dW_t$ to obtain θ.

A.1 Example

As an example, we suppose that σ_ξ, r, ρ are nonrandom, and for some constants a_Y, a_S, we set

$$\mu_{\hat{\xi},t} = -\mu_{\xi,t} + \bar{\sigma}^2_{\xi,t} = a_Y W^Y_t + a_S W^S_t, \ \textit{i.e.}, \ \mu_{\xi,t} = \bar{\sigma}^2_{\xi,t} - (a_Y W^Y_t + a_S W^S_t). \tag{A.32}$$

Moreover, for λ^i, setting $\lambda^i_{1,t} = H_t b^i_{1,t}$ and $\lambda^i_{2,t} = H_t b^i_{2,t}$ (b^i_1, b^i_2: nonrandom), we have

$$\hat{\lambda}_{i,t} = (\rho_t \lambda^i_{1,t} + \hat{\rho}_t \lambda^i_{2,t})(\rho_t, \hat{\rho}_t)^\top = H_t(\rho_t b^i_{1,t} + \hat{\rho}_t b^i_{2,t})(\rho_t, \hat{\rho}_t)^\top, \tag{A.33}$$

where for $\mu^H_t = \mu^H_{1,t} + \mu^H_{2,t} H_t$ ($\mu^H_{1,t}, \mu^H_{2,t}$: nonrandom) and nonrandom σ^H_t, we define

$$dH_t = \mu^H_t dt + \sigma^H_t \cdot dW_t; \ H_0 = h. \tag{A.34}$$

Under this setting where μ_ξ and λ^i are random, we calculate dZ^θ with (A.30) and (A.31) to obtain θ.
Firstly, using

$$\int_t^T W^Y_s ds = \int_t^T \{W^Y_t + (W^Y_s - W^Y_t)\} ds = (T-t)W^Y_t + \int_t^T (T-u) dW^Y_u, \tag{A.35}$$

we have

$$\int_t^T \mu_{\hat{\xi},s} ds = (a_Y W^Y_t + a_S W^S_t)(T-t) + \int_t^T a_Y (T-s) dW^Y_s + \int_t^T a_S (T-s) dW^S_s. \tag{A.36}$$

Then, when we define $\sigma^y_s := (a_Y(T-s), a_S(T-s))^\top$, with $\sigma_{\hat{\xi},s} = (\sigma^{\hat{\xi}}_{1,s}, \sigma^{\hat{\xi}}_{2,s})^\top$ and $\hat{\lambda}_{i,s} = (\rho_s \lambda^i_{1,s} + \hat{\rho}_s \lambda^i_{2,s})(\rho_s, \hat{\rho}_s)^\top$, we express (A.31) as follows:

$$\begin{aligned} C^i_t &:= \mathbf{E}_t \left[\frac{B_T}{B_t} \frac{\hat{\xi}_T}{\hat{\xi}_t} \frac{Z^i_T}{Z^i_t} \right] \\ &= \frac{B_T}{B_t} \exp\{(T-t)(a_Y W^Y_t + a_S W^S_t)\} \mathbf{E}_t \left[\exp\left\{ \int_t^T (-\frac{1}{2}|\sigma_{\hat{\xi},s}|^2 - \frac{1}{2}|\hat{\lambda}_{i,s}|^2) ds + \int_t^T (\sigma^y_s + \sigma_{\hat{\xi},s} + \hat{\lambda}_{i,s}) \cdot dW_s \right\} \right]. \end{aligned} \tag{A.37}$$

Let

$$D^i(t,T) = \exp\left\{ \int_t^T -\frac{1}{2} |\sigma^y_s + \sigma_{\hat{\xi},s} + \hat{\lambda}_{i,s}|^2 ds + \int_t^T (\sigma^y_s + \sigma_{\hat{\xi},s} + \hat{\lambda}_{i,s}) \cdot dW_s \right\}. \tag{A.38}$$

Then,

$$\begin{aligned} dH_t &= \mu^H_t dt + \sigma^H_t \cdot (dW^i_t + (\sigma^y_t + \sigma_{\hat{\xi},t} + \hat{\lambda}_{i,t}) dt) \\ &= [\mu^H_t + \sigma^H_t \cdot (\sigma^y_t + \sigma_{\hat{\xi},t} + \hat{\lambda}_{i,t})] dt + \sigma^H_t \cdot dW^i_t := dH^i_t; \ H^i_0 = h, \end{aligned} \tag{A.39}$$

where we use the fact that W^i is a Brownian motion under $\mathbf{P}^i$, the probability measure induced by $D^i(0,T)$, and

$$dW^i_t = dW_t - (\sigma^y_t + \sigma_{\hat{\xi},t} + \hat{\lambda}_{i,t}) dt. \tag{A.40}$$

Hence, we have

$$\begin{aligned} &\mathbf{E}_t \left[\exp\left\{ \int_t^T (-\frac{1}{2}|\sigma_{\hat{\xi},s}|^2 - \frac{1}{2}|\hat{\lambda}_{i,s}|^2) ds + \int_t^T (\sigma^y_s + \sigma_{\hat{\xi},s} + \hat{\lambda}_{i,s}) \cdot dW_s \right\} \right] \\ &= \mathbf{E}_t \left[D^i(t,T) \exp\left\{ \int_t^T \left(\frac{1}{2}|\sigma^y_s|^2 + \sigma^y_s \cdot \sigma_{\hat{\xi},s} + (\sigma^y_s + \sigma_{\hat{\xi},s}) \cdot \hat{\lambda}_{i,s} \right) ds \right\} \right] \\ &= \exp\left\{ \int_t^T \left(\frac{1}{2}|\sigma^y_s|^2 + \sigma^y_s \cdot \sigma_{\hat{\xi},s} \right) ds \right\} \mathbf{E}^i_t \left[\exp\left\{ \int_t^T (\sigma^y_s + \sigma_{\hat{\xi},s}) \cdot \hat{\lambda}_{i,s} ds \right\} \right], \end{aligned} \tag{A.41}$$

where $\mathbf{E}^i_t[\cdot]$ denotes the conditional expectation operator under $\mathbf{P}^i$.
Since

$$(\sigma^y + \sigma_{\hat{\xi}}) \cdot \hat{\lambda}_i = H^i(\sigma^y + \sigma_{\hat{\xi}}) \cdot (\rho b^i_1 + \hat{\rho} b^i_2)(\rho, \hat{\rho})^\top \tag{A.42}$$

is a Gaussian process under $\mathbf{P}^i$, we know that for some nonrandom function $A^i(t,T)$ and $B^i(t,T)$ in Remark 3 below,

$$\mathbf{E}^i_t \left[\exp\left\{ \int_t^T (\sigma^y_s + \sigma_{\hat{\xi},s}) \cdot \hat{\lambda}_{i,s} ds \right\} \right] = \exp[A^i(t,T) + B^i(t,T) H^i_t] := P^i(t,T), \tag{A.43}$$

$$dP^i(t,T) = P^i(t,T)[-\{(\sigma^y_t + \sigma_{\hat{\xi},t}) \cdot \hat{\lambda}_{i,t}\} dt + B^i(t,T) \sigma^H_t \cdot dW^i_t]. \tag{A.44}$$

Hence, we have

$$C_t^i = \frac{B_T}{B_t} \exp\left\{\int_t^T \left(\frac{1}{2}|\sigma_s^y|^2 + \sigma_s^y \cdot \sigma_{\hat{\xi},s}\right) ds\right\} \exp\{(T-t)(a_Y W_t^Y + a_S W_t^S)\} P^i(t,T). \tag{A.45}$$

Therefore, with

$$Z_t^\theta = \mathbf{E}_t\left[\frac{B_T}{\xi_T}\sum_i x_0^i Z_T^i\right] = B_t\hat{\xi}_t \sum_i x_0^i Z_t^i \mathbf{E}_t\left[\frac{B_T}{B_t}\frac{\hat{\xi}_T}{\hat{\xi}_t}\frac{Z_T^i}{Z_t^i}\right] = B_t\hat{\xi}_t\sum_i x_0^i Z_t^i C_t^i, \tag{A.46}$$

and $\sigma_{p,t}^i := B^i(t,T)\sigma_t^H$, we obtain

$$\begin{aligned} dZ_t^\theta &= B_t\hat{\xi}_t\sum_i x_0^i Z_t^i C_t^i[\sigma_t^y + \sigma_{\hat{\xi},t} + \hat{\lambda}_{i,t} + \sigma_{p,t}^i]\cdot dW_t \\ &= -Z_t^\theta \sum_i \frac{x_0^i Z_t^i C_t^i}{\sum_i x_0^i Z_t^i C_t^i}[-\sigma_t^y - \sigma_{\hat{\xi},t} - \hat{\lambda}_{i,t} - \sigma_{p,t}^i]\cdot dW_t = -Z_t^\theta \theta_t \cdot dW_t, \end{aligned} \tag{A.47}$$

where $\theta = \frac{\mu - r}{\sigma}(\rho,\hat{\rho})^\top$. Hence, with $\hat{\lambda}_i = (\rho\lambda_1^i + \hat{\rho}\lambda_2^i)(\rho,\hat{\rho})^\top$ we have

$$-\sigma_t^y - \sigma_{\hat{\xi},t} - \sum_i \frac{x_0^i Z_t^i C_t^i}{\sum_i x_0^i Z_t^i C_t^i}\left\{(\rho_t\lambda_{1,t}^i + \hat{\rho}_t\lambda_{2,t}^i)(\rho_t,\hat{\rho}_t)^\top - \sigma_{p,t}^i\right\} = \theta_t = \frac{\mu_t - r_t}{\sigma_t}(\rho_t,\hat{\rho}_t)^\top. \tag{A.48}$$

Then, for some nonrandom $\bar{\sigma}_t^H \in \mathcal{R}$, we set

$$\sigma_t^H = \bar{\sigma}_t^H(\rho_t,\hat{\rho}_t)^\top. \tag{A.49}$$

By using $\sigma_t^y = (a_Y(T-t), a_S(T-t))^\top$ and $\sigma_{\hat{\xi},t} = (\sigma_{1,t}^{\hat{\xi}}, \sigma_{2,t}^{\hat{\xi}})^\top$,

$$(-\sigma_t^y - \sigma_{\hat{\xi},t}) = -(a_Y(T-t), a_S(T-t)) - (\sigma_{1,t}^{\hat{\xi}}, \sigma_{2,t}^{\hat{\xi}}), \tag{A.50}$$

and, to satisfy (A.48) it should hold that there exists some nonrandom $k_t \neq 0$ such that

$$(-\sigma_t^y - \sigma_{\hat{\xi},t}) = -(a_Y(T-t), a_S(T-t)) - (\sigma_{1,t}^{\hat{\xi}}, \sigma_{2,t}^{\hat{\xi}}) = k_t(\rho_t,\hat{\rho}_t). \tag{A.51}$$

Specifically, given σ_t^y and $\sigma_{\hat{\xi},t} = -\sigma_{\xi,t}$, we set k_t and $(\rho_t,\hat{\rho}_t)$ as follows:

$$k_t = |(-\sigma_t^y - \sigma_{\hat{\xi},t})|, \tag{A.52}$$

$$(\rho_t,\hat{\rho}_t) = \frac{1}{k_t}(-\sigma_t^y - \sigma_{\hat{\xi},t}). \tag{A.53}$$

Then, we obtain

$$\theta_t = \left\{k_t - \sum_i \frac{x_0^i Z_t^i C_t^i}{\sum_i x_0^i Z_t^i C_t^i}(\rho_t\lambda_{1,t}^i + \hat{\rho}_t\lambda_{2,t}^i + B^i(t,T)\bar{\sigma}_t^H)\right\}(\rho_t,\hat{\rho}_t)^\top, \tag{A.54}$$

$$\frac{\mu_t - r_t}{\sigma_t} = \left\{k_t - \sum_i \frac{x_0^i Z_t^i C_t^i}{\sum_i x_0^i Z_t^i C_t^i}(\rho_t\lambda_{1,t}^i + \hat{\rho}_t\lambda_{2,t}^i + B^i(t,T)\bar{\sigma}_t^H)\right\}. \tag{A.55}$$

Let us recall by (A.1) and (A.24) that

$$\sigma_{S,t}^\top = \sigma_t(\rho_t,\hat{\rho}_t)^\top = \frac{\sum_i x_0^i Z_t^i \hat{\lambda}_{i,t}}{\sum_i x_0^i Z_t^i} + \theta_t = \left\{\frac{\sum_i x_0^i Z_t^i(\rho_t\lambda_{1,t}^i + \hat{\rho}_t\lambda_{2,t}^i)}{\sum_i x_0^i Z_t^i} + \frac{\mu_t - r_t}{\sigma_t}\right\}(\rho_t,\hat{\rho}_t)^\top. \tag{A.56}$$

Thus,

$$\begin{aligned} \sigma_t &= \frac{\sum_i x_0^i Z_t^i\left(\rho_t\lambda_{1,t}^i + \hat{\rho}_t\lambda_{2,t}^i\right)}{\sum_i x_0^i Z_t^i} + \frac{\mu_t - r_t}{\sigma_t} \\ &= \sum_i \frac{x_0^i Z_t^i}{\sum_i x_0^i Z_t^i}\left(\rho_t\lambda_{1,t}^i + \hat{\rho}_t\lambda_{2,t}^i\right) + \left\{k_t - \sum_i \frac{x_0^i Z_t^i C_t^i}{\sum_i x_0^i Z_t^i C_t^i}(\rho_t\lambda_{1,t}^i + \hat{\rho}_t\lambda_{2,t}^i + B^i(t,T)\bar{\sigma}_t^H)\right\}, \end{aligned} \tag{A.57}$$

and given r_t, the expected return μ_t is determend with this σ_t and (A.55) as $\mu_t = r_t + \sigma_t\left(\frac{\mu_t - r_t}{\sigma_t}\right)$. In addition, the equation (A.19) provides $S_0 = \sum_i x_0^i$. Consequently, the equilibrium price of the risky asset is completely specified. We note that in the current model, the source of incompleteness is the expected return $\mu_{\xi,t} = \bar{\sigma}_{\xi,t}^2 - (a_Y W_t^Y + a_S W_t^S)$ of the stochastic process (A.27) of the exogenously given dividend ξ for the risky asset, and that heterogeneity of the agents is reflected in ξ's and hence the risky asset's expected return through $\hat{\lambda}_i$, which determines agent i's specific state dependence (or subjective probability measure) in the expected utility $\mathbf{E}[\eta_T^i \log X_T^i]$ with $\eta_T^i =$

$\exp\left[\int_0^T \lambda_s^i \cdot dW_s - \frac{1}{2}\int_0^T |\lambda_s^i|^2 ds\right]$.

Adopting case (i) in Remark 1, we summarize our equilibrium model in the following:

(1) Given a terminal dividend, $\mathcal{F}_T$-measurable random variable $\xi > 0$ that an agent holding one unit equity at T receives, and ξ is determined by the stochastic process (A.27).

The volatility and expected return of ξ_t are specified as nonrandom process $\sigma_{\xi,t} = (\sigma_{1,t}^{\xi}, \sigma_{2,t}^{\xi})^\top$, and $\mu_{\xi,t} = \bar{\sigma}_{\xi,t}^2 - (a_Y W_t^Y + a_S W_t^S)$ with $\bar{\sigma}_{\xi,t} = \sqrt{(\sigma_{1,t}^{\xi})^2 + (\sigma_{2,t}^{\xi})^2}$ and some $\mathcal{R}$-valued constants $a_Y,\ a_S$, respectively.

(2) Given each agent i's initial wealth $x_0^i > 0$, and the determinat of state dependence (subjective probability) of each agent i's utility,

$\eta_T^i = \exp\left[\int_0^T \lambda_s^i \cdot dW_s - \frac{1}{2}\int_0^T |\lambda_s^i|^2 ds\right]$ with

$\lambda^i = (\lambda_1^i, \lambda_2^i)$, $\lambda_{1,t}^i = H_t b_{1,t}^i$ and $\lambda_{2,t}^i = H_t b_{2,t}^i$ (b_1^i, b_2^i: nonrandom).

$H \in \mathcal{R}$ is a given stochastic process defined by (A.34)

$\mu_t^H = \mu_{1,t}^H + \mu_{2,t}^H H_t$ ($\mu_{1,t}^H, \mu_{2,t}^H$: nonrandom) and

$\sigma_t^H = \bar{\sigma}_t^H (\rho_t, \hat{\rho}_t)^\top$ ($\bar{\sigma}_t^H$: nonrandom).

$(\rho_t, \hat{\rho}_t)$ is specified by (A.52) and (A.53).

(3) The constant equilibrium interest rate r is given by (A.16), where Z^i and $\hat{\lambda}_i$ are given by (A.9) and (A.33), respectively.

(4) The equilibrium market price of risk θ_t is given by (A.54), where C_t^i is obtained by (A.45). In addition, $P^i(t,T) = \exp[A^i(t,T) + B^i(t,T)H_t^i]$ in C_t^i is concretely given in Remark 3 below, and the stochastic process H_t^i is defined in (A.39).

(5) The equilibrium risky asset price process is obtained as follows:

initial price: $S_0 = \sum_i x_0^i$.

volatility: $\sigma_{S,t} = \sigma_t(\rho_t, \hat{\rho}_t)$ with (A.57), (A.52) and (A.53).

expected return μ_t:

$\mu_t = r + \sigma_t \left(\frac{\mu_t - r}{\sigma_t}\right)$ with (A.16), (A.57) and (A.55).

We finally remark that a similar result can be obtained by setting $\mu_{\hat{\xi}}$ (μ_ξ) as a general Gaussian process.

Also, when we adopt case (ii) in Remark 1, the risk-free interest rate r can be exogenously given as a general Gaussian process.

Considering practical applications with numerical experiments will be a future research topic.

Remark 3 Calculation of $A^i(t,T)$ and $B^i(t,T)$ in (A.43) :
First, let us define a nonrandom function f^i as $f_t^i := -(\sigma_t^y + \sigma_{\hat{\xi},t}) \cdot (\rho_t b_{1,t}^i + \hat{\rho}_t b_{2,t}^i)(\rho_t, \hat{\rho}_t)^\top$.

Hence,

$$H_t^i(\sigma_t^y + \sigma_{\hat{\xi},t}) \cdot (\rho_t b_{1,t}^i + \hat{\rho}_t b_{2,t}^i)(\rho_t, \hat{\rho}_t)^\top = -H_t^i f_t^i, \tag{A.58}$$

and we have

$$\mathbf{E}_t^i\left[\exp\left\{\int_t^T (\sigma_s^y + \sigma_{\hat{\xi},s}) \cdot \hat{\lambda}_{i,s} ds\right\}\right] = \mathbf{E}_t^i\left[\exp\left\{-\int_t^T f_s^i H_s^i ds\right\}\right] = \mathbf{E}_t^i\left[\exp\left\{-\int_t^T \hat{H}_s^i ds\right\}\right], \tag{A.59}$$

where we define $\hat{H}^i = f^i H^i$. Then, we obtain

$$\begin{aligned} d\hat{H}_t^i &= (\partial_t f_t^i) H_t^i dt + f_t^i dH_t^i \\ &= \left[f_t^i\{\mu_{1,t}^H + \mu_{2,t}^H H_t^i + \sigma_t^H \cdot (\sigma_t^y + \sigma_{\hat{\xi},t} + \hat{\lambda}_{i,t})\} + (\partial_t f_t^i / f_t^i) f_t^i H_t^i\right] dt + f_t^i \sigma_t^H \cdot dW_t^i \\ &:= \left[\alpha_{\hat{H}^i,t} + \beta_{\hat{H}^i,t} \hat{H}_t^i\right] dt + \sigma_{\hat{H}^i,t} \cdot dW_t^i, \end{aligned} \tag{A.60}$$

where we assume $f_t^i \neq 0$ for all $t \in [0,T]$ and define the following nonrandom functions;

$$\alpha_{\hat{H}^i,t} = f_t^i\{\mu_{1,t}^H + \sigma_t^H \cdot (\sigma_t^y + \sigma_{\hat{\xi},t} + \hat{\lambda}_{i,t})\}, \tag{A.61}$$

$$\beta_{\hat{H}^i,t} = (\partial_t f_t^i / f_t^i + \mu_{2,t}^H), \tag{A.62}$$

$$\sigma_{\hat{H}^i,t} = f_t^i \sigma_t^H. \tag{A.63}$$

Then, straightforward calculation with Gaussianity of $\hat{H}^i_t$ shows that

$$\mathbf{E}^i_t\left[\exp\left\{\int_t^T(\sigma^y_s+\sigma_{\hat{\xi},s})\cdot\hat{\lambda}_{i,s}ds\right\}\right]=\mathbf{E}^i_t\left[\exp\left\{-\int_t^T\hat{H}^i_s ds\right\}\right]=\exp\left[\hat{A}^i(t,T)+\hat{B}^i(t,T)\hat{H}^i_t\right], \tag{A.64}$$

where we defined nonrandom functions

$$\hat{B}^i(t,T)=-\int_t^T\exp\left\{\int_t^s\beta_{\hat{H}^i,u}du\right\}ds, \tag{A.65}$$

$$\hat{A}^i(t,T)=\int_t^T\alpha_{\hat{H}^i,u}\hat{B}^i(u,T)du+\frac{1}{2}\int_t^T|\sigma_{\hat{H}^i,u}|^2(\hat{B}^i(u,T))^2du. \tag{A.66}$$

Finally, noting $\hat{H}^i_t=f^i_tH^i_t$, we obtain

$$\mathbf{E}^i_t\left[\exp\left\{\int_t^T(\sigma^y_s+\sigma_{\hat{\xi},s})\cdot\hat{\lambda}_{i,s}ds\right\}\right]=\exp\left[A^i(t,T)+B^i(t,T)H^i_t\right], \tag{A.67}$$

where

$$A^i(t,T)=\hat{A}^i(t,T), \tag{A.68}$$

$$B^i(t,T)=\hat{B}^i(t,T)f^i_t. \tag{A.69}$$