CARF Working Paper

CARF-F-557

Online supplement for "Equilibrium multi-agent model with heterogeneous views on fundamental risks"

Keisuke Kizaki
Graduate School of Economics, The University of Tokyo

Taiga Saito
Graduate School of Economics, The University of Tokyo

Akihiko Takahashi
Graduate School of Economics, The University of Tokyo

March 15, 2023

CARF is presently supported by Nomura Holdings, Inc., Sumitomo Mitsui Banking Corporation, The Dai-ichi Life Insurance Company, Limited, The Norinchukin Bank, MUFG Bank, Ltd. and Ernst & Young ShinNihon LLC. This financial support enables us to issue CARF Working Papers.

Online supplement for "Equilibrium multi-agent model with heterogeneous views on fundamental risk"

Keisuke Kizaki

Graduate School of Economics, The University of Tokyo , 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan

Taiga Saito*

Graduate School of Economics, The University of Tokyo , 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan

Akihiko Takahashi

Graduate School of Economics, The University of Tokyo , 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan

Abstract

This supplementary file provides an example of the stochastic process ν satisfying Assumptions 3 and 4 in Section 3, details of the proofs for the Proposition 6 and Lemma 8 in Appendix A, the multi-agent equilibrium in the exponential utility case, and a possible extension to the case of stochastic boundaries on agents' views.

For the convenience of reference, the numbering of the equations, Propositions, and Lemmas is subsequent to that of the main text.

1. Example of stochastic process ν

In this section, we present an example of the stochastic process ν satisfying Assumptions 3 and 4 in Section 3. We consider the following form for ν.

Example 1. *We consider ν described as a sum of Ornstein-Uhlenbeck processes as follows. Under the probability measure* $\mathbf{P}$*,*

$$\nu_\tau = \sum_{j=1}^{d} X_{j,\tau},$$

$$X_{j,\tau} = X_{j,0} + \int_0^\tau (a_j - b_j X_{j,s})ds + \int_0^\tau \sigma_j dB_{j,s},$$

*Corresponding author

Email addresses: kizaki-keisuke526@g.ecc.u-tokyo.ac.jp (Keisuke Kizaki), staiga@e.u-tokyo.ac.jp (Taiga Saito)

where $a_j, b_j \in \boldsymbol{R}, \sigma_j > 0$, *which is rewritten as*

$$X_{j,\tau} = X_{j,0} + \int_0^\tau (a_j - b_j X_{j,s})ds + \int_0^\tau \sigma_j (dB_{j,s}^{\boldsymbol{\lambda}^{k,\star}} + \lambda_{j,s}^{k,\star} ds),$$

under $\mathbf{P}^{\boldsymbol{\lambda}^{k,\star}}$. *We confirm that* ν *satisfies Assumption 3 as follows.*

Taking the Malliavin derivative $D_{j,u}^{\boldsymbol{\lambda}^{k,\star}}$,

$$D_{j,u}^{\boldsymbol{\lambda}^{k,\star}} X_{j,\tau} = -\int_u^\tau b_j D_{j,u}^{\boldsymbol{\lambda}^{k,\star}} X_{j,s} ds + \sigma_j,$$

we obtain

$$D_{j,u}^{\boldsymbol{\lambda}^{k,\star}} X_{j,\tau} = \sigma_j e^{-b_j(\tau-u)}, (u \le \tau).$$

Then,

$$\begin{aligned}\int_u^s \mathbf{E}_u^{\boldsymbol{\lambda}^{k,\star}}[D_{j,u}^{\boldsymbol{\lambda}^{k,\star}} \nu_\tau] d\tau &= \int_u^s \mathbf{E}_u^{\boldsymbol{\lambda}^{k,\star}}[D_{j,u}^{\boldsymbol{\lambda}^{k,\star}} X_{j,\tau}] d\tau \\ &= \int_u^s \sigma_j e^{-b_j(\tau-u)} d\tau = \sigma_j \frac{1-e^{-b_j(s-u)}}{b_j} \ge 0,\end{aligned}$$

where we denote $\mathbf{E}^{\boldsymbol{\lambda}^{k,\star}}[\cdot|\mathcal{F}_u]$ *by* $\mathbf{E}_u^{\boldsymbol{\lambda}^{k,\star}}$.

Also, this Ornstein-Uhlenbeck process also satisfies Assumption 4.

We note that

$$X_{j,s} = X_{j,0} + a_j \int_0^s e^{-b_j(s-u)} du + \sigma_j \int_0^s e^{-b_j(s-u)} dB_{j,u}.$$

Under $\mathbf{P}^{\boldsymbol{\lambda}^{k,\star}+\alpha\hat{\boldsymbol{\lambda}}^k}$,

$$\begin{aligned}X_{j,s} &= X_{j,0} + a_j \int_0^s e^{-b_j(s-u)} du \\ &+ \sigma_j \int_0^s e^{-b_j(s-u)} [dB_{j,u}^{\boldsymbol{\lambda}^{k,\star}+\alpha\hat{\boldsymbol{\lambda}}^k} + (\lambda_{j,u}^{k,\star} + \alpha\hat{\lambda}_{j,u}^k) du].\end{aligned}$$

$B^{\boldsymbol{\lambda}^{k,\star}+\alpha\hat{\boldsymbol{\lambda}}^k}$ *under* $\mathbf{P}^{\boldsymbol{\lambda}^{k,\star}+\alpha\hat{\boldsymbol{\lambda}}^k}$ *has the same distribution as* $B^{\boldsymbol{\lambda}^{k,\star}}$ *under* $\mathbf{P}^{\boldsymbol{\lambda}^{k,\star}}$. *Hence, we have*

$$\mathbf{E}^{\boldsymbol{\lambda}^{k,\star}+\alpha\hat{\boldsymbol{\lambda}}^k}\left[\int_0^T \int_0^t \sum_{j=1}^d X_{j,s} ds dt\right] = \mathbf{E}^{\boldsymbol{\lambda}^{k,\star}}\left[\int_0^T \int_0^t \left(\sum_{j=1}^d X_{j,s} + \alpha\sigma_j \int_0^s e^{-b_j(s-u)} \hat{\lambda}_{j,u}^k du\right) ds dt\right].$$

Then,

$$\begin{aligned}&\lim_{\alpha\to 0} \frac{1}{\alpha}\left\{\mathbf{E}^{\boldsymbol{\lambda}^{k,\star}+\alpha\hat{\boldsymbol{\lambda}}^k}\left[\int_0^T \int_0^t \sum_{j=1}^d X_{j,s} ds dt\right] - \mathbf{E}^{\boldsymbol{\lambda}^{k,\star}}\left[\int_0^T \int_0^t \sum_{j=1}^d X_{j,s} ds dt\right]\right\} \\ &= \lim_{\alpha\to 0} \mathbf{E}^{\boldsymbol{\lambda}^{k,\star}}\left[\int_0^T \frac{1}{\alpha}\left\{\int_0^t \left(\sum_{j=1}^d X_{j,s} + \alpha\sigma_j \int_0^s e^{-b_j(s-u)} \hat{\lambda}_{j,u}^k du\right) ds - \int_0^t \sum_{j=1}^d X_{j,s} ds\right\} dt\right] \\ &= \mathbf{E}^{\boldsymbol{\lambda}^{k,\star}}\left[\int_0^T \int_0^t \sum_{j=1}^d \sigma_j \int_0^s e^{-b_j(s-u)} \hat{\lambda}_{j,u}^k du ds dt\right] > 0.\end{aligned}$$

Therefore, the Ornstein-Uhlenbeck process satisfies

$$\lim_{\alpha \to 0} \frac{1}{\alpha} \left\{ \mathbf{E}^{\boldsymbol{\lambda}^{k,\star}+\alpha\hat{\boldsymbol{\lambda}}^k} \left[\int_0^T \int_0^t \nu_s ds dt \right] - \mathbf{E}^{\boldsymbol{\lambda}^{k,\star}} \left[\int_0^T \int_0^t \nu_s ds dt \right] \right\}$$
$$> 0.$$

2. Proof of Proposition 6

In this section, we provide the proof of Proposition 6. Particularly, we use a Malliavin calculus approach. (For Malliavin calculus approaches to optimal portfolio problems, see Ocone and Karatzas [21] and Takahashi and Yoshida [32], for example).

(A) First of all, (11) and (12) in Lemma 1 in the main text hold. In fact, since $c_t^{k,*} = \frac{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \mathbf{E}[\int_0^T H_{0,t}\varepsilon_t^k dt]}{TH_{0,t}}$, we have

$$c_0^{k,*} = \frac{\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right]}{T}, \tag{42}$$

and

$$\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \frac{c_0^{k,*}}{c_t^{k,*}} = H_{0,t}, \tag{43}$$

which corresponds to (11) in the main text in the log utility case. Also,

$$\begin{aligned} &\mathbf{E}\left[\int_0^T H_{0,t} c_t^{k,*} dt\right] \\ &= \frac{\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right]}{T} \int_0^T \mathbf{E}\left[\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\right] dt \\ &= \mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right], \end{aligned} \tag{44}$$

which indicates (12) in the main text.

(B) Thus, by Lemma 1 in the main text, we have only to show (13) in the main text holds, that is, $sgn(Z_j) = 1,\ j = 1, \ldots, d_1$, where

$$\begin{aligned} &dV_t^{k,\hat{\boldsymbol{\lambda}}_1^k,\boldsymbol{\lambda}_2^k} \\ &= -\left(U^k(c_t^{k,*}) - \sum_{j=1}^{d_1} \bar{\lambda}_{j,t}^k |Z_{j,t}| + \sum_{j=d_1+1}^{d_1+d_2} \lambda_{j,t}^k Z_{j,t} \right) dt \\ &+ \sum_{j=1}^{d} Z_{j,t} dB_{j,t},\ V_T^{k,\hat{\boldsymbol{\lambda}}_1^k,\boldsymbol{\lambda}_2^k} = 0. \end{aligned} \tag{45}$$

We note that for $\boldsymbol{\lambda}_1^{k,*} = (-\bar{\lambda}_1^k, \dots, -\bar{\lambda}_{d_1}^k)^\top$, if we show $sgn(Z_j) = 1,\ j = 1, \dots, d_1$, where $Z_j, j = 1, \dots, d_1$ are part of a solution $(V^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}, Z)$ of a BSDE

$$\begin{aligned}
&dV_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \\
&= -\left(U^k(c_t^{k,*}) - \sum_{j=1}^{d_1} \bar{\lambda}_{j,t}^k Z_{j,t} + \sum_{j=d_1+1}^{d_1+d_2} \lambda_{j,t}^k Z_{j,t}\right) dt \\
&+ \sum_{j=1}^{d} Z_{j,t} dB_{j,t}, \\
&= -U^k(c_t^{k,*})dt + \sum_{j=1}^{d} Z_{j,t} dB_{j,t}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k},\ V_T^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} = 0,
\end{aligned} \tag{46}$$

$(V^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}, Z)$ is also a solution of BSDE (45).

Then, by the uniqueness a solution of BSDE (45), it results in $sgn(Z_j) = 1,\ j = 1, \dots, d_1$ for Z in (45).

In the following, we denote $\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k) = (\boldsymbol{\lambda}_1^{k,*\top}, \boldsymbol{\lambda}_2^{k\top}, 0, \dots, 0)^\top$.

Since

$$\begin{aligned}
TY^k &= \mathbf{E}\left[\int_0^T \left(\frac{\sum_{l=1}^K \eta_t^{l,*} Y^l}{\varepsilon_t}\right) \varepsilon_t^k dt\right] \\
&= \frac{\sum_{l=1}^K Y^l}{\varepsilon_0} \mathbf{E}\left[\int_0^T H_{0,t} \varepsilon_t^k dt\right],
\end{aligned} \tag{47}$$

where we used (23) in Proposition 2 in the main text in the first equality and (22) in the main text in the second equality, it follows that

$$\mathbf{E}\left[\int_0^T H_{0,t} \varepsilon_t^k dt\right] = \frac{\varepsilon_0 TY^k}{\sum_{l=1}^K Y^l},\ k = 1, \dots, K.$$

Hence,

$$\begin{aligned}
c_t^{k,*} &= \frac{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \mathbf{E}\left[\int_0^T H_{0,t} \varepsilon_t^k dt\right]}{TH_{0,t}} \\
&= \frac{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} Y^k}{\left(\eta_t^{1,*} + \sum_{l=2}^K \eta_t^{l,*} Y^l\right)} \varepsilon_t.
\end{aligned} \tag{48}$$

In the following, we show $sgn(Z_j) = 1,\ j = 1, \dots, d_1$ in Steps 1-3.

Step1: Representation of $Z_j,\ j = 1, \dots, d_1$

We use a Malliavin calculus-based method to investigate the sign of $Z_j\ j = 1, \dots, d_1$.

Noting (46), we let

$$\mathcal{X}^k(T) := \int_0^T U^k(c_s^{k,*})ds = \int_0^T Z_s^\top dB_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_0^T U^k(c_s^{k,*})ds\right].$$

By taking the Malliavin derivative $D_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$ for the both sides, we have

$$D_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\mathcal{X}^k(T) = \int_u^T [D_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} Z_s]^\top dB_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + Z_u,$$

where $D_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$ is the Malliavin derivative with respect to $B_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$. We first suppose that the conditional expectation $\mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[\int_u^T [D_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} Z_s]^\top dB_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}] = 0$ for all $0 \le u \le T$.

Then, by taking the conditional expectation $\mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$, by which we denote $\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[\cdot|\mathcal{F}_u]$,

$$\begin{aligned} Z_u^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} &= \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[D_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\mathcal{X}^k(T)] \\ &= \int_u^T \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[D_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} U^k(c_s^{k,*})\right] ds. \end{aligned} \tag{49}$$

In the following, we denote Z as $Z_u^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$ for clarity.

In Step 2, we first calculate $Z^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$ by (49) and later confirm the conditional expectation is zero with the calculated $Z^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$.

Step2: Calculation of $Z_j^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$, $j = 1,\dots,d_1$

Here, we recall Assumption 1 and that we assumed any $\boldsymbol{\lambda}_2^k$ to be nonrandom.

By (48),

$$c_t^{k,*} = \frac{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} Y^k}{\left(\eta_t^{1,*} + \sum_{l=2}^K \eta_t^{l,*} Y^l\right)}\varepsilon_t = \frac{\frac{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}}{\eta_t^{1,*}} Y^k}{\left(1 + \sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l\right)}\varepsilon_t.$$

Taking the log of $c_t^{k,*}$, we have

$$\begin{aligned} \log c_t^{k,*} = \log Y^k + \log \frac{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}}{\eta_t^{1,*}} + \log \varepsilon_t \\ - \log\left(1 + \sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l\right). \end{aligned} \tag{50}$$

By (49) and (50), $Z_j^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$ is

$$\begin{aligned}
&Z_{j,u}^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\\
&= \int_u^T \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \log c_s^{k,*}]ds\\
&= \int_u^T \Bigg\{ \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \log \varepsilon_s]\\
&+ \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \log \frac{\eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}}{\eta_s^{1,*}} \right]\\
&- \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \log\left(1+\sum_{l=2}^K \frac{\eta_s^{l,*}}{\eta_s^{1,*}} Y^l\right)\right] \Bigg\} ds. \qquad (51)
\end{aligned}$$

We consider the first term of the integrand in (51). Under $\mathbf{P}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$, by (20) in the main text, $\log \varepsilon_s$ is given by

$$\begin{aligned}
\log \varepsilon_s &= \log \varepsilon_0 + \int_0^s (\nu_\tau - \frac{1}{2}|\rho_\tau|^2)d\tau + \int_0^s \rho_\tau^\top [dB_\tau^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + \boldsymbol{\lambda}_\tau^{k,*}(\boldsymbol{\lambda}_2^k)d\tau]\\
&= \log \varepsilon_0 + \int_0^s (\nu_\tau - \frac{1}{2}|\rho_\tau|^2 + \rho_\tau^\top \boldsymbol{\lambda}_\tau^{k,*}(\boldsymbol{\lambda}_2^k))d\tau + \int_0^s \rho_\tau^\top dB_\tau^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}.
\end{aligned}$$

Since $\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k)$ is nonrandom, taking Malliavin derivative of $\log\varepsilon$ with respect to Brownian motion $B_j^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$,

$$D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \log \varepsilon_s = \int_u^s D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \nu_\tau d\tau + \rho_{j,u}. \qquad (52)$$

For the second term of the integrand in (51), by the definition of $\eta_t^{\boldsymbol{\lambda}^k}$,

$$\eta_t^{\boldsymbol{\lambda}^k} = \exp\left\{ \sum_{j=1}^d \int_0^t \lambda_{j,s}^k dB_{j,s} - \frac{1}{2}\sum_{j=1}^d \int_0^t |\lambda_{j,s}^k|^2 ds \right\}, \qquad (53)$$

we have

$$\begin{aligned}
&\log \frac{\eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}}{\eta_s^{1,*}} = -\frac{1}{2}\int_0^s (|\boldsymbol{\lambda}_\tau^{k,*}(\boldsymbol{\lambda}_2^k)|^2 - |\boldsymbol{\lambda}_\tau^{1,*}|^2)d\tau\\
&+ \int_0^s (\boldsymbol{\lambda}_\tau^{k,*}(\boldsymbol{\lambda}_2^k) - \boldsymbol{\lambda}_\tau^{1,*})^\top [dB_\tau^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + \boldsymbol{\lambda}_\tau^{k,*}(\boldsymbol{\lambda}_2^k)d\tau]\\
&= \frac{1}{2}\int_0^s |\boldsymbol{\lambda}_\tau^{k,*}(\boldsymbol{\lambda}_2^k) - \boldsymbol{\lambda}_\tau^{1,*}|^2 d\tau + \int_0^s (\boldsymbol{\lambda}_\tau^{k,*}(\boldsymbol{\lambda}_2^k) - \boldsymbol{\lambda}_\tau^{1,*})^\top dB_\tau^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}.
\end{aligned}$$

Thus, by nonrandomness of $\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k)$ and $\boldsymbol{\lambda}^{1,*}$,

$$D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \log \frac{\eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}}{\eta_s^{1,*}} = \lambda_{j,u}^{k,*} - \lambda_{j,u}^{1,*},\ j = 1,\ldots,d_1. \qquad (54)$$

For the third term of the integrand in (51),

$$D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\log\left(1+\sum_{l=2}^K\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l\right)=\frac{D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\sum_{l=2}^K\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l}{\left(1+\sum_{l=2}^K\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l\right)}$$
$$=\frac{\sum_{l=2}^K\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^lD_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\log\frac{\eta_s^{l,*}}{\eta_s^{1,*}}}{\left(1+\sum_{l=2}^K\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l\right)}.$$

Similarly, by (53), we have

$$\log\frac{\eta_s^{l,*}}{\eta_s^{1,*}}=-\frac{1}{2}\int_0^s(|\boldsymbol{\lambda}_\tau^{l,*}|^2-|\boldsymbol{\lambda}_\tau^{1,*}|^2)d\tau$$
$$+\int_0^s(\boldsymbol{\lambda}_\tau^{l,*}-\boldsymbol{\lambda}_\tau^{1,*})^\top[dB_\tau^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}+\boldsymbol{\lambda}_\tau^{k,*}(\boldsymbol{\lambda}_2^k)d\tau].$$

By nonrandomness of $\boldsymbol{\lambda}^{1,*},\boldsymbol{\lambda}^{l,*}$ and $\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k)$,

$$D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\log\frac{\eta_s^{l,*}}{\eta_s^{1,*}}=\lambda_{j,u}^{l,*}-\lambda_{j,u}^{1,*},\ j=1,\ldots,d_1.$$

Thus,

$$D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\log\left(1+\sum_{l=2}^K\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l\right)$$
$$=\frac{\sum_{l=2}^K\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l(\lambda_{j,u}^{l,*}-\lambda_{j,u}^{1,*})}{\left(1+\sum_{l=2}^K\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l\right)}. \tag{55}$$

By (52), (54) and (55), (51) is

$$Z_{j,u}^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$$
$$=\int_u^T\left\{\mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_u^sD_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\nu_\tau d\tau\right]+\rho_{j,u}+(\lambda_{j,u}^{k,*}-\lambda_{j,u}^{1,*})-\mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\frac{\sum_{l=2}^K\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l(\lambda_{j,u}^{l,*}-\lambda_{j,u}^{1,*})}{\left(1+\sum_{l=2}^K\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l\right)}\right]\right\}ds$$
$$=\int_u^T\left\{\int_u^s\mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\nu_\tau]d\tau+\rho_{j,u}+\mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\frac{1+\sum_{l=2;l\neq k}^K\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l}{\left(1+\sum_{l=2}^K\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l\right)}(\lambda_{j,u}^{k,*}-\lambda_{j,u}^{1,*})\right]\right.$$
$$\left.-\mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\frac{\sum_{l=2;l\neq k}^K\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l(\lambda_{j,u}^{l,*}-\lambda_{j,u}^{1,*})}{\left(1+\sum_{l=2}^K\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l\right)}\right]\right\}ds$$
$$=\int_u^T\left\{\int_u^s\mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\nu_\tau]d\tau+\rho_{j,u}+\mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\frac{(\lambda_{j,u}^{k,*}-\lambda_{j,u}^{1,*})+\sum_{l=2;l\neq k}^K\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l(\lambda_{j,u}^{k,*}-\lambda_{j,u}^{l,*})}{\left(1+\sum_{l=2}^K\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l\right)}\right]\right\}ds.$$

Using $Y^1 = 1$, we have

$$Z^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}_{j,u} = \int_u^T \left\{ \int_u^s \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\nu_\tau]d\tau + \rho_{j,u} - \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\frac{\sum_{l=1;l\neq k}^K \frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l(\lambda_{j,u}^{l,*}-\lambda_{j,u}^{k,*})}{\left(1+\sum_{l=2}^K \frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l\right)}\right]\right\}ds. \tag{56}$$

Next, with this expression of $Z^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$,

$$Z^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}_{j,s} = \int_s^T \left\{ \int_s^t \mathbf{E}_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[D_{j,s}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\nu_\tau]d\tau + \rho_{j,s} - \mathbf{E}_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\frac{\sum_{l=1;l\neq k}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l(\lambda_{j,s}^{l,*}-\lambda_{j,s}^{k,*})}{\left(1+\sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l\right)}\right]\right\}dt, \tag{57}$$

we will confirm $\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[\int_u^T [D_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} Z_s^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}]^\top dB_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}] = 0$ for all $0 \leq u \leq T$.

By (25) in the main text in Assumption 3 and nonrandomness of ρ_j, we have only to show

$$\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_u^T [D_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \int_s^T \mathbf{E}_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\frac{\sum_{l=1;l\neq k}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l(\lambda_{j,s}^{l,*}-\lambda_{j,s}^{k,*})}{\left(1+\sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l\right)}\right]dt]^\top dB_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\right] = 0. \tag{58}$$

For all $0 \leq u \leq s \leq T$, $m = 1, \ldots, d_1$, noting that

$$\begin{aligned} & D_{m,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \int_s^T \mathbf{E}_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\frac{\sum_{l=1;l\neq k}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l(\lambda_{j,s}^{l,*}-\lambda_{j,s}^{k,*})}{\left(1+\sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l\right)}\right]dt \\ &= \int_s^T \mathbf{E}_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[D_{m,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\frac{\sum_{l=1;l\neq k}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l(\lambda_{j,s}^{l,*}-\lambda_{j,s}^{k,*})}{\left(1+\sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l\right)}\right]dt, \end{aligned} \tag{59}$$

where we exchanged the order between Malliavin derivative and conditional expectation due to boundedness of $\frac{\sum_{l=1;l\neq k}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l(\lambda_{j,s}^{l,*}-\lambda_{j,s}^{k,*})}{\left(1+\sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l\right)}$ as in (65) below, we calculate

$$\begin{aligned} & D_{m,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\frac{\sum_{l=1;l\neq k}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l(\lambda_{j,s}^{l,*}-\lambda_{j,s}^{k,*})}{\left(1+\sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l\right)} \\ &= \frac{\sum_{l=1;l\neq k}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l(\lambda_{j,s}^{l,*}-\lambda_{j,s}^{k,*})}{\left(1+\sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l\right)} D_{m,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left(\log \sum_{l=1;l\neq k}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l(\lambda_{j,s}^{l,*}-\lambda_{j,s}^{k,*}) - \log\left(1+\sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l\right)\right), \end{aligned} \tag{60}$$

$$D_{m,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \log\left(\sum_{l=1;l\neq k}^{K} \frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l(\lambda_{j,s}^{l,*}-\lambda_{j,s}^{k,*})\right) = \frac{\sum_{l=2}^{K} \frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l(\lambda_{j,s}^{l,*}-\lambda_{j,s}^{k,*})(\lambda_{m,u}^{l,*}-\lambda_{m,u}^{1,*})}{\left(1+\sum_{l=2}^{K}\frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l(\lambda_{j,s}^{l,*}-\lambda_{j,s}^{k,*})\right)}, \tag{61}$$

and

$$D_{m,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \log\left(1+\sum_{l=2}^{K} \frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l\right) = \frac{\sum_{l=2}^{K} \frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l(\lambda_{m,u}^{l,*}-\lambda_{m,u}^{1,*})}{\left(1+\sum_{l=2}^{K}\frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l\right)}. \tag{62}$$

Then,

$$\begin{aligned}
&\left| D_{m,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \frac{\sum_{l=1;l\neq k}^{K} \frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l(\lambda_{j,s}^{l,*}-\lambda_{j,s}^{k,*})}{\left(1+\sum_{l=2}^{K}\frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l\right)} \right| \\
&\le 2 \max_{l=1,\ldots,K,0\le s\le T} |\lambda_{j,s}^{l,*}-\lambda_{j,s}^{k,*}| \max_{l=1,\ldots,K,0\le u\le T} |\lambda_{m,u}^{l,*}-\lambda_{m,u}^{1,*}| \\
&< \infty,
\end{aligned} \tag{63}$$

and thus

$$\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[\int_u^T \left[D_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \int_s^T \mathbf{E}_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \left[\frac{\sum_{l=1;l\neq k}^{K} \frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l(\lambda_{j,s}^{l,*}-\lambda_{j,s}^{k,*})}{\left(1+\sum_{l=2}^{K}\frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l\right)} \right] dt]^\top dB_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \right] = 0. \tag{64}$$

Therefore, $\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[\int_u^T [D_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} Z_s]^\top dB_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}] = 0$ for all $0 \le u \le T$.

Step3: $sgn(Z_j^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}) = 1,\ j = 1,\ldots,d_1$

Using $\frac{\sum_{l=1;l\neq k}^{K} \frac{\eta_s^{l,*}}{\eta_s^{1,*}} Y^l}{\left(1+\sum_{l=2}^{K}\frac{\eta_t^{k,*}}{\eta_t^{1,*}} Y^l\right)} \in (0,1)$, we have

$$\begin{aligned}
\frac{\sum_{l=1;l\neq k}^{K} \frac{\eta_s^{l,*}}{\eta_s^{1,*}} Y^l(\lambda_{j,u}^{l,*}-\lambda_{j,u}^{k,*})}{\left(1+\sum_{l=2}^{K}\frac{\eta_s^{l,*}}{\eta_s^{1,*}} Y^l\right)} &\le \max_{l,k\in\{1,\ldots,K\};l\neq k} (\lambda_{j,u}^{l,*}-\lambda_{j,u}^{k,*}) \frac{\sum_{l=1;l\neq k}^{K} \frac{\eta_s^{l,*}}{\eta_s^{1,*}} Y^l}{\left(1+\sum_{l=2}^{K}\frac{\eta_s^{l,*}}{\eta_s^{1,*}} Y^l\right)} \\
&< \max_{l,k\in\{1,\ldots,K\};l\neq k} (\lambda_{j,u}^{l,*}-\lambda_{j,u}^{k,*}).
\end{aligned} \tag{65}$$

By Assumptions 1-3, the right hand side of (56) is positive, that is $Z_{j,u}^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} > 0,\ \forall u \in [0,T]$, and thus $sgn(Z_j^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}) \equiv 1$.

Therefore,

$$-\bar{\lambda}_j^k sgn(Z_j^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}) = -\bar{\lambda}_j^k = \lambda_j^{k,*},\ j = 1,\ldots,d_1.$$

□

3. Proof of Lemma 8

Let $\boldsymbol{\lambda}_2^{l,*} = (\lambda_{d_1+1}^{l,*}, \dots, \lambda_{d_1+d_2}^{l,*})^\top$,

where $\lambda_{d_1+1}^{l,*}, \dots, \lambda_{d_1+d_2}^{l,*}$ are defined in (21) in the main text, and $\bar{\boldsymbol{\lambda}}_2^l = (\bar{\lambda}_{d_1+1}^l, \dots, \bar{\lambda}_{d_1+d_2}^l)^\top$, $l = 1, \dots, K$. Also, we let $\boldsymbol{B}_3 = (B_{d_1+d_2+1}, \dots, B_d)^\top$, $\boldsymbol{\rho}_1 = (\rho_1, \dots, \rho_{d_1})^\top$, $\boldsymbol{\rho}_2 = (\rho_{d_1+1}, \dots, \rho_{d_1+d_2})^\top$, and $\boldsymbol{\rho}_3 = (\rho_{d_1+d_2+1}, \dots, \rho_d)^\top$. By (22) in the main text, we have

$$\log H_{0,t} = \log\left(\eta_t^{1,*} + \sum_{l=2}^K \eta_t^{l,*} Y^l\right) - \log\varepsilon_t + \log\varepsilon_0 - \log\left(1 + \sum_{l=2}^K Y^l\right).$$

Since $\log\varepsilon_0 - \log\left(1 + \sum_{l=2}^K Y^l\right)$ is a constant, we have only to consider

$$\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k}\left[\int_0^T \left\{\log\varepsilon_t - \log\left(\eta_t^{1,*} + \sum_{l=2}^K \eta_t^{l,*} Y^l\right)\right\} dt\right],$$

which is a part of $\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k}\left[\int_0^T -\log H_{0,t} dt\right]$.

We define $F(\boldsymbol{\lambda}_2^k)$ as follows.

$$\begin{aligned}
F(\boldsymbol{\lambda}_2^k) &= F_1(\boldsymbol{\lambda}_2^k) + F_2(\boldsymbol{\lambda}_2^k), \\
F_1(\boldsymbol{\lambda}_2^k) &= \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k}\left[\int_0^T \log\varepsilon_t dt\right], \\
F_2(\boldsymbol{\lambda}_2^k) &= \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k}\left[\int_0^T -\log\left(\eta_t^{1,*} + \sum_{l=2}^K \eta_t^{l,*} Y^l\right) dt\right].
\end{aligned}$$

For any nonrandom $\hat{\boldsymbol{\lambda}}_2^k \le \bar{\boldsymbol{\lambda}}_2^k$ $(0 \le \hat{\lambda}_j^k \le \bar{\lambda}_j^k,\ j = d_1+1, \dots, d_1+d_2)$,

$$\begin{aligned}
&\lim_{\alpha\to 0} \frac{F(\boldsymbol{\lambda}_2^k + \alpha\hat{\boldsymbol{\lambda}}_2^k) - F(\boldsymbol{\lambda}_2^k)}{\alpha} \\
&= \lim_{\alpha\to 0} \frac{F_1(\boldsymbol{\lambda}_2^k + \alpha\hat{\boldsymbol{\lambda}}_2^k) - F_1(\boldsymbol{\lambda}_2^k)}{\alpha} + \lim_{\alpha\to 0} \frac{F_2(\boldsymbol{\lambda}_2^k + \alpha\hat{\boldsymbol{\lambda}}_2^k) - F_2(\boldsymbol{\lambda}_2^k)}{\alpha}.
\end{aligned}$$

Step 1: Calculation of $\lim_{\alpha\to 0} \frac{F_1(\boldsymbol{\lambda}_2^k + \alpha\hat{\boldsymbol{\lambda}}_2^k) - F_1(\boldsymbol{\lambda}_2^k)}{\alpha}$

For $\log\varepsilon_t$,

$$\begin{aligned}
&\log\varepsilon_t \\
&= \log\varepsilon_0 + \int_0^t (\nu_s - \frac{1}{2}|\boldsymbol{\rho}_s|^2) ds + \int_0^t \boldsymbol{\rho}_s^\top dB_s \\
&= \log\varepsilon_0 + \int_0^t (\nu_s - \frac{1}{2}|\boldsymbol{\rho}_s|^2) ds \\
&+ \int_0^t \boldsymbol{\rho}_{1,s}^\top [d\boldsymbol{B}_{1,s}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k + \alpha\hat{\boldsymbol{\lambda}}_2^k} + \boldsymbol{\lambda}_{1,s}^{k,*} ds] \\
&+ \int_0^t \boldsymbol{\rho}_{2,s}^\top [d\boldsymbol{B}_{2,s}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k + \alpha\hat{\boldsymbol{\lambda}}_2^k} + (\boldsymbol{\lambda}_{2,s}^k + \alpha\hat{\boldsymbol{\lambda}}_{2,s}^k) ds] + \int_0^t \boldsymbol{\rho}_{3,s}^\top d\boldsymbol{B}_{3,s}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k + \alpha\hat{\boldsymbol{\lambda}}_2^k}.
\end{aligned}$$

$(\boldsymbol{B}_1^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}, \boldsymbol{B}_2^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}, \boldsymbol{B}_3^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k})$ under $\mathbf{P}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}$ has the same distribution as $(\boldsymbol{B}_1^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}, \boldsymbol{B}_2^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}, \boldsymbol{B}_3^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k})$ under $\mathbf{P}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$. Hence, we have

$$\begin{aligned}&\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}\left[\int_0^T\int_0^t \boldsymbol{\rho}_{1,s}^\top[d\boldsymbol{B}_{1,s}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}+\boldsymbol{\lambda}_{1,s}^{k,*}ds]dt\right]\\&=\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_0^T\int_0^t \boldsymbol{\rho}_{1,s}^\top[d\boldsymbol{B}_{1,s}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}+\boldsymbol{\lambda}_{1,s}^{k,*}ds]dt\right],\end{aligned}$$

and

$$\begin{aligned}&\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}\left[\int_0^T\int_0^t \boldsymbol{\rho}_{2,s}^\top[d\boldsymbol{B}_{2,s}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}+(\boldsymbol{\lambda}_{2,s}^k+\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k)ds]dt\right]\\&=\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_0^T\int_0^t \boldsymbol{\rho}_{2,s}^\top[d\boldsymbol{B}_{2,s}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}+(\boldsymbol{\lambda}_{2,s}^k+\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k)ds]dt\right].\end{aligned}$$

Hence,

$$\begin{aligned}&\lim_{\alpha\to0}\frac{F_1(\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k)-F_1(\boldsymbol{\lambda}_2^k)}{\alpha}\\&=\lim_{\alpha\to0}\frac{1}{\alpha}\left\{\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}\left[\int_0^T\int_0^t\nu_s dsdt\right]-\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_0^T\int_0^t\nu_s dsdt\right]\right\}\\&\quad+\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_0^T\int_0^t\boldsymbol{\rho}_{2,s}^\top\hat{\boldsymbol{\lambda}}_{2,s}^k dsdt\right].\end{aligned}\tag{66}$$

For the term containing ν, by Assumption 4,

$$\lim_{\alpha\to0}\frac{1}{\alpha}\left\{\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}\left[\int_0^T\int_0^t\nu_s dsdt\right]-\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_0^T\int_0^t\nu_s dsdt\right]\right\}\geq 0.\tag{67}$$

Step 2: Calculation of $\lim_{\alpha\to0}\frac{F_2(\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k)-F_2(\boldsymbol{\lambda}_2^k)}{\alpha}$

For $l=1,\dots,K$,

$$\eta_t^{l,*}=\exp\left\{\int_0^t\boldsymbol{\lambda}_{2,s}^{l,*\top}\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k ds\right\}Z_t^l(\boldsymbol{B}_1^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k},\boldsymbol{B}_2^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}),$$

where we set

$$\begin{aligned}&Z_t^l(\boldsymbol{B}_1^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k},\boldsymbol{B}_2^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k})\\&=\exp\left\{\int_0^t\boldsymbol{\lambda}_{1,s}^{l,*\top}(dB_{1,s}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}+\boldsymbol{\lambda}_{1,s}^{k,*}ds)\right.\\&\left.+\int_0^t\boldsymbol{\lambda}_{2,s}^{l,*\top}(d\boldsymbol{B}_{2,s}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}+\boldsymbol{\lambda}_{2,s}^k ds)-\frac{1}{2}\int_0^t|\boldsymbol{\lambda}_s^{l,*}|^2ds\right\}.\end{aligned}$$

Since $Z_t^l(\boldsymbol{B}_1^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k},\boldsymbol{B}_2^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k})$ under $\mathbf{P}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}$ has the same distribution as $\eta_t^{l,*}=\exp\left\{\int_0^t\boldsymbol{\lambda}_{1,s}^{l,*\top}(dB_{1,s}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}+\boldsymbol{\lambda}_{1,s}^{k,*}ds)+\int_0^t\boldsymbol{\lambda}_{2,s}^{l,*\top}(d\boldsymbol{B}_{2,s}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}+\boldsymbol{\lambda}_{2,s}^k ds)-\frac{1}{2}\int_0^t|\boldsymbol{\lambda}_s^{l,*}|^2ds\right\}$ under $\mathbf{P}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$, we have

$$
\begin{aligned}
&F_2(\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k)\\
&=\mathbf{E}^{\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}\left[\int_0^T -\log\left(\eta_t^{1,*}+\sum_{l=2}^K \eta_t^{l,*}Y^l\right)dt\right]\\
&=\mathbf{E}^{\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}\Bigg[\int_0^T -\log\Bigg(\exp\left\{\int_0^t \boldsymbol{\lambda}_{2,s}^{1,*\top}\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k ds\right\} Z_{1,t}(\boldsymbol{B}_1^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k},\boldsymbol{B}_2^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k})\\
&+\sum_{l=2}^K \exp\left\{\int_0^t \boldsymbol{\lambda}_{2,s}^{l,*\top}\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k ds\right\} Z_t^l(\boldsymbol{B}_1^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k},\boldsymbol{B}_2^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k})Y^l\Bigg)dt\Bigg]\\
&=\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_0^T -\log\left(\exp\left\{\int_0^t \boldsymbol{\lambda}_{2,s}^{1,*\top}\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k ds\right\}\eta_t^{1,*}+\sum_{l=2}^K \exp\left\{\int_0^t \boldsymbol{\lambda}_{2,s}^{l,*\top}\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k ds\right\}\eta_t^{l,*}Y^l\right)dt\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&F_2(\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k)-F_2(\boldsymbol{\lambda}_2^k)\\
&=\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_0^T -\log\left(e^{\int_0^t \boldsymbol{\lambda}_{2,s}^{1,*\top}\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k ds}\eta_t^{1,*}+\sum_{l=2}^K e^{\int_0^t \boldsymbol{\lambda}_{2,s}^{l,*\top}\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k ds}\eta_t^{l,*}Y^l\right)dt\right]\\
&-\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_0^T -\log\left(\eta_t^{1,*}+\sum_{l=2}^K \eta_t^{l,*}Y^l\right)dt\right].
\end{aligned}
$$

Thus, we can obtain Gateaux derivative of F_2 as

$$
\begin{aligned}
&\lim_{\alpha\to 0}\frac{F_2(\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k)-F_2(\boldsymbol{\lambda}_2^k)}{\alpha}\\
&=\lim_{\alpha\to 0}\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\Bigg[\int_0^T \frac{1}{\alpha}\Bigg\{-\log\left(e^{\int_0^t \boldsymbol{\lambda}_{2,s}^{1,*\top}\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k ds}\eta_t^{1,*}+\sum_{l=2}^K e^{\int_0^t \boldsymbol{\lambda}_{2,s}^{l,*\top}\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k ds}\eta_t^{l,*}Y^l\right)\\
&+\log\left(\eta_t^{1,*}+\sum_{l=2}^K \eta_t^{l,*}Y^l\right)\Bigg\}dt\Bigg]
\end{aligned}
$$

Noting that $\frac{\eta_t^{l,*}Y^l}{\eta_t^{1,*}+\sum_{l=2}^K \eta_t^{l,*}Y^l}\in(0,1)$, we have

$$
\begin{aligned}
&\left|\frac{d}{d\alpha}\{-\log(e^{\int_0^t \boldsymbol{\lambda}_{2,s}^{1,*\top}\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k ds}\eta_t^{1,*}+\sum_{l=2}^K e^{\int_0^t \boldsymbol{\lambda}_{2,s}^{l,*\top}\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k ds}\eta_t^{l,*}Y^l)\}\Big|_{\alpha=0}\right|\\
&=\left|\frac{(\int_0^t \boldsymbol{\lambda}_{2,s}^{1,*\top}\hat{\boldsymbol{\lambda}}_{2,s}^k ds)\eta_t^{1,*}+\sum_{l=2}^K(\int_0^t \boldsymbol{\lambda}_{2,s}^{l,*\top}\hat{\boldsymbol{\lambda}}_{2,s}^k ds)\eta_t^{l,*}Y^l}{\eta_t^{1,*}+\sum_{l=2}^K \eta_t^{l,*}Y^l}\right|\\
&\le\int_0^t \bar{\boldsymbol{\lambda}}_{2,s}^{1\top}\hat{\boldsymbol{\lambda}}_{2,s}^k ds+\sum_{l=2}^K\int_0^t \bar{\boldsymbol{\lambda}}_{2,s}^{l\top}\hat{\boldsymbol{\lambda}}_{2,s}^k ds<\infty,
\end{aligned}
$$

for all $(\omega, t) \in \Omega \times [0, T]$. Then, by the dominated convergence theorem, we have

$$
\begin{aligned}
&\lim_{\alpha \to 0} \frac{F_2(\boldsymbol{\lambda}_2^k + \alpha \hat{\boldsymbol{\lambda}}_2^k) - F_2(\boldsymbol{\lambda}_2^k)}{\alpha} \\
&= \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k} \Bigg[\int_0^T \lim_{\alpha \to 0} \frac{1}{\alpha} \Bigg\{ - \log \left(e^{\int_0^t \boldsymbol{\lambda}_{2,s}^{1,*\top} \alpha \hat{\boldsymbol{\lambda}}_{2,s}^k ds} \eta_t^{1,*} + \sum_{l=2}^K e^{\int_0^t \boldsymbol{\lambda}_{2,s}^{l,*\top} \alpha \hat{\boldsymbol{\lambda}}_{2,s}^k ds} \eta_t^{l,*} Y^l \right) \\
&+ \log \left(\eta_t^{1,*} + \sum_{l=2}^K \eta_t^{l,*} Y^l \right) \Bigg\} dt \Bigg] \\
&= -\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k} \Bigg[\int_0^T \frac{d}{d\alpha} \log \left(e^{\int_0^t \boldsymbol{\lambda}_{2,s}^{1,*\top} \alpha \hat{\boldsymbol{\lambda}}_{2,s}^k ds} \eta_t^{1,*} + \sum_{l=2}^K e^{\int_0^t \boldsymbol{\lambda}_{2,s}^{l,*\top} \alpha \hat{\boldsymbol{\lambda}}_{2,s}^k ds} \eta_t^{l,*} Y^l \right) \Bigg|_{\alpha=0} dt \Bigg] \\
&= -\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k} \Bigg[\int_0^T \frac{(\int_0^t \boldsymbol{\lambda}_{2,s}^{1,*\top} \hat{\boldsymbol{\lambda}}_{2,s}^k ds) \eta_t^{1,*} + \sum_{l=2}^K (\int_0^t \boldsymbol{\lambda}_{2,s}^{l,*\top} \hat{\boldsymbol{\lambda}}_{2,s}^k ds) \eta_t^{l,*} Y^l}{\eta_t^{1,*} + \sum_{l=2}^K \eta_t^{l,*} Y^l} dt \Bigg] \\
&\geq -\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k} \Bigg[\int_0^T \left\{ \int_0^t \max_{l=1,\dots,K} \left(\boldsymbol{\lambda}_{2,s}^{l,*\top} \hat{\boldsymbol{\lambda}}_{2,s}^k \right) ds \times \frac{\eta_t^{1,*} + \sum_{l=2}^K \eta_t^{l,*} Y^l}{\eta_t^{1,*} + \sum_{l=2}^K \eta_t^{l,*} Y^l} \right\} dt \Bigg] \\
&= -\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k} \Bigg[\int_0^T \int_0^t \max_{l=1,\dots,K} \left(\boldsymbol{\lambda}_{2,s}^{l,*\top} \hat{\boldsymbol{\lambda}}_{2,s}^k \right) ds dt \Bigg]. \qquad (68)
\end{aligned}
$$

Step 3: Calculation of $\lim_{\alpha \to 0} \frac{F(\boldsymbol{\lambda}_2^k + \alpha \hat{\boldsymbol{\lambda}}_2^k) - F(\boldsymbol{\lambda}_2^k)}{\alpha}$

Therefore, by (66), (67), (68), and Assumption 2, we obtain

$$
\lim_{\alpha \to 0} \frac{F(\boldsymbol{\lambda}_2^k + \alpha \hat{\boldsymbol{\lambda}}_2^k) - F(\boldsymbol{\lambda}_2^k)}{\alpha} > 0.
$$

Hence, (A.4) in the main text is increasing with respect to $\boldsymbol{\lambda}_2^k$. □

4. $\mathbf{E}[\int_0^T \log(\bar{c}_t^{k,*})^2 dt] < \infty$ in the proof of Theorem 3

$\mathbf{E}[\int_0^T \log(\bar{c}_t^{k,*})^2 dt] < \infty$ is confirmed as follows.

$$
\begin{aligned}
(\log \bar{c}_t^{k,*})^2 &= \left(\log \eta_t^{k,*} + \log \mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right] - \log H_{0,t}\right)^2 \\
&\leq 3\left((\log \eta_t^{k,*})^2 + (\log H_{0,t})^2 + \left(\log \mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right]\right)^2\right) \\
&\leq 3\Bigg(\left(-\frac{1}{2}\int_0^t |\boldsymbol{\lambda}_s^{k,*}|^2 ds + \int_0^t \boldsymbol{\lambda}_s^{k,*\top} dB_s\right)^2 + \left(-\int_0^t r_s ds - \frac{1}{2}\int_0^t |\boldsymbol{\theta}_s|^2 ds + \int_0^t \boldsymbol{\theta}_s^\top dB_s\right)^2 \\
&\quad + \left(\log \mathbf{E}[\int_0^T H_{0,t}\varepsilon_t^k dt]\right)^2\Bigg) \\
&\leq 9\Bigg(\left(-\frac{1}{2}\int_0^t |\boldsymbol{\lambda}_s^{k,*}|^2 ds\right)^2 + \left(\int_0^t \boldsymbol{\lambda}_s^{k,*\top} dB_s\right)^2 \\
&\quad + \left(\int_0^t r_s ds\right)^2 + \left(-\frac{1}{2}\int_0^t |\boldsymbol{\theta}_s|^2 ds\right)^2 + \left(\int_0^t \boldsymbol{\theta}_s^\top dB_s\right)^2 + \left(\log \mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right]\right)^2\Bigg).
\end{aligned}
\tag{69}
$$

Thus,

$$
\begin{aligned}
\int_0^T \mathbf{E}[(\log \bar{c}_t^{k,*})^2] dt \leq 9\int_0^T \mathbf{E}\Bigg[\Bigg(&\left(-\frac{1}{2}\int_0^t |\boldsymbol{\lambda}_s^{k,*}|^2 ds\right)^2 + \left(\int_0^t \boldsymbol{\lambda}_s^{k,*\top} dB_s\right)^2 \\
&+ \left(\int_0^t r_s ds\right)^2 + \left(-\frac{1}{2}\int_0^t |\boldsymbol{\theta}_s|^2 ds\right)^2 + \left(\int_0^t \boldsymbol{\theta}_s^\top dB_s\right)^2 + \left(\log \mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right]\right)^2\Bigg)\Bigg] dt.
\end{aligned}
\tag{70}
$$

Noting that

$$
\begin{aligned}
\int_0^T \mathbf{E}\left[\left(\int_0^t \boldsymbol{\lambda}_s^{k,*\top} dB_s\right)^2\right] dt &= \int_0^T \int_0^t |\boldsymbol{\lambda}_s^{k,*}|^2 ds dt, \\
\int_0^T \mathbf{E}\left[\left(\int_0^t \boldsymbol{\theta}_s^\top dB_s\right)^2\right] dt &= \int_0^T \int_0^t \mathbf{E}\left[|\boldsymbol{\theta}_s|^2\right] ds dt,
\end{aligned}
\tag{71}
$$

$$
\begin{aligned}
\int_0^T \mathbf{E}\left[\left(\int_0^t r_s ds\right)^2\right] dt &\leq T\int_0^T \int_0^t \mathbf{E}\left[r_s^2\right] ds dt \\
= T\int_0^T \int_0^t \mathbf{E}&\left[\left(\nu_s - |\boldsymbol{\rho}_s|^2 + \boldsymbol{\rho}_s^\top \left[\sum_{k=1}^K \left(\frac{Y^k \eta_s^{k,*}}{\sum_{l=1}^K Y^l \eta_s^{l,*}}\right)\boldsymbol{\lambda}_s^{k,*}\right]\right)^2\right] ds dt \\
\leq 2T\int_0^T \int_0^t &\left(\mathbf{E}[\nu_s^2] + \mathbf{E}\left[\left(|\boldsymbol{\rho}_s|^2 + \boldsymbol{\rho}_s^\top \left[\sum_{k=1}^K \left(\frac{Y^k \eta_s^{k,*}}{\sum_{l=1}^K Y^l \eta_s^{l,*}}\right)\boldsymbol{\lambda}_s^{k,*}\right]\right)^2\right]\right) ds dt,
\end{aligned}
\tag{72}
$$

since $\left(\frac{Y^k \eta_s^{k,*}}{\sum_{l=1}^K Y^l \eta_s^{l,*}}\right) \in (0,1)$, $\boldsymbol{\lambda}^{k,*}$ and $\boldsymbol{\rho}$ are bounded, and $\int_0^T \int_0^t \mathbf{E}[\nu_s^2] ds dt < \infty$, we have $\int_0^T \mathbf{E}[\log(\bar{c}_t^{k,*})^2] dt < \infty$.

5. Exponential utility case

In the exponential utility case, we consider a market where each agent has a exponential-utility function U^k given by $U^k(x) = -\frac{e^{-\gamma^k x}}{\gamma^k}$, $0 < \gamma^k < \infty$, and the aggregate endowment process ε satisfies an SDE $d\varepsilon_t = \nu_t dt + \boldsymbol{\rho}_t^\top dB_t$, where ν is a $\mathbf{R}$-valued $\{\mathcal{F}_t\}$-progressively measurable process with $\int_0^T |\nu_t| dt < \infty$ a.s. and $\mathbf{E}[\int_0^T \exp(-\frac{4}{\sum_{m=1}^K \frac{1}{\gamma^m}} \int_0^t \nu_s ds) dt] < \infty$, and $\boldsymbol{\rho}_t = (\rho_{1,t}, \ldots, \rho_{d,t})^\top$ is a nonrandom process satisfying Assumption 1.

The state-price density process in equilibrium H_0 is searched by first solving the individual optimization problems (5) as the optimal consumption problems presupposing a form of the conservative and aggressive views of the agents (21) in Section 3 in the main text and then by imposing the market clearing conditions, particularly the clearing on the commodity market (17).

In the following, we first provide H_0 obtained in the above way and confirm that the state-price density process is in fact in equilibrium. That is, given the state-price density process H_0, we first solve the individual optimization problems (5) in Proposition 9, and then show that the market is in equilibrium in Proposition 13, namely the solutions of the individual optimization problems satisfy the market clearing conditions (17)-(19).

Hereafter, we assume that ν is driven by the following Ornstein-Uhlenbeck processes. For $a_j, b_j \in \boldsymbol{R}, \sigma_j > 0$, $\nu_t = \sum_{j=1}^d X_{j,t}$, where $X_{j,t} = X_{j,0} + \int_0^t (a_j - b_j X_{j,s}) ds + \int_0^t \sigma_j dB_{j,s}$.

We further assume the following. Let $\Delta = \sum_{m=1}^K \frac{1}{\gamma^m}$.

Assumption 5. *For $j = 1, \ldots, d$,*

$$\rho_{j,u} - \Delta \max\left[\max_{l,k\in\{1,\ldots,K\};l\neq k} (\lambda_{j,u}^{l,*} - \lambda_{j,u}^{k,*}), \max_{l\in\{1,\ldots,K\}} \lambda_{j,u}^{l,*}\right] > 0, \ \forall u \in [0,T].$$

Then, the state-price density process H_0 in equilibrium in the exponential utility case is given by

$$H_{0,t} = \exp\left(-\frac{\varepsilon_t - \varepsilon_0}{\Delta}\right) \prod_{k=1}^K \left(\eta_t^{k,*}\right)^{\frac{1}{\gamma^k \Delta}}, \tag{73}$$

where $\eta^{k,*} = \eta^{\boldsymbol{\lambda}^{k,*}}$, and $\boldsymbol{\lambda}^{k,*}$ is given by (21), i.e. $\boldsymbol{\lambda}^{k,*} = (\lambda_1^{k,*}, \ldots, \lambda_d^{k,*})^\top$, where

$$\lambda_{j,t}^{k,*} = \begin{cases} -\bar{\lambda}_{j,t}^k, j \in \mathcal{J}_1^k \\ +\bar{\lambda}_{j,t}^k, j \in \mathcal{J}_2^k \\ 0, j \in \mathcal{J}_3^k, \end{cases}, \ 0 \leq t \leq T.$$

Namely, the following propositions hold.

Proposition 9. *Under Assumptions 1 and 5, given* H_0 *in (73),* $\lambda_j^{k,*}, j \in \mathcal{J}_1^k, \mathcal{J}_2^k$ *of* $\boldsymbol{\lambda}^{k,*}$ *in (21) and* $(\bar{c}^{k,*}, \boldsymbol{\pi}^{k,*})$ *with* $\bar{c}_t^{k,*} = -\frac{1}{\gamma^k}\log\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right) + \bar{c}_0^{k,*}$,

where $\bar{c}_0^{k,*} = \frac{1}{\mathbf{E}\left[\int_0^T H_{0,t}dt\right]}\left\{\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right] + \mathbf{E}\left[\int_0^T H_{0,t}\frac{\log\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right)}{\gamma^k}dt\right]\right\}$*, and* $\boldsymbol{\pi}^{k,*}$ *in (15) attain the individual optimization problem (5), i.e.,*

$$\sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k}\ \sup_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_2^k}\ \inf_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k}$$
$$\mathbf{E}\left[\int_0^T \eta_t^{\boldsymbol{\lambda}^k}U^k(c_t^k)dt\right]\left(=\mathbf{E}^{\boldsymbol{P}^{\boldsymbol{\lambda}^k}}\left[\int_0^T U^k(c_t^k)dt\right]\right). \tag{74}$$

Proof.

Hereafter, we assume $\mathcal{J}_1^k = \{1,\dots,d_1\}$, $\mathcal{J}_2^k = \{d_1+1,\dots,d_1+d_2\}$, $\mathcal{J}_3^k = \{d_1+d_2+1,\dots,d\}$, without loss of generality.

Thus, we consider the following individual optimization problem

$$\begin{aligned}
&\sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k}\ \sup_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_2^k}\ \inf_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k}\mathbf{E}\left[\int_0^T \eta_t^{\boldsymbol{\lambda}^k}U^k(c_t^k)dt\right]\\
&=\sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k}\ \sup_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_2^k}\ \inf_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k} J^k(c^k,\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k)\\
&=\sup_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_2^k}\ \sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k}\ \inf_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k} J^k(c^k,\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k),
\end{aligned} \tag{75}$$

where we set $J^k(c^k,\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k) = \mathbf{E}\left[\int_0^T \eta_t^{\boldsymbol{\lambda}^k}U^k(c_t^k)dt\right]$.

In the following, we first consider $\sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k}\inf_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k} J^k(c^k,\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k)$ for given $\boldsymbol{\lambda}_2^k$ satisfying $|\lambda_j^k|\le\bar{\lambda}_j^k$, $j\in\mathcal{J}_2^k$ in Lemma 10, then show that $\boldsymbol{\lambda}_2^{k*}$ attains $\sup_{\boldsymbol{\lambda}_2^k,|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_2^k} J^k(c^{k*},\boldsymbol{\lambda}_1^{k*},\boldsymbol{\lambda}_2^k)$, where c^{k*}, $\boldsymbol{\lambda}_1^{k*}$ attains the first part, i.e., $\sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k}\inf_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k} J^k(c^k,\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k)$, in Lemma 11.

First, the following lemma holds.

Lemma 10. *For given* $\boldsymbol{\lambda}_2^k$ *satisfying* $|\lambda_j^k|\le\bar{\lambda}_j^k$, $j = d_1+1,\dots,d_1+d_2$, $\boldsymbol{\lambda}_1^{k,*} = (-\bar{\lambda}_1^k,\dots,-\bar{\lambda}_{d_1}^k)^\top$ *and* $c_t^{k,*} = -\frac{1}{\gamma^k}\log\left(H_{0,t}/\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\right) + c_0^{k,*}$ *with* $\boldsymbol{\lambda}_1^{k,*} = (-\bar{\lambda}_1^k,\dots,-\bar{\lambda}_{d_1}^k)^\top$ *attain the sup-inf problem below:*

$$\sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k}\ \inf_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k} J^k(c^k,\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k) = J^k(c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k). \tag{76}$$

Proof.

In the exponential-utility case, since

$$U^{k\prime}(c^k) = e^{-\gamma^k c^k},$$

and $c_t^{k,*}$ satisfying the first order condition in Lemma 1 in the main text

$$H_{0,t} = \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \frac{U^{k'}(c_t^{k,*})}{U^{k'}(c_0^{k,*})},$$

is given by

$$H_{0,t} = \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} e^{-\gamma^k c_t^{k,*}} / U^{k'}(c_0^{k,*}),$$

taking log, we have

$$\begin{aligned} c_t^{k,*} &= -\frac{1}{\gamma^k}\left(\log H_{0,t} - \log \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + \log U^{k'}(c_0^{k,*})\right) \\ &= -\frac{1}{\gamma^k}\log\left(H_{0,t}/\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\right) + c_0^{k,*}, \end{aligned} \tag{77}$$

where $c_0^{k,*}$ is obtained as

$$c_0^{k,*} = \frac{1}{\mathbf{E}\left[\int_0^T H_{0,t}dt\right]}\left\{\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right] + \mathbf{E}\left[\int_0^T H_{0,t}\frac{\log\left(H_{0,t}/\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\right)}{\gamma^k}dt\right]\right\},$$

due to the budget constraint

$$\mathbf{E}\left[\int_0^T H_{0,t}c_t^{k,*}dt\right] = \mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right].$$

By Lemma1 in the main text, we have only to show $sgn(Z_j^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}) = 1,\ j = 1,\dots,d_1$ for $\boldsymbol{\lambda}_1^{k,*} = (-\bar{\lambda}_1^k,\dots,-\bar{\lambda}_{d_1}^k)^\top$.

First, we note that the Ornstein-Uhlenbeck process X_j is solved as

$$\begin{aligned} X_{j,s} &= X_{j,0}e^{-b_j s} + a_j\int_0^s e^{-b_j(s-\tau)}d\tau + \sigma_j\int_0^s e^{-b_j(s-\tau)}dB_{j,\tau} \\ &= \phi_{j,s} + \sigma_j\int_0^s e^{-b_j(s-\tau)}dB_{j,\tau}, \end{aligned}$$

where we set the deterministic term $\phi_{j,s}$ as follows:

$$\phi_{j,s} = X_{j,0}e^{-b_j s} + a_j\int_0^s e^{-b_j(s-\tau)}d\tau.$$

Then, ε is expressed as

$$\begin{aligned} \varepsilon_t &= \varepsilon_0 + \sum_{j=1}^d\int_0^t \phi_{j,s}ds + \sum_{j=1}^d\int_0^t \sigma_j\int_0^s e^{-b_j(s-\tau)}dB_{j,\tau}ds + \sum_{j=1}^d\int_0^t \rho_{j,s}dB_{j,s} \\ &= \varepsilon_0 + \sum_{j=1}^d\int_0^t \phi_{j,s}ds + \sum_{j=1}^d\int_0^t\left(\rho_{j,s} + \sigma_j\int_s^t e^{-b_j(\tau-s)}d\tau\right)dB_{j,s}. \end{aligned}$$

In the following, we denote $\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k) = (\boldsymbol{\lambda}_1^{k,*\top}, \boldsymbol{\lambda}_2^{k\top}, 0, \ldots, 0)^\top$.

Step1: Calculation of $Z_j^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$

We note

$$\begin{aligned} Z_{j,u}^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} &= \int_u^T \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} U^k(c_s^{k,*})\right] ds \\ &= \int_u^T \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[e^{-\gamma^k c_s^{k,*}} D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} c_s^{k,*}\right] ds. \end{aligned}$$

By

$$H_{0,t} = \exp\left(-\frac{\varepsilon_t - \varepsilon_0}{\Delta}\right) \prod_{l=1}^K \left(\eta_t^{l,*}\right)^{\frac{1}{\gamma^l \Delta}},$$

taking log of H_0, we have

$$\log H_{0,t} = -\frac{\varepsilon_t - \varepsilon_0}{\Delta} + \sum_{l=1}^K \frac{1}{\gamma^l \Delta} \log \eta_t^{l,*}.$$

Due to

$$\begin{aligned} dB_{j,t} &= dB_{j,t}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + \lambda_{j,t}^{k,*} dt \ (j = 1, \ldots, d_1), \\ dB_{j,t} &= dB_{j,t}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + \lambda_{j,t}^{k} dt \ (j = d_1+1, \ldots, d_1+d_2), \\ dB_{j,t} &= dB_{j,t}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \ (j = d_1+d_2+1, \ldots, d), \end{aligned}$$

with Brownian motions under $P^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$, by (93), Malliavin derivative with respect to $B_j^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$, $j = 1, \ldots, d_1$ is

$$\begin{aligned} D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} c_s^{k,*} &= \frac{1}{\gamma^k \Delta}\left(\rho_{j,u} + \sigma_j \int_u^s e^{-b_j(\tau-u)} d\tau\right) - \sum_{l=1}^K \frac{1}{\gamma^k \gamma^l \Delta} \lambda_{j,u}^{l,*} + \frac{1}{\gamma^k} \lambda_{j,u}^{k,*} \\ &= \frac{1}{\gamma^k \Delta}\left\{\left(\rho_{j,u} + \sigma_j \int_u^s e^{-b_j(\tau-u)} d\tau\right) - \sum_{l=1}^K \frac{1}{\gamma^l} \lambda_{j,u}^{l,*} + \Delta \lambda_{j,u}^{k,*}\right\} \\ &= \frac{1}{\gamma^k \Delta}\left\{\left(\rho_{j,u} + \sigma_j \int_u^s e^{-b_j(\tau-u)} d\tau\right) - \sum_{l=1}^K \frac{1}{\gamma^l} \lambda_{j,u}^{l,*} + \left(\sum_{l=1}^K \frac{1}{\gamma^l}\right) \lambda_{j,u}^{k,*}\right\} \\ &= \frac{1}{\gamma^k \Delta}\left\{\left(\rho_{j,u} + \sigma_j \int_u^s e^{-b_j(\tau-u)} d\tau\right) - \sum_{l=1; l\neq k}^K \frac{1}{\gamma^l} (\lambda_{j,u}^{l,*} - \lambda_{j,u}^{k,*})\right\}. \end{aligned}$$

Hence, we have

$$Z_{j,u}^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} = \int_u^T \left[\frac{e^{-\gamma^k c_s^{k,*}}}{\gamma^k \Delta}\left\{\left(\rho_{j,u} + \sigma_j \int_u^s e^{-b_j(\tau-u)} d\tau\right) - \sum_{l=1; l\neq k}^K \frac{1}{\gamma^l} (\lambda_{j,u}^{l,*} - \lambda_{j,u}^{k,*})\right\}\right] ds. \quad (78)$$

Step2: Determination of $sgn(Z_j^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k})$

Since

$$\sigma_j \int_u^s e^{-b_j(\tau-u)} d\tau = \sigma_j \frac{1-e^{-b_j(s-u)}}{b_j} \geq 0 \text{ (equality holds at } s=u),$$

by Assumptions 1 and 5, the right hand side of (78) is positive, that is $Z_{j,u}^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} > 0,\ \forall u \in [0,T]$, and thus $sgn(Z_j^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}) \equiv 1$. Hence, since $\lambda_j^{k,*} = -\bar{\lambda}_j^k$, we have

$$-\bar{\lambda}_j^k sgn(Z_j^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}) = -\bar{\lambda}_j^k = \lambda_j^{k,*}.$$

□

Next, we solve the following problem on $\boldsymbol{\lambda}_2^k$ for $(c^{k,*}(\boldsymbol{\lambda}_2^k), \boldsymbol{\lambda}_1^{k,*}(\boldsymbol{\lambda}_2^k))$:

$$\sup_{\boldsymbol{\lambda}_2^k, |\lambda_j^k| \leq \bar{\lambda}_j^k, j=d_1+1,\ldots,d_1+d_2} \mathbf{E}\left[\int_0^T \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} U^k(c_t^{k,*}(\boldsymbol{\lambda}_2^k))dt\right]. \tag{79}$$

Lemma 11. $\boldsymbol{\lambda}_2^{k,*} = (\bar{\lambda}_{d_1+1}^k, \ldots, \bar{\lambda}_{d_1+d_2}^k)^\top$ *is optimal in (79).*

Proof. Since the optimal consumption for fixed $\boldsymbol{\lambda}_2^k$ is (93), $\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} U^k(c_t^{k,*}(\boldsymbol{\lambda}_2^k))$ is

$$\begin{aligned}
\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} U^k(c_t^{k,*}(\boldsymbol{\lambda}_2^k)) &= \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \times -\frac{\exp\left(\log\left(U^{k\prime}(c_0^{k,*}(\boldsymbol{\lambda}_2^k))\frac{H_{0,t}}{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}}\right)\right)}{\gamma^k} \\
&= \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \times -\frac{1}{\gamma^k} U^{k\prime}(c_0^{k,*}(\boldsymbol{\lambda}_2^k)) \frac{H_{0,t}}{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}} \\
&= -\frac{1}{\gamma^k} U^{k\prime}(c_0^{k,*}(\boldsymbol{\lambda}_2^k)) H_{0,t}.
\end{aligned}$$

Substituting (93) to

$$\mathbf{E}\left[\int_0^T H_{0,t} c_t^{k,*} dt\right] = \mathbf{E}\left[\int_0^T H_{0,t} \varepsilon_t^k dt\right],$$

we have

$$\mathbf{E}\left[\int_0^T H_{0,t} \left\{-\frac{\log\left(H_{0,t}/\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\right)}{\gamma^k} + c_0^{k,*}\right\} dt\right] = \mathbf{E}\left[\int_0^T H_{0,t} \varepsilon_t^k dt\right].$$

Hence, we obtain

$$c_0^{k,*}(\boldsymbol{\lambda}_2^k) = \frac{1}{\mathbf{E}\left[\int_0^T H_{0,t} dt\right]} \left\{\mathbf{E}\left[\int_0^T H_{0,t} \varepsilon_t^k dt\right] + \mathbf{E}\left[\int_0^T H_{0,t} \frac{\log\left(H_{0,t}/\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\right)}{\gamma^k} dt\right]\right\}. \tag{80}$$

Since $c_0^{k,*}(\boldsymbol{\lambda}_2^k)$ is (80), the objective function of $\boldsymbol{\lambda}_2^k$ is

$$
\begin{aligned}
&\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} U^k(c_t^{k,*}(\boldsymbol{\lambda}_2^k)) \\
&= -\frac{H_{0,t}}{\gamma^k} \exp\left(-\gamma^k \frac{\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right]}{\mathbf{E}\left[\int_0^T H_{0,t} dt\right]}\right) \times \exp\left(-\frac{\mathbf{E}\left[\int_0^T H_{0,t}\left\{\log\left(H_{0,t}/\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\right)\right\} dt\right]}{\mathbf{E}\left[\int_0^T H_{0,t} dt\right]}\right) \\
&= -\Gamma_t \exp\left(\frac{\mathbf{E}\left[\int_0^T H_{0,t}\log\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} dt\right]}{\mathbf{E}\left[\int_0^T H_{0,t} dt\right]}\right),
\end{aligned}
$$

where we defined Γ_t independent of $\boldsymbol{\lambda}_2^k$ as

$$
\Gamma_t = \frac{H_{0,t}}{\gamma^k} \exp\left(-\gamma^k \frac{\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right]}{\mathbf{E}\left[\int_0^T H_{0,t} dt\right]}\right) \times \exp\left(-\frac{\mathbf{E}\left[\int_0^T H_{0,t}\log H_{0,t} dt\right]}{\mathbf{E}\left[\int_0^T H_{0,t} dt\right]}\right) > 0.
$$

Moreover, using

$$
\begin{aligned}
&\log \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \\
&= \sum_{j=1}^{d_1}\left(\int_0^t \lambda_{j,s}^{k,*} dB_{j,s} - \frac{1}{2}\int_0^t (\lambda_{j,s}^{k,*})^2 ds\right) + \sum_{j=d_1+1}^{d_1+d_2}\left(\int_0^t \lambda_{j,s}^{k} dB_{j,s} - \frac{1}{2}\int_0^t \lambda_{j,s}^{k,2} ds\right) \\
&= \int_0^t \boldsymbol{\lambda}_{1,s}^{k,*\top} d\boldsymbol{B}_{1,s} - \frac{1}{2}\int_0^t |\boldsymbol{\lambda}_{1,s}^{k,*}|^2 ds + \int_0^t \boldsymbol{\lambda}_{2,s}^{k\top} d\boldsymbol{B}_{2,s} - \frac{1}{2}\int_0^t |\boldsymbol{\lambda}_{2,s}^{k}|^2 ds,
\end{aligned}
$$

we rewrite

$$
\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} U^k(c_t^{k,*}(\boldsymbol{\lambda}_2^k)) = -\Gamma_t F_1(\boldsymbol{\lambda}_1^{k,*}) F_2(\boldsymbol{\lambda}_2^k),
$$

where

$$
F_1(\boldsymbol{\lambda}_1^{k,*}) = \exp\Bigg(\frac{1}{\mathbf{E}\left[\int_0^T H_{0,t} dt\right]}\mathbf{E}\Bigg[\int_0^T H_{0,t}\Bigg\{\int_0^t \boldsymbol{\lambda}_{1,s}^{k,*\top} d\boldsymbol{B}_{1,s} - \frac{1}{2}\int_0^t |\boldsymbol{\lambda}_{1,s}^{k,*}|^2 ds\Bigg\} dt\Bigg]\Bigg),
$$

and

$$
F_2(\boldsymbol{\lambda}_2^{k}) = \exp\Bigg(\frac{1}{\mathbf{E}\left[\int_0^T H_{0,t} dt\right]}\mathbf{E}\Bigg[\int_0^T H_{0,t}\Bigg\{\int_0^t \boldsymbol{\lambda}_{2,s}^{k\top} d\boldsymbol{B}_{2,s} - \frac{1}{2}\int_0^t |\boldsymbol{\lambda}_{2,s}^{k}|^2 ds\Bigg\} dt\Bigg]\Bigg).
$$

The optimization problem is equivalent to

$$
\sup_{\boldsymbol{\lambda}_2^k, |\lambda_j^k| \le \bar{\lambda}_j^k, j=d_1+1,\dots,d_1+d_2} \mathbf{E}\left[\int_0^T -\Gamma_t F_1(\boldsymbol{\lambda}_1^{k,*}) F_2(\boldsymbol{\lambda}_2^k) dt\right]. \tag{81}
$$

Since $\Gamma_t F_1(\boldsymbol{\lambda}_1^{k,*})$ is positive and independent of $\boldsymbol{\lambda}_2^k$, we only have to investigate $F_2(\boldsymbol{\lambda}_2^k)$. If $F_2(\boldsymbol{\lambda}_2^k)$ is minimized, the expected utility is maximized with respect to $\boldsymbol{\lambda}_2^k$. Thus, we have to confirm that

$F_2(\boldsymbol{\lambda}_2^k)$ is decreasing as a functional of the deterministic process $\boldsymbol{\lambda}_2^k = (\lambda_{d_1+1}^k, \ldots, \lambda_{d_1+d_2}^k)^\top$, and minimized at $\boldsymbol{\lambda}_2^{k,*}$. Note that H_0 is

$$H_{0,t} = \exp\left(-\frac{1}{\Delta}\left\{ \sum_{j=1}^{d} \int_0^t \phi_{j,s} ds + \sum_{j=1}^{d} \int_0^t \left(\rho_{j,s} + \sigma_j \int_s^t e^{-b_j(\tau-s)} d\tau \right) dB_{j,s} \right\}\right)$$
$$\times \prod_{l=1}^{K} \exp\left(\frac{1}{\gamma^l \Delta}\left\{ \sum_{j=1}^{d} \left(\int_0^t \lambda_{j,s}^{l,*} dB_{j,s} - \frac{1}{2}\int_0^t (\lambda_{j,s}^{l,*})^2 ds \right)\right\}\right).$$

Set $\boldsymbol{\xi} = (\xi_{d_1+1}, \ldots, \xi_{d_1+d_2})^\top$, where

$$\xi_{j,s} = -\frac{1}{\Delta}\left(\rho_{j,s} + \sigma_j \int_s^t e^{-b_j(\tau-s)} d\tau \right) + \sum_{l=1}^{K} \frac{1}{\gamma^l \Delta} \lambda_{j,s}^{l,*}, \tag{82}$$

$j = d_1+1, \ldots, d_1+d_2$. Then, we rewrite H_0 as follows:

$$\begin{aligned}
H_{0,t} &= \exp\left(\int_0^t \boldsymbol{\xi}_s^\top d\boldsymbol{B}_{2,s} - \frac{1}{2}\int_0^t |\boldsymbol{\xi}_s|^2 ds \right) \exp\left(\frac{1}{2}\int_0^t |\boldsymbol{\xi}_s|^2 ds \right) \\
&\quad \times \exp\left(-\frac{1}{\Delta}\left\{ \sum_{i=1}^{d} \int_0^t \phi_{i,s} ds + \sum_{i=1; i\notin \mathcal{J}_2^k}^{d} \int_0^t \left(\rho_{i,s} + \sigma_i \int_s^t e^{-b_i(\tau-s)} d\tau \right) dB_{i,s} \right\}\right) \\
&\quad \times \prod_{l=1}^{K} \exp\left(\frac{1}{\gamma^l \Delta}\left\{ \sum_{i=1; i\notin \mathcal{J}_2^k}^{d} \left(\int_0^t \lambda_{i,s}^{l,*} dB_{i,s} - \frac{1}{2}\int_0^t (\lambda_{i,s}^{l,*})^2 ds \right)\right\} - \sum_{j=1}^{d} \frac{1}{2}\int_0^t (\lambda_{j,s}^{l,*})^2 ds \right) \\
&= \exp\left(\int_0^t \boldsymbol{\xi}_s^\top d\boldsymbol{B}_{2,s} - \frac{1}{2}\int_0^t |\boldsymbol{\xi}_s|^2 ds \right) H(t, \boldsymbol{B}_2^-),
\end{aligned}$$

where we set

$$\begin{aligned}
&H(t, \boldsymbol{B}_2^-) \\
&= \exp\left(\frac{1}{2}\int_0^t |\boldsymbol{\xi}_s|^2 ds \right) \exp\left(-\frac{1}{\Delta}\left\{ \sum_{i=1}^{d} \int_0^t \phi_{i,s} ds + \sum_{i=1; i\notin \mathcal{J}_2^k}^{d} \int_0^t \left(\rho_{i,s} + \sigma_i \int_s^t e^{-b_i(\tau-s)} d\tau \right) dB_{i,s} \right\}\right) \\
&\times \prod_{l=1}^{K} \exp\left(\frac{1}{\gamma^l \Delta}\left\{ \sum_{i=1; i\notin \mathcal{J}_2^k}^{d} \left(\int_0^t \lambda_{i,s}^{l,*} dB_{i,s} - \frac{1}{2}\int_0^t (\lambda_{i,s}^{l,*})^2 ds \right)\right\} - \sum_{j=1}^{d} \frac{1}{2}\int_0^t (\lambda_{j,s}^{l,*})^2 ds \right) > 0.
\end{aligned}$$

We define a probability measure $\tilde{\mathbf{P}}$ with a positive martingale as

$$Z_t^\xi = \exp\left(\int_0^t \boldsymbol{\xi}_s^\top d\boldsymbol{B}_{2,s} - \frac{1}{2}\int_0^t |\boldsymbol{\xi}_s|^2 ds \right).$$

Then, we obtain Brownian motions under $\tilde{\mathbf{P}}$ as follows:

$$\begin{aligned}
\tilde{B}_{i,t} &= B_{i,t} \ (i \notin \mathcal{J}_2^k), \\
\tilde{B}_{i,t} &= B_{i,t} - \int_0^t \xi_{i,s} ds \ (i \in \mathcal{J}_2^k).
\end{aligned}$$

Set $\tilde{\boldsymbol{B}}_2 = (\tilde{B}_{d_1+1}, \ldots, \tilde{B}_{d_1+d_2})^\top$. Then,

$$
\begin{aligned}
&F_2(\boldsymbol{\lambda}_2^k) \\
&= \exp\left(\frac{1}{\mathbf{E}\left[\int_0^T H_{0,t}dt\right]}\left\{\mathbf{E}^{\tilde{\mathbf{P}}}\left[\int_0^T H(t,\tilde{\boldsymbol{B}}_2^-)\left(\int_0^t \boldsymbol{\lambda}_{2,s}^{k\top}(d\tilde{\boldsymbol{B}}_{2,s}+\boldsymbol{\xi}_s ds) - \int_0^t \frac{|\boldsymbol{\lambda}_{2,s}^k|^2}{2}ds\right)dt\right]\right\}\right).
\end{aligned}
$$

Since $\tilde{\boldsymbol{B}}_2^-$ and $\tilde{\boldsymbol{B}}_2$ are independent and

$$
\mathbf{E}^{\tilde{\mathbf{P}}}\left[\int_0^t \boldsymbol{\lambda}_{2,s}^{k\top} d\tilde{\boldsymbol{B}}_{2,s}\right] = 0,
$$

we have

$$
\begin{aligned}
&F_2(\boldsymbol{\lambda}_2^k) \\
&= \exp\left(\frac{1}{\mathbf{E}\left[\int_0^T H_{0,t}dt\right]}\left\{\mathbf{E}^{\tilde{\mathbf{P}}}\left[\int_0^T H(t,\tilde{\boldsymbol{B}}_2^-)\left(\int_0^t \boldsymbol{\lambda}_{2,s}^{k\top}\boldsymbol{\xi}_s ds - \frac{1}{2}\int_0^t |\boldsymbol{\lambda}_{2,s}^k|^2 ds\right)dt\right]\right\}\right).
\end{aligned}
$$

Since $H(t, \tilde{\boldsymbol{B}}_2^-) > 0$, we only have to consider

$$
f_{2,t}(\boldsymbol{\lambda}_2^k) = \int_0^t \left(\boldsymbol{\lambda}_{2,s}^{k\top}\boldsymbol{\xi}_s - \frac{1}{2}|\boldsymbol{\lambda}_{2,s}^k|^2\right) ds.
$$

For any nonrandom $\hat{\boldsymbol{\lambda}}_2^k = (\hat{\lambda}_{d_1+1}, \ldots, \hat{\lambda}_{d_1+d_2})^\top$ with $0 < \hat{\lambda}_j^k \le \bar{\lambda}_j^k,\ j = d_1+1, \ldots, d_1+d_2$, we calculate

$$
\lim_{\alpha\to 0} \frac{f_{2,t}(\boldsymbol{\lambda}_2^k + \alpha\hat{\boldsymbol{\lambda}}_2^k) - f_{2,t}(\boldsymbol{\lambda}_2^k)}{\alpha}.
$$

Since

$$
\begin{aligned}
&f_{2,t}(\boldsymbol{\lambda}_2^k + \alpha\hat{\boldsymbol{\lambda}}_2^k) - f_{2,t}(\boldsymbol{\lambda}_2^k) \\
&= \int_0^t \left((\boldsymbol{\lambda}_{2,s}^k + \alpha\hat{\boldsymbol{\lambda}}_{2,s}^k)^\top\boldsymbol{\xi}_s - \frac{|\boldsymbol{\lambda}_{2,s}^k + \alpha\hat{\boldsymbol{\lambda}}_{2,s}^k|^2}{2}\right) ds - \int_0^t \left(\boldsymbol{\lambda}_{2,s}^{k\top}\boldsymbol{\xi}_s - \frac{|\boldsymbol{\lambda}_{2,s}^k|^2}{2}\right) ds \\
&= \int_0^t \left(\alpha\hat{\boldsymbol{\lambda}}_{2,s}^{k\top}\boldsymbol{\xi}_s - \alpha\hat{\boldsymbol{\lambda}}_{2,s}^{k\top}\boldsymbol{\lambda}_{2,s}^k - \frac{|\hat{\boldsymbol{\lambda}}_{2,s}^k|^2}{2}\alpha^2\right) ds,
\end{aligned}
$$

$$
\lim_{\alpha\to 0} \frac{f_{2,t}(\boldsymbol{\lambda}_2^k + \alpha\hat{\boldsymbol{\lambda}}_2^k) - f_{2,t}(\boldsymbol{\lambda}_2^k)}{\alpha} = \int_0^t \left(\hat{\boldsymbol{\lambda}}_{2,s}^{k\top}\boldsymbol{\xi}_s - \hat{\boldsymbol{\lambda}}_{2,s}^{k\top}\boldsymbol{\lambda}_{2,s}^k\right) ds.
$$

If $\boldsymbol{\xi}_s < 0$ for all $0 \le s \le t$, then $f_{2,t}(\boldsymbol{\lambda}_2^k)$ is decreasing in $\boldsymbol{\lambda}_2^k$ since the integrand is decreasing in $\boldsymbol{\xi}_s \le \boldsymbol{\lambda}_{2,s}^k \le \bar{\boldsymbol{\lambda}}_{2,s}^k$. Hence, $f_{2,t}(\boldsymbol{\lambda}_2^k)$ is minimized at $\boldsymbol{\lambda}_2^k = \bar{\boldsymbol{\lambda}}_2^k$, and thus the expected utility is maximized. In fact, by Assumption 5, $\boldsymbol{\xi}_s < 0$ holds for all $0 \le s \le t$. $\square$

The admissibility of $(\bar{c}^{k,*}, \pi^{k,*})$ follows in the same manner as in the proof of Theorem 3 in the main text. Particularly, $\mathbf{E}[\int_0^T U^k(\bar{c}_t^{k,*})^2 dt] < \infty$ follows from the nonrandomness of $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}_s^{l,*}$ and $\mathbf{E}[\int_0^T \exp(-\frac{4}{\Delta}\int_0^t \nu_s ds)dt] < \infty$ (see Remark 6 below for details). Thus, by Lemmas 10 and 11, Proposition 9 holds. □

Remark 6. *For the admissibility of* $(\bar{c}^{k,*}, \pi^{k,*})$*,* $\mathbf{E}[\int_0^T U^k(\bar{c}_t^{k,*})^2 dt] < \infty$ *is confirmed as follows. First, we note that since*

$$U^k(\bar{c}_t^{k,*}) = -\frac{1}{\gamma^k}\exp(-\gamma^k \bar{c}_t^{k,*}),$$
$$\bar{c}_t^{k,*} = -\frac{1}{\gamma^k}\log\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right) + \bar{c}_0^{k,*}, \tag{83}$$

we have

$$U^k(\bar{c}_t^{k,*}) = -\frac{K}{\gamma^k}\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right), \tag{84}$$

where $K = \exp(-\gamma^k \bar{c}_0^{k,*})$*. Then,*

$$U^k(\bar{c}_t^{k,*})^2 = \frac{K^2}{(\gamma^k)^2}\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right)^2. \tag{85}$$

Here,

$$\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right)^2 = \exp\left(-\frac{2(\varepsilon_t-\varepsilon_0)}{\Delta}\right)\prod_{l=1}^{K}(\eta_t^{l,*})^{\frac{2}{\gamma^l\Delta}}/(\eta_t^{k,*})^2$$
$$= \exp\left(-\frac{2(\varepsilon_t-\varepsilon_0)}{\Delta} + \sum_{l=1}^{K}\left(-\frac{1}{\gamma^l\Delta}\int_0^t|\boldsymbol{\lambda}_s^{l,*}|^2 ds + \frac{2}{\gamma^l\Delta}\int_0^t \boldsymbol{\lambda}_s^{l,*\top}dB_s\right) + \int_0^t|\boldsymbol{\lambda}_s^{k,*}|^2 ds - 2\int_0^t \boldsymbol{\lambda}_s^{k,*\top}dB_s\right), \tag{86}$$

where

$$\varepsilon_t = \varepsilon_0 + \int_0^t \nu_s ds + \int_0^t \boldsymbol{\rho}_s^\top dB_s. \tag{87}$$

Hence,

$$\mathbf{E}\left[\int_0^T U^k(\bar{c}_t^{k,*})^2 dt\right] \le \frac{K^2}{(\gamma^k)^2}\sqrt{\mathbf{E}\left[\int_0^T \exp\left(-\frac{4}{\Delta}\int_0^t \nu_s ds\right)dt\right]}\sqrt{\mathbf{E}\left[\int_0^T \exp(A_t)dt\right]}, \tag{88}$$

where

$$A_t = -\frac{4}{\Delta}\int_0^t \boldsymbol{\rho}_s^\top dB_s + \sum_{l=1}^{K}\left(-\frac{2}{\gamma^l\Delta}\int_0^t|\boldsymbol{\lambda}_s^{l,*}|^2 ds + \frac{4}{\gamma^l\Delta}\int_0^t \boldsymbol{\lambda}_s^{l,*\top}dB_s\right) + 2\int_0^t|\boldsymbol{\lambda}_s^{k,*}|^2 ds - 4\int_0^t \boldsymbol{\lambda}_s^{k,*\top}dB_s. \tag{89}$$

Since $\boldsymbol{\rho}$ *and* $\boldsymbol{\lambda}_s^{l,*}$ *are nonrandom and* $\mathbf{E}[\int_0^T \exp(-\frac{4}{\Delta}\int_0^t \nu_s ds)dt] < \infty$*, we obtain* $\mathbf{E}[\int_0^T U^k(\bar{c}_t^{k,*})^2 dt] < \infty$*.*

$\square$

Finally, we show that the clearing conditions (17)-(19) are satisfied.

Proposition 12. *Under Assumptions 1 and 5, given* H_0 *in (73), the clearing conditions (17)-(19) hold.*

Proof.

First, we confirm the clearing condition of the consumption goods market. Since

$$\bar{c}_t^{k,*} = -\frac{1}{\gamma^k}\log\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right) + \bar{c}_0^{k,*},$$

where $\bar{c}_0^{k,*} = \frac{1}{\mathbf{E}\left[\int_0^T H_{0,t}dt\right]}\left\{\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right] + \mathbf{E}\left[\int_0^T H_{0,t}\frac{\log\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right)}{\gamma^k}dt\right]\right\}$,

$$\sum_{k=1}^K \bar{c}_t^{k,*} = -\sum_{k=1}^K \frac{1}{\gamma^k}\log\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right) + \sum_{k=1}^K \bar{c}_0^{k,*}.$$

Note that

$$\begin{aligned}
-\sum_{k=1}^K \frac{1}{\gamma^k}\log\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right) &= -\sum_{k=1}^K \frac{1}{\gamma^k}(\log H_{0,t} - \log \eta_t^{k,*}) \\
&= -\Delta \log H_{0,t} + \sum_{k=1}^K \frac{1}{\gamma^k}\log \eta_t^{k,*} \\
&= \varepsilon_t - \varepsilon_0,
\end{aligned}$$

since

$$\Delta \log H_{0,t} = -(\varepsilon_t - \varepsilon_0) + \sum_{k=1}^K \frac{1}{\gamma^k}\eta_t^{k,*}.$$

Moreover,

$$\begin{aligned}
\sum_{k=1}^K \bar{c}_0^{k,*} &= \frac{\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t dt\right] + \mathbf{E}\left[\int_0^T H_{0,t}\sum_{k=1}^K \frac{\log\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right)}{\gamma^k}dt\right]}{\mathbf{E}\left[\int_0^T H_{0,t}dt\right]} \\
&= \frac{\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t dt\right] + \mathbf{E}\left[\int_0^T H_{0,t}(-\varepsilon_t + \varepsilon_0)dt\right]}{\mathbf{E}\left[\int_0^T H_{0,t}dt\right]} = \varepsilon_0.
\end{aligned}$$

Hence, we have

$$\sum_{k=1}^K \bar{c}_t^{k,*} = \varepsilon_t,\ t \in [0,T].$$

The clearing conditions for the security market and the money market also follow in the same way as in the proof of Theorem 4 in the main text. □

□

Furthermore, by applying Ito's formula to (73) and comparing the result with (2), we have the following expressions for the interest rate and the market price of risk with heterogeneous views on the fundamental risks in the exponential utility case.

Proposition 13. *The interest rate* r *and the market price of risk* $-\boldsymbol{\theta}$ *in equilibrium are given by* $r_t = \frac{\nu_t}{\Delta} - \frac{|\boldsymbol{\rho}_t|^2}{2\Delta^2} + \sum_{k=1}^K \frac{1}{2\gamma^k\Delta^2}\left(\sum_{m\neq k}^K \frac{|\boldsymbol{\lambda}_t^{k,*}|^2}{\gamma^m} + 2\boldsymbol{\rho}_t^\top \boldsymbol{\lambda}_t^{k,*}\right)$, *and* $-\boldsymbol{\theta}_t = \frac{1}{\Delta}\boldsymbol{\rho}_t - \sum_{k=1}^K \frac{1}{\gamma^k\Delta}\boldsymbol{\lambda}_t^{k,*}$.

Proof.

By $H_{0,t} = \exp\left(-\frac{\varepsilon_t - \varepsilon_0}{\Delta}\right)\prod_{k=1}^K \left(\eta_t^{k,*}\right)^{\frac{1}{\gamma^k\Delta}}$,$\log H_0$ is expressed as

$$\log H_0 = -\frac{\varepsilon_t - \varepsilon_0}{\Delta} + \sum_{k=1}^K \frac{1}{\gamma^k\Delta}\log \eta_t^{k,*}.$$

Thus, we have

$$\begin{aligned}
d\log H_{0,t} &= -\frac{1}{\Delta}d\varepsilon_t + \sum_{k=1}^K \frac{1}{\gamma^k\Delta} d\log \eta_t^{k,*} \\
&= -\frac{1}{\Delta}(\nu_t dt + \boldsymbol{\rho}_t^\top dB_t) + \sum_{k=1}^K \frac{1}{\gamma^k\Delta}(\boldsymbol{\lambda}_t^{k,*\top} dB_t - \frac{1}{2}|\boldsymbol{\lambda}_t^{k,*}|^2 dt) \\
&= \left[-\frac{1}{\Delta}\nu_t - \frac{1}{2}\sum_{k=1}^K \frac{1}{\gamma^k\Delta}|\boldsymbol{\lambda}_t^{k,*}|^2\right] dt + \left[-\frac{1}{\Delta}\boldsymbol{\rho}_t + \sum_{k=1}^K \frac{1}{\gamma^k\Delta}\boldsymbol{\lambda}_t^{k,*}\right]^\top dB_t.
\end{aligned}$$

On the other hand,

$$dH_{0,t} = H_{0,t}[-r_t dt + \boldsymbol{\theta}_t^\top dB_t], \tag{90}$$

and thus

$$d\log H_{0,t} = \left[-r_t - \frac{1}{2}|\boldsymbol{\theta}_t|^2\right] dt + \boldsymbol{\theta}_t^\top dB_t.$$

By these representations of $d\log H_{0,t}$, the market price of risks $-\theta$ and the interest rate r are given by

$$-\boldsymbol{\theta}_t = \frac{1}{\Delta}\boldsymbol{\rho}_t - \sum_{k=1}^K \frac{1}{\gamma^k\Delta}\boldsymbol{\lambda}_t^{k,*},$$

and

$$
\begin{aligned}
r_t &= \frac{\nu_t}{\Delta} + \frac{1}{2}\sum_{k=1}^{K}\frac{1}{\gamma^k\Delta}|\boldsymbol{\lambda}_t^{k,*}|^2 - \frac{1}{2}|\boldsymbol{\theta}_t|^2 \\
&= \frac{\nu_t}{\Delta} + \sum_{k=1}^{K}\frac{|\boldsymbol{\lambda}_t^{k,*}|^2}{2\gamma^k\Delta} - \frac{1}{2}\Big|\frac{1}{\Delta}\boldsymbol{\rho}_t - \sum_{k=1}^{K}\frac{1}{\gamma^k\Delta}\boldsymbol{\lambda}_t^{k,*}\Big|^2 \\
&= \frac{\nu_t}{\Delta} - \frac{|\boldsymbol{\rho}_t|^2}{2\Delta^2} + \sum_{k=1}^{K}\frac{|\boldsymbol{\lambda}_t^{k,*}|^2}{2\gamma^k\Delta}\Big(1 - \frac{1}{\gamma^k\Delta}\Big) + \sum_{k=1}^{K}\frac{\boldsymbol{\rho}_t^\top\boldsymbol{\lambda}_t^{k,*}}{\gamma^k\Delta^2} \\
&= \frac{\nu_t}{\Delta} - \frac{|\boldsymbol{\rho}_t|^2}{2\Delta^2} + \sum_{k=1}^{K}\frac{1}{2\gamma^k\Delta^2}\left(\sum_{m\neq k}^{K}\frac{|\boldsymbol{\lambda}_t^{k,*}|^2}{\gamma^m} + 2\boldsymbol{\rho}_t^\top\boldsymbol{\lambda}_t^{k,*}\right).
\end{aligned}
$$

□

Remark 7. *Since $\boldsymbol{\rho}_t$ and $\boldsymbol{\lambda}_t^{k,*}$ are nonrandom, $\mathcal{Z}_t = \exp\left(\int_0^t \boldsymbol{\theta}_s^\top dB_s - \frac{1}{2}\int_0^t |\boldsymbol{\theta}_s|^2 ds\right),\ t \in [0,T]$ is in fact a martingale in the exponential case.*

6. Possible extension to an exponential utility case with stochastic boundaries

In this section, we show a possible extension of the model to the case where the boundaries of the views $\bar{\lambda}_j^k,\ j \in \mathcal{J}_1, \mathcal{J}_2$ are stochastic, namely, we assume $\bar{\lambda}_j^k$ are positive $\{\mathcal{F}_t\}$-progressively measurable processes.

In the exponential utility case, we consider a market where each agent has a exponential-utility function U^k given by $U^k(x) = -\frac{e^{-\gamma^k x}}{\gamma^k},\ 0 < \gamma^k < \infty$, and the aggregate endowment process ε is a normal type stochastic process expressed as

$$
\varepsilon_t = \varepsilon_0 + \int_0^t \nu_\tau d\tau + \int_0^t \boldsymbol{\rho}_\tau^\top dB_\tau, \tag{91}
$$

where $\nu, \boldsymbol{\rho}$ are $\{\mathcal{F}_t\}$-progressively measurable processes with $\mathbf{E}[\int_0^T |\nu_\tau| d\tau] < \infty$, $\mathbf{E}[\int_0^T |\boldsymbol{\rho}_\tau|^2 d\tau] < \infty$, and each element of $\boldsymbol{\rho}$ being positive, i.e. $\rho_j > 0\ (j = 1, \cdots, d)$.

Then, for a given state-price density process H_0 in (73), namely, $H_{0,t} = \exp\left(-\frac{\varepsilon_t - \varepsilon_0}{\Delta}\right)\prod_{k=1}^K \left(\eta_t^{k,*}\right)^{\frac{1}{\gamma^k\Delta}}$, we consider the following individual optimization problem for the k-th agent

$$
\begin{aligned}
&\sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k}\ \sup_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_2^k}\ \inf_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k} \\
&\mathbf{E}\left[\int_0^T \eta_t^{\boldsymbol{\lambda}^k} U^k(c_t^k)dt\right]\left(= \mathbf{E}^{\boldsymbol{P}^{\boldsymbol{\lambda}^k}}\left[\int_0^T U^k(c_t^k)dt\right]\right) \\
= &\sup_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_2^k}\ \sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k}\ \inf_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k} \mathbf{E}\left[\int_0^T \eta_t^{\boldsymbol{\lambda}^k} U^k(c_t^k)dt\right].
\end{aligned} \tag{92}
$$

Then, as in the discussion in Section 5, under certain conditions, $\lambda_{j,t}^{k,*} = -\bar{\lambda}_j^k, j \in \mathcal{J}_1^k$, $\lambda_{j,t}^{k,*} = \bar{\lambda}_j^k, j \in \mathcal{J}_2^k$, $\bar{c}_t^{k,*} = -\frac{1}{\gamma^k}\log\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right) + \bar{c}_0^{k,*}$ with $\bar{c}_0^{k,*} = \frac{1}{\mathbf{E}\left[\int_0^T H_{0,t}dt\right]}\left\{\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right] + \mathbf{E}\left[\int_0^T H_{0,t}\frac{\log\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right)}{\gamma^k}dt\right]\right\}$, attain $\sup_{|\lambda_j^k|\leq\bar{\lambda}_j^k,\ j\in\mathcal{J}_2^k}\sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k}\inf_{|\lambda_j^k|\leq\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k} J^k(c^k,\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k)$ and satisfy the clearing conditions (17)-(19) as in Proposition 12 of Section 5.

Hereafter, we assume $\mathcal{J}_1^k = \{1,\ldots,d_1\}$, $\mathcal{J}_2^k = \{d_1+1,\ldots,d_1+d_2\}$, $\mathcal{J}_3^k = \{d_1+d_2+1,\ldots,d\}$, without loss of generality.

In the following, we particularly investigate the conditions under which the individual optimization problem is solved as mentioned. We first consider conditions where for given $\boldsymbol{\lambda}_2^k$, $sgn(Z_j) = +1, j \in j = 1,\ldots,d_1$ for Z_j in (14) of Lemma 1 and then examine conditions where maximization with respect to $\boldsymbol{\lambda}_2^k$ is attained at $\boldsymbol{\lambda}_2^{k,*}$.

6.1. $\sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k}\inf_{|\lambda_j^k|\leq\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k} J^k(c^k,\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k)$ *for given* $\boldsymbol{\lambda}_2^k$

First, for given $\boldsymbol{\lambda}_2^k, |\lambda_j^k| \leq \bar{\lambda}_j^k,\ j \in \mathcal{J}_2^k$, we consider $\sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k}\inf_{|\lambda_j^k|\leq\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k} J^k(c^k,\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k)$. As in Lemma 10 of Section 5,

$$c_t^{k,*} = -\frac{1}{\gamma^k}\log\left(H_{0,t}/\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\right) + c_0^{k,*}, \tag{93}$$

where $c_0^{k,*}$ and $\log H_{0,t}$ are given respectively as

$$c_0^{k,*} = \frac{1}{\mathbf{E}\left[\int_0^T H_{0,t}dt\right]}\left\{\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right] + \mathbf{E}\left[\int_0^T H_{0,t}\frac{\log\left(H_{0,t}/\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\right)}{\gamma^k}dt\right]\right\},$$

$$\log H_{0,t} = \frac{1}{\Delta}\left(-(\varepsilon_t - \varepsilon_0) + \sum_{l=1}^K \frac{1}{\gamma^l}\log\eta_t^{l,*}\right),$$

where $\eta^{l,*} = \eta^{\boldsymbol{\lambda}^{l,*}}$ with $\boldsymbol{\lambda}^{l,*}$ given by (21), and $\boldsymbol{\lambda}_1^{k*} = (-\bar{\lambda}_1^k,\ldots,-\bar{\lambda}_{d_1}^k)$ attain $\sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k}\inf_{|\lambda_j^k|\leq\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k} J^k(c^k,\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k)$ if $sgn(Z_{j,u}^k) = +1,\ j = 1,\ldots,d_1$ where we denote $Z_{j,t}$ of agent k in (14) as $Z_{j,t}^k$.

Also, we will use the result shown in Appendix A. Namely, $Z_{j,u}^k$ is calculated as follows:

$$Z_{j,u}^k = \int_u^T \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\left(\frac{1}{\gamma^k}e^{-\gamma^k c_s^{k,*}}\right)\left(\lambda_{j,u}^{k,*} + \left\{\gamma^k D_u^j(c_s^{k,*}) - D_u^j(\log\eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k})\right\}\right)\right]ds, \tag{94}$$

$j = 1,\ldots,d_1$.

To know $sgn(Z^k_{j,u})$, as $\frac{1}{\gamma^k}e^{-\gamma^k c^{k,*}_s} > 0$, we only have to examine the sign of

$$\lambda^{k,*}_{j,u} + \left\{ \gamma^k D^j_u(c^{k,*}_s) - D^j_u(\log \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}) \right\}. \tag{95}$$

Hereafter, we consider a special case of $K = 2$, $d = 1$. Also, we assume that agent 1 is conservative against the risk $B \equiv B_1$, and agent 2 is aggressive against the risk $B \equiv B_1$. Thus, we note $\eta_t^{\boldsymbol{\lambda}_1^{1,*},\boldsymbol{\lambda}_2^1} = \eta_t^{\boldsymbol{\lambda}_1^{1,*},0} = \eta_t^{1,*}$.

Hence, for agent 1, noting that in Lemma 10,

$$c^{1,*}_t = -\frac{1}{\gamma^1}\log\left(H_{0,t}/\eta_t^{\boldsymbol{\lambda}_1^{1,*},0}\right) + c^{1,*}_0, \tag{96}$$

we have

$$c^{1,*}_s = \left(\frac{1}{\gamma^1+\gamma^2}\right)\left(\gamma^2\varepsilon_s + \log\frac{\eta^{1,*}_s}{\eta^{2,*}_s}\right) + \left\{ c^{1,*}_0 - \left(\frac{\gamma^2}{\gamma^1+\gamma^2}\right)\varepsilon_0 \right\}. \tag{97}$$

Next, suppose that the aggregate endowment process is given as

$$\varepsilon_t = \varepsilon_0 + \int_0^t \nu_\tau d\tau + \int_0^t \rho_\tau dB_\tau, \tag{98}$$

$$\nu_t = \nu_0 + \int_0^t (a - b\nu_s)ds + \int_0^t \sigma dB_s. \tag{99}$$

Then, we have

$$\gamma^1 D_u c^{1,*}_s = \left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right) D_u\varepsilon_s + \left(\frac{\gamma^1}{\gamma^1+\gamma^2}\right)\left(D_u\log\eta^{1,*}_s - D_u\log\eta^{2,*}_s\right), \tag{100}$$

and thus,

$$\gamma^1 D_u c^{1,*}_s - D_u\log\eta^{1,*}_s = \left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right) D_u\varepsilon_s - \left(\frac{\gamma^2}{\gamma^1+\gamma^2}\right) D_u\log\eta^{1,*}_s - \left(\frac{\gamma^1}{\gamma^1+\gamma^2}\right) D_u\log\eta^{2,*}_s. \tag{101}$$

We also calculate

$$\begin{aligned}
&\left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right) D_u\varepsilon_s \\
&= \int_u^s \left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right) D_u\nu_\tau d\tau + \int_u^s \left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right) D_u\rho_\tau dB_\tau + \left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right)\rho_u.
\end{aligned} \tag{102}$$

Hence, plugging those into

$$\lambda^{1,*}_u + \left\{\gamma^1 D_u(c^{1,*}_s) - D_u(\log\eta^{1,*}_s)\right\}, \tag{103}$$

we obtain the following:

$$
\begin{aligned}
&\lambda_u^{1,*} + \gamma^1 D_u c_s^{1,*} - D_u \log \eta_s^{1,*} \\
&= \int_u^s \left\{ \left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2} \right) D_u \nu_\tau + \left(\frac{\gamma^2}{\gamma^1+\gamma^2} \right) D_u \frac{1}{2} |\lambda_\tau^{1,*}|^2 + \left(\frac{\gamma^1}{\gamma^1+\gamma^2} \right) D_u \frac{1}{2} |\lambda_\tau^{2,*}|^2 \right\} d\tau \\
&+ \int_u^s \left\{ \left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2} \right) D_u \rho_\tau - \left(\frac{\gamma^2}{\gamma^1+\gamma^2} \right) D_u \lambda_\tau^{1,*} - \left(\frac{\gamma^1}{\gamma^1+\gamma^2} \right) D_u \lambda_\tau^{2,*} \right\} dB_\tau \\
&+ \left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2} \right) \rho_u + \left(\frac{\gamma^1}{\gamma^1+\gamma^2} \right) \lambda_u^{1,*} - \left(\frac{\gamma^1}{\gamma^1+\gamma^2} \right) \lambda_u^{2,*}. \qquad (104)
\end{aligned}
$$

Then, to have $-\bar{\lambda}_1^1$ to be optimal for the conservative agent 1 in the random boundary case, that is, to get $sgn(Z^1) = 1$ as in Lemma 10 of Section 5, we need the following conditions:

$$D_u \nu_\tau + D_u \frac{1}{2\gamma^1} |\lambda_\tau^{1,*}|^2 + D_u \frac{1}{2\gamma^2} |\lambda_\tau^{2,*}|^2 > 0, ((\text{integrand of integration w.r.t } d\tau \text{ in (104) }) > 0), \qquad (105)$$

$$D_u \rho_\tau = \left(\frac{1}{\gamma^1} \right) D_u \lambda_\tau^{1,*} + \left(\frac{1}{\gamma^2} \right) D_u \lambda_\tau^{2,*}, ((\text{integrand of integration w.r.t } dB_\tau \text{ in (104)}) = 0), \quad (106)$$

$$\rho_u > \left(\frac{1}{\gamma^2} \right) \left(\lambda_u^{2,*} - \lambda_u^{1,*} \right), (\text{the last term in (104)} > 0), \qquad (107)$$

6.2. $\sup_{\boldsymbol{\lambda}_2^k, |\lambda_j^k| \le \bar{\lambda}_j^k,\ j \in \mathcal{J}_2^k} J^k(c^{k,*}, \boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k)$

Next, we consider $\sup_{\boldsymbol{\lambda}_2^k, |\lambda_j^k| \le \bar{\lambda}_j^k,\ j \in \mathcal{J}_2^k} J^k(c^{k,*}, \boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k)$ for $c^{k,*}$ and $\boldsymbol{\lambda}_1^{k,*}$ that attain the first $\sup_{(c^k, \boldsymbol{\pi}^k) \in \mathcal{A}^k} \inf_{|\lambda_j^k| \le \bar{\lambda}_j^k,\ j \in \mathcal{J}_1^k} J^k(c^k, \boldsymbol{\lambda}_1^k, \boldsymbol{\lambda}_2^k)$ part for given $\boldsymbol{\lambda}_2^k$.

As in Lemma 11 of Section 5, by using (82) with $d = 1$, $d_1 = 0$, $d_2 = 1$ and $K = 2$, we need the following condition for $\bar{\lambda}_2^2$ to be optimal for the aggressive agent 2 in the random boundary case:

$$\rho_s + D_s \nu_t = \rho_s + \sigma \int_s^t e^{-b(\tau - s)} d\tau > \frac{1}{\gamma^1} \lambda_s^{1,*} + \frac{1}{\gamma^2} \lambda_s^{2,*}, \qquad (108)$$

where we note $D_s \nu_t = \sigma \int_s^t e^{-b(\tau-s)} d\tau \ge 0$. Hence, the next condition is sufficient:

$$\rho_s > \frac{1}{\gamma^1} \lambda_s^{1,*} + \frac{1}{\gamma^2} \lambda_s^{2,*} \qquad (109)$$

Moreover, noting $\lambda_u^{1,*} < 0$ and $\lambda_u^{2,*} > 0$, the condition $\rho_u > \left(\frac{1}{\gamma^2} \right) \left(\lambda_u^{2,*} - \lambda_u^{1,*} \right)$ is more stringent than $\rho_s > \frac{1}{\gamma^1} \lambda_s^{1,*} + \frac{1}{\gamma^2} \lambda_s^{2,*}$. Thus, if (107) holds, (109) follows.

6.3. Sufficient conditions and implications

Noting $D_s \nu_t = \sigma \int_s^t e^{-b(\tau-s)} d\tau \ge 0$, the next condition is sufficient for (105):

$$D_u \frac{1}{2\gamma^1} |\lambda_\tau^{1,*}|^2 + D_u \frac{1}{2\gamma^2} |\lambda_\tau^{2,*}|^2 > 0. \qquad (110)$$

Also, the following condition is sufficient for (106):

$$\rho_\tau - c_{\rho,\tau} = \left(\frac{1}{\gamma^1}\right)\lambda_\tau^{1,*} + \left(\frac{1}{\gamma^2}\right)\lambda_\tau^{2,*};\ c_{\rho,\tau} > 0,\ \text{nonrandom.} \tag{111}$$

Then, both conditions, (107) and (111) imply that

$$\left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right)c_\rho > -\lambda^{1,*} > 0,\ \text{equivalently},\ \lambda^{1,*} > -\left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right)c_\rho. \tag{112}$$

Namely, $|\lambda^{1,*}|$, the magnitude of agent 1's random conservative view/sentiment should be bounded by the nonrandom process $\left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right)c_\rho$.

In sum, the equations (110), (111) and (112) are sufficient conditions for $\lambda^{1,*} = -\bar{\lambda}^1$ and $\lambda^{2,*} = \bar{\lambda}^2$.

Furthermore, (111) with (112) indicates that

$$\rho - \left(\frac{\gamma^1}{\gamma^1+\gamma^2}\right)c_\rho > \left(\frac{1}{\gamma^2}\right)\lambda^{2,*} > 0. \tag{113}$$

We also note that for a given aggregate endowment's random volatility ρ, we can interpret that those conditions (112) and (113) specify ranges where the agents' views/sentiments vary. That is, given $\rho,\ c_\rho > 0$, the equations (112) and (113) provide lower and upper limits for the range of agent 1's conservative and agent 2's aggressive views/sentiments, respectively.

Moreover, we remark that as in Proposition 13 of Section 5, we can obtain the equilibrium interest rate r and market price of risk $-\theta$. Namely, let us recall that

$$\begin{aligned} d\log H_{0,t} &= -\frac{1}{\Delta}\left[\nu_t + \frac{1}{2}\sum_{k=1}^{2}\frac{1}{\gamma^k}|\boldsymbol{\lambda}_t^{k,*}|^2\right]dt - \frac{1}{\Delta}\left[\boldsymbol{\rho}_t - \sum_{k=1}^{2}\frac{1}{\gamma^k}\boldsymbol{\lambda}_t^{k,*}\right]^\top dB_t \\ &= -\left[r_t + \frac{1}{2}|\boldsymbol{\theta}_t|^2\right]dt + \boldsymbol{\theta}_t^\top dB_t. \end{aligned}$$

Hence, in the current setting with (111) we have the following:

$-\theta = c_\rho/\Delta = \left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right)c_\rho$, and

$$\begin{aligned} r_t &= \frac{1}{\Delta}\left[\nu_t + \frac{1}{2}\sum_{k=1}^{2}\frac{1}{\gamma^k}|\lambda_t^{k,*}|^2\right] - \frac{1}{2}|\theta_t|^2 \\ &= \left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right)\left[\nu_t + \frac{1}{2}\sum_{k=1}^{2}\frac{1}{\gamma^k}|\bar{\lambda}_t^k|^2\right] - \frac{1}{2}\left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right)^2 c_{\rho,t}^2. \end{aligned} \tag{114}$$

As an example, given each agent's ARA parameter $\gamma^k > 0$ $(k = 1, 2)$ and a nonrandom process $c_\rho > 0$, let $M_1 := \left(\frac{\gamma^2}{\gamma^1+\gamma^2}\right)c_\rho - c > 0$ for an arbitrary small constant $c > 0$. We also define each $Y^k > 0$, $k = 1, 2$ as a mean-reverting square-root process:

$$dY_t^k = (a_y^k - b_y^k\, Y_t^k)dt + \sigma_y^k\sqrt{Y_t^k}dB_t;\ Y_0^k > 0,\ a_y^k,\ b_y^k > 0,\ a_y^k > (\sigma_y^k)^2/2, \tag{115}$$

where $D_u\{\sqrt{Y_\tau^k}\} > 0$ and $D_u\{Y_\tau^k\} > 0$ $(\tau > u)$ thanks to Proposition 4.1 and Corollary 4.2 in Alos and Ewald (2008).

Moreover, let $f(y)$ be a smoothly modified function of $\min\{M_1, \sqrt{y}\}$ $(y > 0)$ to define $f'(y) \geq 0$ for all $y > 0$ including $y = (M_1)^2$. Then, we set the aggregate endowment volatility ρ as

$$\rho = c_\rho - f(Y^1) + \sqrt{Y^2} > 0. \tag{116}$$

We finally put $\bar{\lambda}^1 = \gamma_1 f(Y^1)$ and $\bar{\lambda}^2 = \gamma_2\sqrt{Y^2}$. Using those $\bar{\lambda}^1$ and $\bar{\lambda}^2$ with (99), the equation (114) explicitly gives us the equilibrium interest rate r.

(**Reference**) Alos, E., and Ewald, C. O. (2008). Malliavin differentiability of the Heston volatility and applications to option pricing. Advances in Applied Probability, 40(1), 144-162.

Appendix A. Derivation of $Z^k_{j,u}$ in (94)

Firstly, for $\boldsymbol{Z}_s^k$ and $\boldsymbol{Z}_s^{p,k}$ in the martingale representations,

$$\int_0^T U^k(c_s^{k,*})ds = \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_0^T U^k(c_s^{k,*})ds\right] + \int_0^T \boldsymbol{Z}_s^{k\top} dB_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}, \tag{A.1}$$

and

$$\int_0^T \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} U^k(c_s^{k,*})ds = \mathbf{E}\left[\int_0^T \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} U^k(c_s^{k,*})ds\right] + \int_0^T \boldsymbol{Z}_s^{p,k\top} dB_s, \tag{A.2}$$

the relation

$$\boldsymbol{Z}_u^k = \frac{\boldsymbol{Z}_u^{p,k}}{\eta_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}} - V_u^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k)_u, \tag{A.3}$$

holds, which is shown as follows. We note the notation $\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k) = (\boldsymbol{\lambda}_1^{k,*\top}, \boldsymbol{\lambda}_2^{k\top}, 0, \ldots, 0)^\top$.

First, let us recall the definition:

$$V_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} = \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_t^T U^k(c_s^{k,*})ds \big| \mathcal{F}_t\right]. \tag{A.4}$$

Then,

$$\begin{aligned} V_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} &= \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_t^T U^k(c_s^{k,*})ds \big| \mathcal{F}_t\right] \\ &= \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_0^T U^k(c_s^{k,*})ds \big| \mathcal{F}_t\right] - \int_0^t U^k(c_s^{k,*})ds \\ &= \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_0^T U^k(c_s^{k,*})ds\right] + \int_0^t \boldsymbol{Z}_s^{k\top} dB_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} - \int_0^t U^k(c_s^{k,*})ds. \end{aligned} \tag{A.5}$$

In the second equality, we used the martingale representation theorem:

$$\int_0^T U^k(c_s^{k,*})ds = \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_0^T U^k(c_s^{k,*})ds\right] + \int_0^T \boldsymbol{Z}_s^{k\top} dB_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}. \tag{A.6}$$

Thus,

$$\begin{aligned} dV_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} &= \boldsymbol{Z}_t^{k\top} dB_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} - U^k(c_t^{k,*})dt \\ &= \boldsymbol{Z}_t^{k\top}(dB_t - \boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k)_t dt) - U^k(c_t^{k,*})dt \\ &= \boldsymbol{Z}_t^{k\top} dB_t - (U^k(c_t^{k,*}) + \boldsymbol{Z}_t^{k\top}\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k)_t)dt. \end{aligned} \tag{A.7}$$

On the other hand, we note

$$V_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} = \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_t^T U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right] = \mathbf{E}\left[\int_t^T \frac{\eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}}{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}} U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right]. \tag{A.8}$$

This is derived as follows:

$$\begin{aligned} V_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} &= \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_t^T U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right] \\ &= \frac{1}{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}}\mathbf{E}\left[\eta_T^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_t^T U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right] \\ &= \frac{1}{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}}\mathbf{E}\left[\int_t^T \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right]. \end{aligned} \tag{A.9}$$

The third equality holds as follows.

$$\begin{aligned} \mathbf{E}\left[\eta_T^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_t^T U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right] &= \mathbf{E}\left[\eta_T^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_0^T U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right] - \mathbf{E}\left[\eta_T^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_0^t U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right] \\ &= \mathbf{E}\left[\eta_T^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_0^T U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right] - \mathbf{E}\left[\eta_T^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\big|\mathcal{F}_t\right]\int_0^t U^k(c_s^{k,*})ds \\ &= \mathbf{E}\left[\eta_T^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_0^T U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right] - \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_0^t U^k(c_s^{k,*})ds. \end{aligned} \tag{A.10}$$

Here,

$$d\left(\eta_v^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_0^v U^k(c_s^{k,*})ds\right) = d\eta_v^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left(\int_0^v U^k(c_s^{k,*})ds\right) + \eta_v^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_v^{k,*})dv. \tag{A.11}$$

Integrating from 0 to t,

$$\begin{aligned} \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_0^t U^k(c_s^{k,*})ds &= \int_0^t\left(\int_0^v U^k(c_s^{k,*})ds\right)d\eta_v^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + \int_0^t \eta_v^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_v^{k,*})dv \\ &= \int_0^t\left(\int_0^v U^k(c_s^{k,*})ds\right)\eta_v^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k)_v^\top dB_v + \int_0^t \eta_v^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_v^{k,*})dv, \end{aligned} \tag{A.12}$$

where we used $d\eta_v^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} = \eta_v^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k)_v^\top dB_v$. Therefore, when we note that the first term is a stochastic integral with Brownian motion, we obtain

$$\begin{aligned}\mathbf{E}\left[\eta_T^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_t^T U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right] &= \mathbf{E}\left[\eta_T^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_0^T U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right] - \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_0^t U^k(c_s^{k,*})ds \\ &= \mathbf{E}\left[\int_0^T \eta_v^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_v^{k,*})dv\big|\mathcal{F}_t\right] - \int_0^t \eta_v^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_v^{k,*})dv \\ &= \mathbf{E}\left[\int_t^T \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right]. \end{aligned} \tag{A.13}$$

Now, we use the martingale representation theorem:

$$\int_0^T \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_s^{k,*})ds = \mathbf{E}\left[\int_0^T \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_s^{k,*})ds\right] + \int_0^T \boldsymbol{Z}_s^{p,k\top}dB_s. \tag{A.14}$$

Thus,

$$\mathbf{E}\left[\int_0^T \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right] = \mathbf{E}\left[\int_0^T \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_s^{k,*})ds\right] + \int_0^t \boldsymbol{Z}_s^{p,k\top}dB_s. \tag{A.15}$$

Here, since

$$\begin{aligned}V_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} &= \frac{1}{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}}\mathbf{E}\left[\int_t^T \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right] \\ &= \frac{1}{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}}\left\{\mathbf{E}\left[\int_0^T \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right] - \int_0^t \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_s^{k,*})ds\right\}, \end{aligned} \tag{A.16}$$

we have

$$\mathbf{E}\left[\int_0^T \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right] = \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}V_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + \int_0^t \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_s^{k,*})ds. \tag{A.17}$$

We calculate

$$\begin{aligned}&d\left(\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}V_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + \int_0^t \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_s^{k,*})ds\right) \\ &= V_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}d\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}dV_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + d\langle\eta^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}, V^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\rangle_t + \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_t^{k,*})dt \\ &= V_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k)_t^\top dB_t + \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}(\boldsymbol{Z}_t^{k\top}dB_t - (U^k(c_t^{k,*}) + \boldsymbol{Z}_t^{k\top}\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k)_t)dt) \\ &+ \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\boldsymbol{Z}_t^{k\top}\boldsymbol{\lambda}_t^{k,*}(\boldsymbol{\lambda}_2^k)dt + \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_t^{k,*})dt \\ &= \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}(V_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k)_t + \boldsymbol{Z}_t^k)^\top dB_t. \end{aligned} \tag{A.18}$$

Since this volatility coefficient must equal $\boldsymbol{Z}_t^{p,k}$, we obtain

$$\boldsymbol{Z}_t^k = \frac{\boldsymbol{Z}_t^{p,k}}{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}} - V_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k)_t. \tag{A.19}$$

Then, we have the following expressions for $Z^{p,k}_{j,u}$ and $-V^{k,\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u \boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}^k_2)_u$:

$$\begin{aligned}
Z^{p,k}_{j,u} &= \int_u^T \mathbf{E}_u\left[D^j_u\left(\eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s \frac{-1}{\gamma^k} e^{-\gamma^k c^{k,*}_s}\right)\right] ds \\
&= \int_u^T \mathbf{E}_u\left[\eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s e^{-\gamma^k c^{k,*}_s} D^j_u(c^{k,*}_s) + \frac{-1}{\gamma^k} e^{-\gamma^k c^{k,*}_s} D^j_u(\eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s)\right] ds \\
&= \int_u^T \mathbf{E}_u\left[\eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s e^{-\gamma^k c^{k,*}_s} \left\{D^j_u(c^{k,*}_s) - \frac{1}{\gamma^k} D^j_u(\log \eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s)\right\}\right] ds, \qquad \text{(A.20)} \\
-V^{k,\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2} \boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}^k_2)_u &= -\mathbf{E}_u\left[\int_u^T \frac{\eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s}{\eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u}\left(\frac{-1}{\gamma^k} e^{-\gamma^k c^{k,*}_s}\right) ds\right] \boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}^k_2)_u \\
&= \int_u^T \mathbf{E}^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u\left[\left(\frac{1}{\gamma^k} e^{-\gamma^k c^{k,*}_s}\right) ds\right] \boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}^k_2)_u, \qquad \text{(A.21)}
\end{aligned}$$

where $\mathbf{E}^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u[\cdot] := \mathbf{E}^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}[\cdot|\mathcal{F}_u]$ and the expression for $Z^{p,k}_{j,u}/\eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u$ as

$$\begin{aligned}
Z^{p,k}_{j,u}/\eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u &= \gamma^k \int_u^T \mathbf{E}_u\left[\frac{\eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s}{\eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u}\left(\frac{1}{\gamma^k} e^{-\gamma^k c^{k,*}_s}\right)\left\{D^j_u(c^{k,*}_s) - \frac{1}{\gamma^k} D^j_u(\log \eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s)\right\}\right] ds \\
&= \gamma^k \int_u^T \mathbf{E}^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u\left[\left(\frac{1}{\gamma^k} e^{-\gamma^k c^{k,*}_s}\right)\left\{D^j_u(c^{k,*}_s) - \frac{1}{\gamma^k} D^j_u(\log \eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s)\right\}\right] ds. \qquad \text{(A.22)}
\end{aligned}$$

Hence, we obtain a simple form of $Z^k_{j,u}, j = 1, \ldots, d_1$ as follows:

$$\begin{aligned}
Z^k_{j,u} &= Z^{p,k}_{j,u}/\eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u - V^{k,\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u \lambda^{k,*}_{j,u} \\
&= \int_u^T \mathbf{E}^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u\left[\left(\frac{1}{\gamma^k} e^{-\gamma^k c^{k,*}_s}\right)\left(\lambda^{k,*}_{j,u} + \left\{\gamma^k D^j_u(c^{k,*}_s) - D^j_u(\log \eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s)\right\}\right)\right] ds.
\end{aligned} \qquad \text{(A.23)}$$