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Mean-field equilibrium price formation with exponential utility *

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Abstract

In this paper, we study a problem of equilibrium price formation among many investors with exponential utility. The investors are heterogeneous in their initial wealth, risk-averseness parameter, as well as stochastic liability at the terminal time. We characterize the equilibrium risk-premium process of the risky stocks in terms of the solution to a novel mean-field backward stochastic differential equation (BSDE), whose driver has quadratic growth both in the stochastic integrands and in their conditional expectations. We prove the existence of a solution to the mean-field BSDE under several conditions and show that the resultant risk-premium process actually clears the market in the large population limit.

Keywords: market clearing, equilibrium price formation, exponential utility, optimal martingale principle, McKean-Vlasov type

1 Introduction

Since the pioneering works of Lasry & Lions [36, 37, 38] and Huang et al. [27, 28, 29], the great developments in the mean-field game (MFG) theory have made us possible to understand some of the long-standing issues of multi-agent games. If the interactions among the agents are symmetric, then the MFG techniques can render, in the large population limit, a very complex problem of solving a large coupled system of equations that characterizes a Nash equilibrium feasible by transforming it into separate and simpler problems of the optimization for a representative agent and of finding a fixed point. The resultant solution in the mean field equilibrium is known to provide an ϵ-Nash equilibrium for the original game with a large but a finite number of agents. The details of the MFG theory and many applications can be found, for example, in two volumes by Carmona & Delarue [5, 6] and in a lecture note by Cardaliaguet [4].

In this paper, we consider a problem of equilibrium price formation with exponential utility via a MFG approach. We seek the price processes of risky stocks (more precisely, the associated risk-premium process) *endogenously* so that they balance the demand and supply among a large number of investors facing the market-wide common noise as well as their own idiosyncratic noise. Unfortunately, this *market clearing condition* does not fit well to the concept of Nash equilibrium. Actually, if we change a trading strategy of one agent away from her equilibrium solution while keeping the other agents' strategies unchanged, then the balance of demand and supply will inevitably be broken down. Since the MFG theory has been developed primarily for the Nash games,

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most of its applications have not treated the market-clearing equilibrium. In fact, in many of the existing examples, their primary interests are not in the price formation and the asset price process is typically assumed exogenously. For example, let us refer to [8, 11, 12, 13, 34] as related works dealing with optimal investment problems with exponential and power utilities. All of these works concern with the Nash equilibrium among the investors competing in a relative performance criterion, while the relevant price processes are given exogenously. In particular, the associated risk-premium process is assumed to be bounded.

Recently, there also has been progress in the MFG theory for the problem of equilibrium price formation under the market clearing condition. Gomes & Saúde [22] present a deterministic model of electricity price. Its extension with random supply is given by Gomes et al. [20]. The same authors also study, in [21], a price formation problem of a commodity whose production is subject to random fluctuations. Evangelista et al. [9] investigate the price formation of an asset being traded in a limit order book and show promising numerical results using the actual high-frequency data of the listed stocks in several exchanges. Shrivats et al. [40] deal with a price formation problem for the solar renewable energy certificate (SREC) by solving forward-backward stochastic differential equations (FBSDEs) of McKean-Vlasov type, and Firoozi et al. [10] deal with a principal-agent problem in the associated emission market. Fujii & Takahashi [16] solve a stochastic mean-field price model of securities in the presence of stochastic order flows, and in [17], the same authors prove the strong convergence to the mean-field limit from the setup with finite number of agents. Fujii & Takahashi [18] extend the above model to the presence of a major player. Recently, Fujii [19] develops a model that allows the co-presence of cooperative and non-cooperative populations to learn how the price process is formed when a group of agents act in a coordinated manner.

Interestingly, there exist two important restrictions in all of these papers given above: firstly, the relevant control of each agent is interpreted as the trading rate that is absolutely continuous with respect to the Lebesgue measure i.e. dt; secondly, the cost function of each agent consists of terms representing some penalties on the trading speed and on the inventory size of the assets. In particular, if we want to apply their techniques to financial securities markets, we cannot deal with the general self-financing trading strategies nor the cost (or utility) function given directly in terms of the associated wealth process of portfolios. [1]

In this paper, we investigate an equilibrium price formation problem of the risky stocks by addressing the above two concerns. Our goal is to construct the risk-premium process endogenously so that the demand and supply of the associated stocks always balance among a large number of investors (agents) who are allowed to deploy general self-financing trading strategies. We assume that the agents have a common type of preference specified by exponential utility with respect to their wealth. The agents are allowed to have heterogeneities in their initial wealth, the size of risk-averseness parameter as well as the stochastic liability at the terminal time. In particular, the liability can depend on the common as well as the idiosyncratic noise, and hence can be associated with, for example, financial markets, individual endowment, consumption, and/or some budgetary target imposed on each agent by her manager. For solving the optimization problem of each agent, we adopt the optimal martingale principle developed by Hu, Imkeller & Müller [24], instead of the Pontryagin's maximal principle. We will see that, in general, it is necessary to relax the boundedness assumption on the risk-premium process, which is often used in the existing literature. Although this fact makes us unable to use the standard results on quadratic-growth BSDEs (qg-BSDEs) [33] as in [24], the special form of the relevant BSDE inherited from the exponential utility allows us to solve the problem. We then characterize the mean-field equilibrium in terms of a novel mean-field BSDE whose driver has quadratic growth both in the stochastic integrands and in their conditional

[1] See also a very recent work by Lavigne & Tankov [35] which adopts a very different approach to investigate the mean-field equilibrium of the carbon emissions among the firms.

expectations. The new BSDE is interesting in its own right since the same type of equations may become relevant for similar applications of mean-field equilibrium to other utility functions. We show the existence of a solution to this novel mean-field BSDE under several conditions, and then prove that the risk-premium process expressed by its solution actually clears the market in the large population limit.

The organization of the paper is as follows. After explaining some notations in Section 2, we solve the optimization problem of each agent with exponential utility in Section 3. In particular, a special attention is paid to allow the unbounded risk-premium process. In Section 4, we derive a novel mean-field BSDE and prove its existence of a solution under several conditions. We show that the risk-premium process characterized by the solution of this BSDE actually clears the market in the large population limit in Section 5. We concludes the paper by Section 6.

2 Notations

Throughout this work $T > 0$ denotes a given time horizon. For random variables and stochastic processes defined on a given complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, we use the following conventions to represent frequently used function spaces:

- $\mathbb{H}^2(\mathbb{P}, \mathbb{F}; \mathbb{R}^{1\times d})$ (or simply $\mathbb{H}^2$) denotes the set of $\mathbb{R}^{1\times d}$-valued $\mathbb{F}$-progressively measurable processes Z satisfying
$$\|Z\|_{\mathbb{H}^2} := \mathbb{E}^{\mathbb{P}}\Big[\int_0^T |Z_t|^2 dt\Big]^{\frac{1}{2}} < \infty.$$

- $\mathbb{H}^2_{\mathrm{BMO}}(\mathbb{P}, \mathbb{F}; \mathbb{R}^{1\times d})$ (or simply $\mathbb{H}^2_{\mathrm{BMO}}$) is a subset of $\mathbb{H}^2(\mathbb{P}, \mathbb{F}; \mathbb{R}^{1\times d})$ satisfying
$$\|Z\|^2_{\mathbb{H}^2_{\mathrm{BMO}}} := \sup_{\tau\in\mathcal{T}}\Big\|\mathbb{E}^{\mathbb{P}}\Big[\int_\tau^T |Z_t|^2 dt|\mathcal{F}_\tau\Big]\Big\|_\infty < \infty. \tag{2.1}$$
Here, $\mathcal{T}$ is the set of $\mathbb{F}$-stopping times with values in $[0, T]$, and $\|\cdot\|_\infty$ denotes the $\mathbb{P}$-essential supremum over Ω. In this case, thanks to the result of Kazamaki [32], the Dolean-Dade exponential $\mathcal{E}\Big(\int_0^\cdot Z_s dW_s\Big)_t$, $t \in [0, T]$ is known to be uniformly integrable (i.e. of class D), and $\Big(\int_0^\cdot Z_s dW_s\Big)_{t\in[0,T]}$ is called a BMO-martingale. Here, W is a d-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$. As an important property of $Z \in \mathbb{H}^2_{\mathrm{BMO}}$, let us mention the so-called energy inequality: For every $n \in \mathbb{N}$, the next inequality holds;
$$\mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T |Z_s|^2 ds\Big)^n\Big] \le n!\big(\|Z\|^2_{\mathbb{H}^2_{\mathrm{BMO}}}\big)^n. \tag{2.2}$$
See [7][Lemma 9.6.5].

- $\mathbb{S}^2(\mathbb{P}, \mathbb{F}; \mathbb{R}^d)$ (or simply $\mathbb{S}^2$) is the set of $\mathbb{R}^d$-valued $\mathbb{F}$-adapted continuous processes X satisfying
$$\|X\|_{\mathbb{S}^2} := \mathbb{E}^{\mathbb{P}}\big[\sup_{t\in[0,T]} |X_t|^2\big]^{\frac{1}{2}} < \infty.$$

- $\mathbb{S}^\infty(\mathbb{P},\mathbb{F};\mathbb{R}^d)$ (or simply $\mathbb{S}^\infty$) is a subset of $\mathbb{S}^2(\mathbb{P},\mathbb{F};\mathbb{R}^d)$ satisfying

$$\|X\|_{\mathbb{S}^\infty} := \Big\| \sup_{t\in[0,T]} |X_t| \Big\|_\infty < \infty.$$

When there is no risk of confusion, the specification such as $(\mathbb{P},\mathbb{F},\mathbb{R}^d)$ or part of it will be often omitted for notional simplicity.

The relevant probability spaces used in the first part of this work are given below.

- $(\Omega^0,\mathcal{F}^0,\mathbb{P}^0)$ is a complete probability space with a complete and right-continuous filtration $\mathbb{F}^0 := (\mathcal{F}^0_t)_{t\geq 0}$ generated by d_0-dimensional standard Brownian motion $W^0 := (W^0_t)_{t\geq 0}$. This space is used to model common noise and information.
- $(\Omega^1,\mathcal{F}^1,\mathbb{P}^1)$ is a complete probability space with a complete and right-continuous filtration $\mathbb{F}^1 := (\mathcal{F}^1_t)_{t\geq 0}$ generated by d-dimensional standard Brownian motion $W^1 := (W^1_t)_{t\geq 0}$ and a σ-algebra $\sigma(\xi^1,\gamma^1)$, which defines $\mathcal{F}^1_0$, generated by a bounded $\mathbb{R}$-valued random variable ξ^1 and a strictly positive bounded random variable γ^1. This space is used to model idiosyncratic noise and information for an agent (agent-1). In later sections, we create copies $(\Omega^i,\mathcal{F}^i,\mathbb{P}^i)_{i\in\mathbb{N}}$ of this space endowed with $\mathbb{F}^i := (\mathcal{F}^i_t)_{t\geq 0}$ to model information for a large number of agents, (agent-i, $i\in\mathbb{N}$).
- $(\Omega^{0,1},\mathcal{F}^{0,1},\mathbb{P}^{0,1})$ is a probability space defined on the product set $\Omega^{0,1} := \Omega^0\times\Omega^1$ with $(\mathcal{F}^{0,1},\mathbb{P}^{0,1})$ the completion of $(\mathcal{F}^0\otimes\mathcal{F}^1,\mathbb{P}^0\otimes\mathbb{P}^1)$. $\mathbb{F}^{0,1} := (\mathcal{F}^{0,1}_t)_{t\geq 0}$ denotes the complete and right-continuous augmentation of $(\mathcal{F}^0_t\otimes\mathcal{F}^1_t)_{t\geq 0}$. A generic element of $\Omega^{0,1}$ is denoted by $\omega := (\omega^0,\omega^1)\in\Omega^0\times\Omega^1$.
- $\mathcal{T}^{0,1}$ is the set of all $\mathbb{F}^{0,1}$-stopping times with values in $[0,T]$.
- $\mathcal{T}^0$ is the set of all $\mathbb{F}^0$-stopping times with values in $[0,T]$.

Throughout this work, we do not distinguish a random variable defined on a marginal probability space with its trivial extension to a bigger one for notational simplicity. For example, we will use a same symbol X for a random variable $X(\omega^0)$ defined on the space $(\Omega^0,\mathcal{F}^0,\mathbb{P}^0)$ and for its trivial extension $X(\omega^0,\omega^1) := X(\omega^0)$ defined on the product space $(\Omega^{0,1},\mathcal{F}^{0,1},\mathbb{P}^{0,1})$.

3 Exponential utility optimization for a given agent

In this section, we consider the optimization problem for an agent (agent-1) whose preference is given by exponential utility. We characterize her optimal trading strategy in terms of the quadratic-growth BSDE by the approach proposed by Hu, Imkeller & Müller [24]. In particular, however, in order to deal with the mean-field price formation as in [16], we need to relax their assumption on the boundedness of the risk-premium process $\theta := (\theta_t)_{t\in[0,T]}$ to the unbounded one in $\mathbb{H}^2_{\mathrm{BMO}}$.

3.1 The market and utility

The market dynamics and the (agent-1)'s idiosyncratic environment are modeled on the filtered probability space $(\Omega^{0,1},\mathcal{F}^{0,1},\mathbb{P}^{0,1};\mathbb{F}^{0,1})$ defined in the last section. In this section, the expectation with respect to $\mathbb{P}^{0,1}$ is simply denoted by $\mathbb{E}[\cdot]$. The financial market is specified as follows.

Assumption 3.1. (i) *The risk-free interest rate is zero.*
(ii) *There are $n \in \mathbb{N}$ non-dividend paying risky stocks whose price dynamics is given by*

$$S_t = S_0 + \int_0^t \mathrm{diag}(S_s)(\mu_s ds + \sigma_s dW_s^0), \ t \in [0,T]$$

where $S_0 \in \mathbb{R}^n_{++}$ is the initial stock price, $\mu := (\mu_t)_{t\in[0,T]}$ is an $\mathbb{R}^n$-valued, $\mathbb{F}^0$-progressively measurable process belonging to $\mathbb{H}^2_{\mathrm{BMO}}(\mathbb{P}^{0,1}, \mathbb{F}^0)$ [2]. *$\sigma := (\sigma_t)_{t\in[0,T]}$ is an $\mathbb{R}^{n\times d_0}$-valued, bounded, and $\mathbb{F}^0$-progressively measurable process such that there exist positive constants $0 < \underline{\lambda} < \overline{\lambda}$ satisfying*

$$\underline{\lambda} I_n \le (\sigma_t \sigma_t^\top) \le \overline{\lambda} I_n, \quad dt \otimes \mathbb{P}^0\text{-}a.e.$$

Here, I_n denotes $n \times n$ identity matrix.

Since the interest rate is zero, the risk-premium process $\theta := (\theta_t)_{t\in[0,T]}$ is defined by $\theta_t := \sigma_t^\top (\sigma_t\sigma_t^\top)^{-1}\mu_t$. Hence, for any $t \ge 0$, $\theta_t \in \mathrm{Range}(\sigma_t^\top) = \mathrm{Ker}(\sigma_t)^\perp$. Note that θ is in $\mathbb{H}^2_{\mathrm{BMO}}$ due to the boundedness of the process σ. By the regularity of $(\sigma\sigma^\top)$, we have $\mathrm{rank}(\sigma) = n$. The financial market is incomplete in general since $n \le d_0$.

Definition 3.1. *For each $s \in [0,T]$, let us denote by*

$$L_s := \{u^\top \sigma_s; u \in \mathbb{R}^n\}$$

the linear subspace of $\mathbb{R}^{1\times d_0}$ spanned by the n row vectors of σ_s. For any $z \in \mathbb{R}^{1\times d_0}$, $\Pi_s(z)$ denotes the orthogonal projection of z onto the linear subspace L_s.

Notice that $\theta_s^\top \in L_s$ for every $s \in [0,T]$ by its construction.

The idiosyncratic characters of the agent-1 are modeled by a triple (ξ^1, γ^1, F^1).

Assumption 3.2. (i) *ξ^1 is an $\mathbb{R}$-valued, bounded, and $\mathcal{F}_0^1$-measurable random variable denoting the initial wealth for the agent-1.*
(ii) *γ^1 is an $\mathbb{R}$-valued, bounded, and $\mathcal{F}_0^1$-measurable random variable, satisfying*

$$\underline{\gamma} \le \gamma^1 \le \overline{\gamma},$$

with some positive constants $0 < \underline{\gamma} \le \overline{\gamma}$. γ^1 denotes the size of risk-averseness of the agent-1.
(iii) *F^1 is an $\mathbb{R}$-valued, bounded, and $\mathcal{F}_T^{0,1}$-measurable random variable denoting the liability of the agent-1 at time T.*
(iv) *The agent-1 has a negligible market share and hence her trading activities have no impact on the stock prices, i.e., she is a price taker.*

Remark 3.1. *Notice that the liability F^1 is subject to common shocks from $\mathcal{F}_T^0$ as well as idiosyncratic shocks from $\mathcal{F}_T^1$.*

The wealth process of the agent-1 using the trading strategy π is given by

$$\mathcal{W}_t^{1,\pi} = \xi^1 + \sum_{j=1}^n \int_0^t \frac{\pi_{j,s}}{S_s^j} dS_s^j = \xi^1 + \int_0^t \pi_s^\top \sigma_s (dW_s^0 + \theta_s ds).$$

[2] Clearly, μ is also in $\mathbb{H}^2_{\mathrm{BMO}}(\mathbb{P}^{0,1}, \mathbb{F}^{0,1})$. In fact, additional information from $\mathcal{F}^1$ cannot increase the $\mathbb{H}^2_{\mathrm{BMO}}$-norm (2.1) since μ is $\mathbb{F}^0$-adapted which is independent from $\mathbb{F}^1$.

Here, $\pi := (\pi_t)_{t\in[0,T]}$ is an $\mathbb{R}^n$-valued, $\mathbb{F}^{0,1}$-progressively measurable process representing the invested amount of money in each of the n stocks. The problem of the agent-1 is to solve

$$\sup_{\pi\in\mathbb{A}^1} U^1(\pi),$$

where the functional U^1 is called exponential utility (for the agent-1). It is defined by

$$U^1(\pi) := \mathbb{E}\Big[-\exp\Big(-\gamma^1\Big(\xi^1 + \int_0^T \pi_s^\top \sigma_s(dW_s^0 + \theta_s ds) - F^1\Big)\Big)\Big]. \tag{3.1}$$

It means that the low performance in the sense of $\mathcal{W}_T^{1,\pi} - F^1 < 0$ is punished heavily and the high performance $\mathcal{W}_T^{1,\pi} - F^1 > 0$ is only weakly valued.

In this paper, we will not delve into the concrete modeling of the liability, which can include common as well as idiosyncratic shocks associated with, for example, financial market, endowment, consumption, local price of commodities, and/or budgetary target imposed on the agent by her manager.

Definition 3.2. *The admissible space $\mathbb{A}^1$ is the set of all $\mathbb{R}^n$-valued, $\mathbb{F}^{0,1}$-progressively measurable trading strategies π that satisfy $\mathbb{E}\Big[\int_0^T |\pi_s^\top\sigma_s|^2 ds\Big] < \infty$, and such that*

$$\big\{\exp(-\gamma^1\mathcal{W}_\tau^{1,\pi}); \tau\in\mathcal{T}^{0,1}\big\}$$

is uniformly integrable (i.e. of class D). We also define $\mathcal{A}^1 := \{p = \pi^\top\sigma; \pi\in\mathbb{A}^1\}$.

Note that p is $\mathbb{R}^{1\times d_0}$-valued and satisfies $p_s \in L_s$ for any $s\in[0,T]$. The problem for the agent-1 can be equivalently said to find the value function:

$$V^{1,*} := \sup_{p\in\mathcal{A}^1} \mathbb{E}\Big[-\exp\Big(-\gamma^1\Big(\xi^1 + \int_0^T p_s(dW_s^0 + \theta_s ds) - F^1\Big)\Big)\Big].$$

3.2 Characterization of the optimal trading strategy

Thanks to the work [24], we can characterize the optimal trading strategy by using a solution to a certain BSDE (instead of FBSDEs) in a rather straightforward way. When a utility (or equivalently cost) function has a special homothetic form as in the current case, their method often provides much simpler description of the optimal strategy than in the case where the Pontryagin's maximal principle is applied.

We try to construct a family of stochastic processes $\{R^p := (R_t^p)_{t\in[0,T]}, p\in\mathcal{A}^1\}$ satisfying the following properties:

Definition 3.3. *(Condition-R)*
(i) $R_T^p = -\exp\big(-\gamma^1(\mathcal{W}_T^{1,p} - F^1)\big)$ *a.s. for all* $p\in\mathcal{A}^1$.
(ii) $R_0^p = R_0$ *a.s. for all* $p\in\mathcal{A}^1$ *for some* $\mathcal{F}_0^1$ $(=\mathcal{F}_0^{0,1})$*-measurable random variable* R_0.
(iii) R^p *is an* $(\mathbb{F}^{0,1},\mathbb{P}^{0,1})$*-supermartingale for all* $p\in\mathcal{A}^1$*, and there exists some* $p^*\in\mathcal{A}^1$ *such that* R^{p^*} *is an* $(\mathbb{F}^{0,1},\mathbb{P}^{0,1})$*-martingale.*

In fact, if we can find a such a family $\{R^p\}$, then for any $p\in\mathcal{A}^1$, we have

$$\mathbb{E}\big[-\exp\big(-\gamma^1(\mathcal{W}_T^{1,p} - F^1)\big)\big] \le \mathbb{E}[R_0] = \mathbb{E}\big[-\exp\big(-\gamma^1(\mathcal{W}_T^{1,p^*} - F^1)\big)\big],$$

and hence p^* is an optimal strategy for the agent-1.

In order to construct the family $\{R^p\}$, we try to find an appropriate process $Y := (Y_t)_{t\in[0,T]}$ with which the process R^p is given by

$$R_t^p = -\exp\big(-\gamma^1(\mathcal{W}_t^{1,p} - Y_t)\big), \quad t \in [0,T],\ p \in \mathcal{A}^1. \tag{3.2}$$

Here, the triple (Y, Z^0, Z^1), which is a $(\mathbb{R}, \mathbb{R}^{1\times d_0}, \mathbb{R}^{1\times d})$-valued process, is an $\mathbb{F}^{0,1}$-adapted solution to the BSDE

$$Y_t = F^1 + \int_t^T f(s, Z_s^0, Z_s^1)ds - \int_t^T Z_s^0 dW_s^0 - \int_t^T Z_s^1 dW_s^1, \quad t \in [0,T].$$

The concrete form of the driver f is to be determined below so that $\{R^p\}$ satisfies the desired properties.

Under the hypothesis of (3.2), we get, by Itô formula,

$$\begin{aligned} dR_t^p &= R_t^p\Big(-\gamma^1 d(\mathcal{W}_t^{1,p} - Y_t) + \frac{(\gamma^1)^2}{2} d\langle \mathcal{W}^{1,p} - Y\rangle_t\Big) \\ &= R_t^p\Big(-\gamma^1(p_t\theta_t + f(t, Z_t^0, Z_t^1)) + \frac{(\gamma^1)^2}{2}(|p_t - Z_t^0|^2 + |Z_t^1|^2)\Big)dt \\ &\quad + R_t^p\big(-\gamma^1(p_t - Z_t^0)dW_t^0 + \gamma^1 Z_t^1 dW_t^1\big), \quad t \in [0,T]. \end{aligned}$$

In order to guess an appropriate form of f, let us formally solve it as

$$\begin{aligned} R_t^p = &-\exp\big(-\gamma^1(\xi^1 - Y_0)\big)\exp\Big(\int_0^t \big[-\gamma^1(p_s\theta_s + f(s, Z_s^0, Z_s^1)) + \frac{(\gamma^1)^2}{2}(|p_s - Z_s^0|^2 + |Z_t^1|^2)\big]ds\Big) \\ &\times \mathcal{E}\Big(\int_0^\cdot \big[-\gamma^1(p_s - Z_s^0)dW_s^0 + \gamma^1 Z_s^1 dW_s^1\big]\Big)_t. \end{aligned}$$

We want to set the driver $f(s, Z_s^0, Z_s^1)$ so that, for all $s \in [0,T]$,

- $-\gamma^1(p_s\theta_s + f(s, Z_s^0, Z_s^1)) + \frac{(\gamma^1)^2}{2}(|p_s - Z_s^0|^2 + |Z_s^1|^2) \geq 0$ for all $p \in \mathcal{A}^1$,
- $\exists p^* \in \mathcal{A}^1$ such that $-\gamma^1(p_s^*\theta_s + f(s, Z_s^0, Z_s^1)) + \frac{(\gamma^1)^2}{2}(|p_s^* - Z_s^0|^2 + |Z_s^1|^2) = 0$.

The above conditions suggest that

$$\begin{aligned} f(s, Z_s^0, Z_s^1) &= \inf_{p\in L_s}\Big\{-p\theta_s + \frac{\gamma^1}{2}(|p - Z_s^0|^2 + |Z_s^1|^2)\Big\} \\ &= \inf_{p\in L_s}\Big\{\frac{\gamma^1}{2}\Big|p_s - \Big(Z_s^0 + \frac{\theta_s^\top}{\gamma^1}\Big)\Big|^2 - Z_s^0\theta_s - \frac{1}{2\gamma^1}|\theta_s|^2 + \frac{\gamma^1}{2}|Z_s^1|^2\Big\}. \end{aligned}$$

This is a special case treated by [24][Section 2] with a trading constraint $\pi_t \in \widetilde{C}$ by a general closed subset $\widetilde{C} \subset \mathbb{R}^n$, which is now replaced by the whole space $\mathbb{R}^n$. A candidate of the optimal strategy p^* is then given by

$$p_t^* = Z_t^{0\|} + \frac{\theta_t^\top}{\gamma^1},\ t \in [0,T]. \tag{3.3}$$

Here, for notational simplicity, we have written $Z_s^{0\|} := \Pi_s(Z_s^0)$ and $Z_s^{0\perp} := Z_s^0 - \Pi_s(Z_s^0)$. They are orthogonal each other and $|Z_s^0|^2 = |Z_s^{0\|}|^2 + |Z_s^{0\perp}|^2$. Recall that $\Pi_s(\theta_s^\top) = \theta_s^\top$ for every s. With this convention, we have

$$f(s, Z_s^0, Z_s^1) = -Z_s^0\theta_s - \frac{1}{2\gamma^1}|\theta_s|^2 + \frac{\gamma^1}{2}(|Z_s^{0\perp}|^2 + |Z_s^1|^2) \\ = -Z_s^{0\|}\theta_s - \frac{1}{2\gamma^1}|\theta_s|^2 + \frac{\gamma^1}{2}(|Z_s^{0\perp}|^2 + |Z_s^1|^2) \tag{3.4}$$

where we used the fact $\theta_s^\top \in L_s$ and hence $Z_s^{0\perp}\theta_s = 0$ in the second equality.

Therefore, the associated qg-BSDE is given by

$$Y_t = F^1 + \int_t^T \Big(-Z_s^{0\|}\theta_s - \frac{|\theta_s|^2}{2\gamma^1} + \frac{\gamma^1}{2}(|Z_s^{0\perp}|^2 + |Z_s^1|^2)\Big)ds - \int_t^T Z_s^0 dW_s^0 - \int_t^T Z_s^1 dW_s^1,\ t \in [0,T]. \tag{3.5}$$

In order to make its appearance simpler, we rewrite the equation with $G^1 := \gamma^1 F^1$, and $(y, z^0, z^1) := (\gamma^1 Y, \gamma^1 Z^0, \gamma^1 Z^1)$. Then, we can equivalently work on the normalized BSDE,

$$y_t = G^1 + \int_t^T \Big(-z_s^{0\|}\theta_s - \frac{1}{2}|\theta_s|^2 + \frac{1}{2}(|z_s^{0\perp}|^2 + |z_s^1|^2)\Big)ds - \int_t^T z_s^0 dW_s^0 - \int_t^T z_s^1 dW_s^1,\ t \in [0,T]. \tag{3.6}$$

In this case, p^* is given by $\gamma^1 p_t^* = z_t^{0\|} + \theta_t^\top$. There should be no confusion which BSDE is being discussed by checking the terminal function and the presence of γ^1. In order to conclude that they are actually what we want, we need to verify that the resultant family $\{R^p\}$ (3.2) and the process p^* (3.3) satisfy (Condition-R).

3.3 Solution of the BSDE and its verification

We emphasize that, in contrast to the work [24], our risk-premium process $\theta \in \mathbb{H}^2_{\mathrm{BMO}}$ is unbounded. As we will see in later sections, this generalization is necessary to handle the mean-field market clearing equilibrium. Due to this unbounded risk-premium process, we cannot apply the standard results on qg-BSDEs given by Kobylanski [33]. Moreover, since the exponential integrability of $(|\theta_t|^2, t \in [0,T])$ is not guaranteed in general, we cannot apply the extensions on the qg-BSDE theories such as [2, 3, 26], either. In particular, the case $(|\theta_t|^{1+\alpha}, \alpha < 1)$ is covered by the result in [26] but not the case where $|\theta_t|^2$-term is contained in the driver. Fortunately, thanks to the special form of its driver inherited from the exponential utility, we can show the existence of a unique solution (y, z^0, z^1) to the BSDE (3.6) (and equivalently (3.5)) in the space $\mathbb{S}^\infty \times \mathbb{H}^2_{\mathrm{BMO}} \times \mathbb{H}^2_{\mathrm{BMO}}$ by a simple modification of the standard approach [33].

Lemma 3.1. *Let Assumptions 3.1 and 3.2 be in force. If there exists a bounded (with respect to the y-component) solution, i.e.* $(y, z^0, z^1) \in \mathbb{S}^\infty \times \mathbb{H}^2 \times \mathbb{H}^2$, *to the BSDE* (3.6), *then* $z := (z^0, z^1)$ *is in* $\mathbb{H}^2_{\mathrm{BMO}}$.

Proof. By Itô formula, we have,

$$\begin{aligned} de^{2y_t} &= e^{2y_t}\big(2z_t^{0\|}\theta_t + |\theta_t|^2 - \big(|z_t^{0\perp}|^2 + |z_t^1|^2\big) + 2(|z_t^0|^2 + |z_t^1|^2)\big)dt \\ &\quad + 2e^{2y_t}(z_t^0 dW_t^0 + z_t^1 dW_t^1) \\ &\geq e^{2y_t}\big(-|z_t^{0\|}|^2 - \big(|z_t^{0\perp}|^2 + |z_t^1|^2\big) + 2(|z_t^0|^2 + |z_t^1|^2)\big)dt \\ &\quad + 2e^{2y_t}(z_t^0 dW_t^0 + z_t^1 dW_t^1) \\ &\geq e^{2y_t}(|z_t^0|^2 + |z_t^1|^2)dt + 2e^{2y_t}(z_t^0 dW_t^0 + z_t^1 dW_t^1) \end{aligned}$$

and thus, for any $t \in [0,T]$,

$$e^{2y_T} - e^{2y_t} \geq \int_t^T e^{2y_s}(|z_s^0|^2 + |z_s^1|^2)ds + \int_t^T 2e^{2y_s}(z_s^0 dW_s^0 + z_s^1 dW_s^1).$$

Thus, it is easy to obtain $\|z\|^2_{\mathbb{H}^2_{\mathrm{BMO}}} := \sup_{\tau \in \mathcal{T}^{0,1}} \Big\| \mathbb{E}\Big[\int_\tau^T (|z_s^0|^2 + |z_s^1|^2)ds | \mathcal{F}_\tau^{0,1}\Big]\Big\|_\infty \leq e^{4\|y\|_{\mathbb{S}^\infty}}$. □

The above lemma is now used to guarantee the uniqueness of solution if $y \in \mathbb{S}^\infty$.

Theorem 3.1. *Let Assumptions 3.1 and 3.2 be in force. If the solution to* (3.6) *is bounded, i.e.* $(y, z^0, z^1) \in \mathbb{S}^\infty \times \mathbb{H}^2 \times \mathbb{H}^2$*, then such a solution is unique.*

Proof. Suppose that there are two bounded solutions (y, z^0, z^1) and $(\acute{y}, \acute{z}^0, \acute{z}^1)$. By Lemma 3.1, we know that (z^0, z^1) and $(\acute{z}^0, \acute{z}^1)$ are actually in $\mathbb{H}^2_{\mathrm{BMO}}$. Let us put;

$$\Delta y_t := y_t - \acute{y}_t, \quad \Delta z_t^0 := z_t^0 - \acute{z}_t^0, \quad \Delta z_t^1 := z_t^1 - \acute{z}_t^1.$$

From the orthogonality between $z^{0\|}$ and $z^{0\perp}$, we have

$$\begin{aligned}
\Delta y_t &= \int_t^T \Big(-\Delta z_s^{0\|}\theta_s + \frac{1}{2}\Delta(z_s^{0\perp})(z_s^{0\perp} + \acute{z}_s^{0\perp})^\top + \frac{1}{2}\Delta z_s^1 (z_s^1 + \acute{z}_s^1)^\top\Big)ds - \int_t^T \Delta z_s^0 dW_s^0 - \int_t^T \Delta z_s^1 dW_s^1 \\
&= \int_t^T \Big(-\Delta z_s^0\Big(\theta_s - \frac{1}{2}(z_s^{0\perp} + \acute{z}_s^{0\perp})^\top\Big) + \frac{1}{2}\Delta z_s^1 (z_s^1 + \acute{z}_s^1)^\top\Big)ds - \int_t^T \Delta z_s^0 dW_s^0 - \int_t^T \Delta z_s^1 dW_s^1 \\
&= -\int_t^T \Delta z_s^0 \Big(dW_s^0 + \big(\theta_s - \frac{1}{2}(z_s^{0\perp} + \acute{z}_s^{0\perp})^\top\big)ds\Big) - \int_t^T \Delta z_s^1 \Big(dW_s^1 - \frac{1}{2}(z_s^1 + \acute{z}_s^1)^\top ds\Big) \\
&= -\int_t^T \Delta z_s^0 d\widetilde{W}_s^0 - \int_t^T \Delta z_s^1 d\widetilde{W}_s^1,
\end{aligned}$$

where we have defined a new measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}^{0,1}$ by

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^{0,1}}\Big|_{\mathcal{F}_t^{0,1}} := M_t := \mathcal{E}\Big(-\int_0^\cdot \big(\theta_s^\top - \frac{1}{2}(z_s^{0\perp} + \acute{z}_s^{0\perp})\big)dW_s^0 + \int_0^\cdot \frac{1}{2}(z_s^1 + \acute{z}_s^1)dW_s^1\Big)_t,$$

and $(\widetilde{W}^0, \widetilde{W}^1)$ denote the standard Brownians under $\widetilde{\mathbb{P}}$. This measure change is well-defined since $(\theta^\top, z^{0\perp} + \acute{z}^{0\perp}, z^1 + \acute{z}^1)$ are in $\mathbb{H}^2_{\mathrm{BMO}}$ and hence M is an uniformly integrable martingale. By the result of Kazamaki [31] and [32][Remark 3.1], the following so-called reverse Hölder inequality holds:

$$\mathbb{E}\big[M_T^r | \mathcal{F}_t^{0,1}\big] \leq C M_t^r,$$

where $C > 0$ and $r > 1$ are some constants depending only on the $\mathbb{H}^2_{\mathrm{BMO}}$-norm of $(\theta^\top, z^{0\perp} + \acute{z}^{0\perp}, z^1 + \acute{z}^1)$. With $q = \frac{r}{r-1} > 1$ and $j = 0, 1$, Hölder and the energy inequality (2.2) imply

$$\begin{aligned}
\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\int_0^T |\Delta z_s^j|^2 ds\Big] &= \mathbb{E}\Big[M_T\Big(\int_0^T |\Delta z_s^j|^2 ds\Big)\Big] \\
&\leq \mathbb{E}[M_T^r]^{\frac{1}{r}} \mathbb{E}\Big[\Big(\int_0^T |\Delta z_s^j|^2 ds\Big)^q\Big]^{\frac{1}{q}} < \infty.
\end{aligned}$$

Thus Δy is an $(\mathbb{F}^{0,1}, \widetilde{\mathbb{P}})$-martingale. Thus we can conclude that $\Delta y = 0$ and so are $(\Delta z^0, \Delta z^1)$. □

Since θ is in $\mathbb{H}^2_{\mathrm{BMO}}$, it is natural to change the measure to absorb the term $(-z^{0\|}\theta)$ $(= -z^0\theta)$ in the driver of (3.6). Let us define the measure $\mathbb{Q}$ $(\sim \mathbb{P}^{0,1})$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}^{0,1}}\Big|_{\mathcal{F}^{0,1}_t} := \mathcal{E}\Big(-\int_0^{\cdot} \theta_s^\top dW_s^0\Big)_t,$$

where the standard Brownian motions under $\mathbb{Q}$ are given by

$$\widetilde{W}^0 = W_t^0 + \int_0^t \theta_s ds, \quad \widetilde{W}^1 = W_t^1, \quad t \in [0,T].$$

Therefore, instead of (3.6), we can equivalently work on the BSDE defined on $(\Omega^{0,1}, \mathcal{F}^{0,1}, \mathbb{Q}; \mathbb{F}^{0,1})$ endowed with the Brownian motions $(\widetilde{W}^0, \widetilde{W}^1)$;

$$y_t = G^1 + \int_t^T \Big(-\frac{1}{2}|\theta_s|^2 + \frac{1}{2}(|z_s^{0\perp}|^2 + |z_s^1|^2)\Big)ds - \int_t^T z_s^0 d\widetilde{W}^0 - \int_t^T z_s^1 d\widetilde{W}_s^1, \ t \in [0,T]. \tag{3.7}$$

Although in general, the filtration $\mathbb{F}^{0,1}$ is bigger than those generated by $(\widetilde{W}^0, \widetilde{W}^1)$, we can still apply the standard technique of BSDEs thanks to the stability of the martingale representation property under the absolutely continuous measure changes. See [23][Theorem 13.12] for general case and [30][Section 1.7.7] for Brownian case. Moreover, by Kazamaki [32][Theorem 3.3], θ is still in $\mathbb{H}^2_{\mathrm{BMO}}(\mathbb{Q}, \mathbb{F}^{0,1})$. Obviously, BSDE (3.6) (and equivalently (3.5)) has a bounded solution if and only if BSDE (3.7) has a bounded solution.

We now provide our first main result.

Theorem 3.2. *Let Assumptions 3.1 and 3.2 be in force. Then there is a unique bounded solution* $(y, z^0, z^1) \in \mathbb{S}^\infty(\mathbb{Q}, \mathbb{F}^{0,1}) \times \mathbb{H}^2_{\mathrm{BMO}}(\mathbb{Q}, \mathbb{F}^{0,1}) \times \mathbb{H}^2_{\mathrm{BMO}}(\mathbb{Q}, \mathbb{F}^{0,1})$ *to the BSDE* (3.7).

Proof. Since we work under measure $\mathbb{Q}$ throughout this proof, we write $\mathbb{E}[\cdot]$ instead of $\mathbb{E}^{\mathbb{Q}}[\cdot]$ for notational simplicity. Firstly, for each $n \in \mathbb{N}$, we consider the truncated BSDE;

$$y_t^n = G^1 + \int_t^T \Big(-\frac{1}{2}(|\theta_s|^2 \wedge n) + \frac{1}{2}(|z_s^{n,0\perp}|^2 + |z_s^{n,1}|^2)\Big)ds - \int_t^T z_s^{n,0} d\widetilde{W}_s^0 - \int_t^T z_s^{n,1} d\widetilde{W}_s^1, \quad t \in [0,T]. \tag{3.8}$$

Clearly, the truncated BSDE (3.8) has a unique bounded solution $(y^n, z^{n,0}, z^{n,1}) \in \mathbb{S}^\infty \times \mathbb{H}^2_{\mathrm{BMO}} \times \mathbb{H}^2_{\mathrm{BMO}}$ by the standard result of [33]. Moreover, by the comparison principle obtained in the same work, we have $y^{n+1} \le y^n$ for all $n \in \mathbb{N}$ [3]. In particular, uniformly in $n \in \mathbb{N}$, the solution y^n is bounded from above as $y^n \le \overline{y}$, where $\overline{y}$ is the solution to another quadratic-growth BSDE;

$$\overline{y}_t = G^1 + \int_t^T \frac{1}{2}(|\overline{z}_s^0|^2 + |\overline{z}_s^1|^2)ds - \int_t^T \overline{z}_s^0 d\widetilde{W}_s^0 - \int_t^T \overline{z}_s^1 d\widetilde{W}_s^1, \quad t \in [0,T].$$

It also has a unique bounded solution $(\overline{y}, \overline{z}^0, \overline{z}^1) \in \mathbb{S}^\infty \times \mathbb{H}^2_{\mathrm{BMO}} \times \mathbb{H}^2_{\mathrm{BMO}}$ with $\|\overline{y}\|_{\mathbb{S}^\infty} \le \|G^1\|_\infty$ by the standard result.

Once again, by the comparison principle, y^n is also bounded from below uniformly in $n \in \mathbb{N}$ as

[3] Here, we use $|z_s^{n,0\perp}|^2 - |z_s^{n+1,0\perp}|^2 = \Delta(z_s^{n,0\perp})(z_s^{n,0\perp} + z_s^{n+1,0\perp})^\top = \Delta z_s^{n,0}(z_s^{n,0\perp} + z_s^{n+1,0\perp})^\top$ to absorb it into the stochastic integral.

$\underline{y} \le y^n$, where $\underline{y}$ is the solution to the next simple BSDE;

$$\underline{y}_t = G^1 + \int_t^T \Big(-\frac{1}{2}|\theta_s|^2\Big)ds - \int_t^T \underline{z}^0_s d\widetilde{W}^0_s - \int_t^T \underline{z}^1_s d\widetilde{W}^1_s, \quad t \in [0,T].$$

Obviously, it has a unique solution $(\underline{y}, \underline{z}^0, \underline{z}^1) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2$. Moreover, for any $t \in [0,T]$,

$$\begin{aligned}\underline{y}_t &= \mathbb{E}[G^1|\mathcal{F}^{0,1}_t] - \frac{1}{2}\mathbb{E}\Big[\int_t^T |\theta_s|^2 ds|\mathcal{F}^{0,1}_t\Big] \\ &\ge -\Big(\|G^1\|_\infty + \frac{1}{2}\|\theta\|^2_{\mathbb{H}^2_{\mathrm{BMO}}}\Big) > -\infty.\end{aligned}$$

Hence we conclude that, uniformly in $n \in \mathbb{N}$, y^n satisfies the following bound,

$$-\Big(\|G^1\|_\infty + \frac{1}{2}\|\theta\|^2_{\mathbb{H}^2_{\mathrm{BMO}}}\Big) \le y^n \le \|G^1\|_\infty. \tag{3.9}$$

Since $\{y^n\}$ is bounded from below and it is monotonically decreasing in $n \in \mathbb{N}$, we can define the process $y := (y_t)_{t\in[0,T]}$ by

$$y = \lim_{n\to\infty} y^n.$$

Moreover, by repeating the proof of Lemma 3.1, we get from the estimate (3.9)

$$\forall n \in \mathbb{N}, \quad \|(z^{n,0}, z^{n,1})\|^2_{\mathbb{H}^2_{\mathrm{BMO}}} \le \exp\big(4\|G^1\|_\infty + 2\|\theta\|^2_{\mathbb{H}^2_{\mathrm{BMO}}}\big).$$

In particular, $(z^{n,0}, z^{n,1})_{n\in\mathbb{N}}$ are weakly relatively compact in $\mathbb{H}^2$ and hence, under an appropriate subsequence (still denoted by n), we have $\exists (z^0, z^1) \in \mathbb{H}^2 \times \mathbb{H}^2$, such that

$$z^{n,0} \rightharpoonup z^0, \quad z^{n,1} \rightharpoonup z^1 \quad \text{weakly in } \mathbb{H}^2 \text{ as } n \to \infty.$$

The remaining procedures to show the triple (y, z^0, z^1) actually solves (3.7) are the same as those in [33]. See also [7][Section 9.6]. For readers convenience, we shall give the details below.

Let us define a smooth convex function $\phi : \mathbb{R} \to \mathbb{R}_+$ satisfying

$$\phi(0) = 0, \quad \phi'(0) = 0,$$

whose concrete form is to be determined later. We consider $m, n \in \mathbb{N}$ such that $m \ge n$ and put

$$\Delta y^{n,m} := y^n - y^m, \quad \Delta z^{n,m;0} := z^{n,0} - z^{m,0}, \quad \Delta z^{n,m;1} := z^{n,1} - z^{m,1}.$$

Note that $\Delta y^{n,m} \ge 0$ since $m \ge n$. From Itô formula, we obtain for any $t \in [0,T]$,

$$\begin{aligned}&\phi(\Delta y^{n,m}_t) + \int_t^T \frac{1}{2}\phi''(\Delta y^{n,m}_s)(|\Delta z^{n,m;0}_s|^2 + |\Delta z^{n,m;1}_s|^2)ds \\ &= \int_t^T \phi'(\Delta y^{n,m}_s)\big[-\frac{1}{2}(|\theta_s|^2 \wedge n) + \frac{1}{2}(|z^{n,0\perp}|^2 + |z^{n,1}_s|^2) + \frac{1}{2}(|\theta_s|^2 \wedge m) - \frac{1}{2}(|z^{m,0\perp}_s|^2 + |z^{m,1}_s|^2)\big]ds \\ &\quad - \int_t^T \phi'(\Delta y^{n,m}_s)(\Delta z^{n,m;0}_s d\widetilde{W}^0_s + \Delta z^{n,m;1}_s d\widetilde{W}^1_s).\end{aligned}$$

Since $\phi(y), \phi'(y) \geq 0, \forall y \geq 0$, we get

$$\begin{aligned}
&\mathbb{E}\int_0^T \frac{1}{2}\phi''(\Delta y_s^{n,m})(|\Delta z_s^{n,m;0}|^2 + |\Delta z_s^{n,m;1}|^2)ds \\
&\quad \leq \mathbb{E}\int_0^T \frac{1}{2}\phi'(\Delta y_s^{n,m})\big(|\theta_s|^2 + |z_s^{n,0}|^2 + |z_s^{n,1}|^2\big)ds \\
&\quad \leq \mathbb{E}\int_0^T \phi'(\Delta y_s^{n,m})\big(|\theta_s|^2 + |z_s^{n,0} - z_s^0|^2 + |z_s^{n,1} - z_s^1|^2 + |z_s^0|^2 + |z_s^1|^2\big)ds.
\end{aligned} \tag{3.10}$$

We now choose the function ϕ as

$$\phi(y) := \frac{1}{8}[e^{4y} - 4y - 1],$$

which gives $\phi'(y) = \frac{1}{2}[e^{4y} - 1]$ and $\phi''(y) = 2e^{4y}$. In particular, we have $\phi''(y) = 4\phi'(y) + 2$. This yields, from (3.10),

$$\begin{aligned}
&\mathbb{E}\int_0^T [2\phi'(\Delta y^{n,m}) + 1](|\Delta z_s^{n,m;0}|^2 + |\Delta z_s^{n,m;1}|^2)ds \\
&\quad \leq \mathbb{E}\int_0^T \phi'(\Delta y_s^{n,m})(|\theta_s|^2 + |z_s^{n,0} - z_s^0|^2 + |z_s^{n,1} - z_s^1|^2 + |z_s^0|^2 + |z_s^1|^2)ds.
\end{aligned} \tag{3.11}$$

Note that, since $(\Delta y^{n,m})_{m\geq n}$ are bounded and strongly convergent $\Delta y^{n,m} \to y^n - y$ as $m \to \infty$, we also have, under an appropriate subsequence (still denoted by m), the following weak convergence in $\mathbb{H}^2$;

$$\sqrt{2\phi'(\Delta y^{n,m}) + 1}\Delta z^{n,m;j} \rightharpoonup \sqrt{2\phi'(y^n - y) + 1}(z^{n,j} - z^j), \quad \text{as } m \to \infty,$$

with $j = 0, 1$. Hence, by [1][Proposition 3.5] and monotone convergence, we obtain from (3.11),

$$\begin{aligned}
&\mathbb{E}\int_0^T [2\phi'(y_s^n - y_s) + 1](|z_s^{n,0} - z_s^0|^2 + |z_s^{n,1} - z_s^1|^2) \\
&\quad \leq \liminf_{m\to\infty} \mathbb{E}\int_0^T [2\phi'(\Delta y_s^{n,m}) + 1](|\Delta z_s^{n,m;0}|^2 + |\Delta z_s^{n,m;1}|^2)ds \\
&\quad \leq \mathbb{E}\int_0^T \phi'(y_s^n - y_s)(|\theta_s|^2 + |z_s^{n,0} - z_s^0|^2 + |z_s^{n,1} - z_s^1|^2 + |z_s^0|^2 + |z_s^1|^2)ds.
\end{aligned}$$

By rearranging the $|z^{n,j} - z^j|^2$-terms ($j = 0, 1$), we get

$$\begin{aligned}
&\mathbb{E}\int_0^T [\phi'(y_s^n - y_s) + 1](|z_s^{n,0} - z_s^0|^2 + |z_s^{n,1} - z_s^1|^2) \\
&\quad \leq \mathbb{E}\int_0^T \phi'(y_s^n - y_s)(|\theta_s|^2 + |z_s^0|^2 + |z_s^1|^2)ds.
\end{aligned}$$

Since the right-hand side converges to zero as $n \to \infty$ by the monotone convergence theorem, we obtain

$$z^{n,0} \to z^0, \quad z^{n,1} \to z^1, \quad \text{strongly in } \mathbb{H}^2.$$

Then, from Burkholder-Davis-Gundy (BDG) inequality [4], it implies that the following convergence

[4]See, for example, [39][Thorem 48 in IV].

holds for $(j=0,1)$ under an appropriate subsequence,

$$\sup_{t\in[0,T]}\Big|\int_t^T (z_s^{n,j}-z_s^j)d\widetilde{W}_s^j\Big|\to 0,\quad \mathbb{Q}\text{-a.s. as } n\to\infty$$

so is the case for $\sup_{t\in[0,T]}|y_t^n-y_t|$. It is now easy to see $(y,z^0,z^1)\in\mathbb{S}^\infty\times\mathbb{H}^2_{\mathrm{BMO}}\times\mathbb{H}^2_{\mathrm{BMO}}$ actually solves the BSDE (3.7). The uniqueness of the solution follows exactly in the same way as in Theorem 3.1. □

Corollary 3.1. *Let Assumptions 3.1 and 3.2 be in force. Then, the BSDE* (3.5) (*resp.* (3.6)) *has a unique bounded solution* (Y,Z^0,Z^1) (*resp.* (y,z^0,z^1)) *in* $\mathbb{S}^\infty(\mathbb{P}^{0,1},\mathbb{F}^{0,1})\times\mathbb{H}^2_{\mathrm{BMO}}(\mathbb{P}^{0,1},\mathbb{F}^{0,1})\times\mathbb{H}^2_{\mathrm{BMO}}(\mathbb{P}^{0,1},\mathbb{F}^{0,1})$.

We are now ready to verify the Condition-R.

Theorem 3.3. *Let Assumptions 3.1 and 3.2 be in force. The family of processes* $\{R^p,p\in\mathcal{A}^1\}$ *defined by* (3.2) *with the process* Y *as the unique bounded solution to the BSDE* (3.5) *satisfies the Condition-R, and the process* p^* *given by* (3.3) *gives the unique* (*up to* $dt\otimes\mathbb{P}^{0,1}$*-null set*) *optimal trading strategy for the agent-1.*

Proof. From (3.2), we have

$$R_t^p=-\exp\big(-\gamma^1(\mathcal{W}_t^{1,p}-Y_t)\big)=-\exp\big(-\gamma^1\mathcal{W}_t^{1,p}+y_t\big),\ t\in[0,T],$$

and

$$R_0^p=-\exp(-\gamma^1\xi^1+y_0)$$

for all $p\in\mathcal{A}^1$. Here, $y:=(y_t)_{t\in[0,T]}$ is the solution to (3.6). Since $y\in\mathbb{S}^\infty$, $(R_t^p,t\in[0,T])$ is clearly of class D for any $p\in\mathcal{A}^1$ by the definition of admissibility $\mathcal{A}^1$.

Let us choose $p=p^*$ as in (3.3). Then we have

$$\begin{aligned} dR_t^{p^*}&=R_t^{p^*}\big(-\gamma^1(p_t^*-Z_t^0)dW_t^0+\gamma^1Z_t^1dW_t^1\big)\\ &=R_t^{p^*}\big(-(\theta_t^\top-z_t^{0\perp})dW_t^0+z_t^1dW_t^1\big),\ t\in[0,T], \end{aligned}$$

and hence, for any $t\in[0,T]$,

$$\begin{aligned} R_t^{p*}&=-\exp\big(-\gamma^1\mathcal{W}_t^{1,p^*}+y_t\big)\\ &=-\exp\big(-\gamma^1\xi^1+y_0\big)\mathcal{E}\Big(-\int_0^\cdot(\theta_s^\top-z_s^{0\perp})dW_s^0+\int_0^\cdot z_t^1dW_t^1\Big)_t. \end{aligned}$$

Since $(\theta^\top-z^{0\perp},z^1)$ are in $\mathbb{H}^2_{\mathrm{BMO}}$ and (γ^1,ξ^1,y_0) are all bounded, R^{p*} is a uniformly integrable martingale. Uniform integrability of R^{p*} and the boundedness of y then imply that $(\exp(-\gamma^1\mathcal{W}_t^{1,p^*}))_{t\in[0,T]}$ is also uniformly integrable. Therefore, we obtain the admissibility $p^*\in\mathcal{A}^1$. The uniqueness of p^* follows from the strict convexity of $-\gamma^1(p\theta+f(s,z^0,z^1))+\frac{(\gamma^1)^2}{2}(|p-z^0|^2+|z^1|^2)$ with respect to p, which induces a strictly negative drift for R^p if $p\neq p^*$. Since we know R^p is of class D, its supermartingale property is now obvious. □

Remark 3.2. *It is important to note that the optimal trading strategy* π^* *(or equivalently* p^**) is independent from the initial wealth* ξ^1*. It is a well-known characteristic of exponential-type utilities.*

This fact, combined with the specification of $U^1(\pi)$ *in* (3.1)*, the problem for the agent-1 and her optimal trading strategy are invariant under the following transformation:*

$$\begin{aligned} \xi^1 &\longrightarrow (\xi^1 - \mathbb{E}[F^1|\mathcal{F}_0^1]), \\ F^1 &\longrightarrow (F^1 - \mathbb{E}[F^1|\mathcal{F}_0^1]). \end{aligned} \tag{3.12}$$

4 Mean-field equilibrium model

We are now going to investigate a financial market being participated by many agents, who are interacting each other through the price process of risky stocks. Recall that, our final goal of this paper is to find an risk-premium process $\theta = (\theta_t)_{t\geq 0}$ of the n risky stocks *endogenously* by imposing *the market-clearing condition*, which requires the demand and supply of the risky stocks to be always balanced among the agents. In this section, we shall propose a novel mean-field BSDE with a quadratic-growth driver, which is expected to provide, at least intuitively, the characterization of the desired equilibrium in the large population limit.

4.1 Heuristic derivation of the mean-field BSDE

Suppose that there are $N \in \mathbb{N}$ agents (agent-i, $1 \leq i \leq N$) participating in the same financial market given in Assumption 3.1. For each $1 \leq i \leq N$, the information set of agent-i is provided by the probability space $(\Omega^{0,i}, \mathcal{F}^{0,i}, \mathbb{P}^{0,i})$ which is a completion of the product space $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0) \otimes (\Omega^i, \mathcal{F}^i, \mathbb{P}^i)$. The associated filtration $\mathbb{F}^{0,i} := (\mathcal{F}_t^{0,i})_{t\geq 0}$ is the complete and right-continuous augmentation of $(\mathcal{F}_t^0 \otimes \mathcal{F}_t^i)_{t\geq 0}$. $\mathcal{T}^{0,i}$ is the set of $\mathbb{F}^{0,i}$-measurable stopping times with values in $[0,T]$. Here, for each i, the filtered probability space $(\Omega^i, \mathcal{F}^i, \mathbb{P}^i; \mathbb{F}^i)$ is an independent copy of $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{F}^1)$ constructed exactly in the same way as in Section 2 with (ξ^i, γ^i, W^i) instead of (ξ^1, γ^1, W^1).

In order to model all the agents in a common probability space, we define the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ such that

- $\Omega := \Omega^0 \times \prod_{i=1}^N \Omega^i$ and $(\mathcal{F}, \mathbb{P})$ is the completion of $\big(\mathcal{F}^0 \otimes \mathcal{F}^1 \otimes \cdots \otimes \mathcal{F}^N, \mathbb{P}^0 \otimes \mathbb{P}^1 \otimes \cdots \otimes \mathbb{P}^N\big)$. $\mathbb{F}$ denotes the complete and the right-continuous augmentation of $(\mathcal{F}_t^0 \otimes \mathcal{F}_t^1 \otimes \cdots \otimes \mathcal{F}_t^N)_{t\geq 0}$. $\mathbb{E}[\cdot]$ denotes the expectation with respect to $\mathbb{P}$.

We also introduce the liability F^i of the agent-i, $1 \leq i \leq N$. Each agent-i is assumed to face the optimization problem

$$\sup_{\pi \in \mathbb{A}^i} U^i(\pi),$$

where the utility functional U^i is defined by $U^i(\pi) := \mathbb{E}\Big[-\exp\Big(-\gamma^i\big(\mathcal{W}_T^{i,\pi} - F^i\big)\Big)\Big]$, with $\mathcal{W}_t^{i,\pi} := \xi^i + \int_0^t \pi_s^\top \sigma_s (dW_s^0 + \theta_s ds)$. The admissible space $\mathbb{A}^i$ (and $\mathcal{A}^i$) is defined by the same way as in Definition 3.2 with all the superscripts "1" replaced by "i".

Definition 4.1. *The admissible space* $\mathbb{A}^i$ *is the set of all* $\mathbb{R}^n$*-valued,* $\mathbb{F}^{0,i}$*-progressively measurable trading strategies* π *that satisfy* $\mathbb{E}\Big[\int_0^T |\pi_s^\top \sigma_s|^2 ds\Big] < \infty$*, and such that*

$$\big\{\exp(-\gamma^i \mathcal{W}_\tau^{i,\pi}); \tau \in \mathcal{T}^{0,i}\big\}$$

is uniformly integrable (i.e. of class D). We also define $\mathcal{A}^i := \big\{p = \pi^\top \sigma; \pi \in \mathbb{A}^i\big\}$.

We work under the following assumption.

Assumption 4.1. (i) *The statements in Assumption 3.2 hold with "1" replaced by "i", $1 \le i \le N$.*
(ii) $\{(\xi^i, \gamma^i), 1 \le i \le N\}$ *have the same distribution. In other words, they are independently and identically distributed (i.i.d.) on* $(\Omega, \mathcal{F}, \mathbb{P})$.
(iii) $\{F^i, 1 \le i \le N\}$ *are* $\mathcal{F}^0$*-conditionally i.i.d.*

We want to find the risk-premium process θ that clears the market. Let us first derive, heuristically, the relevant mean-field BSDE, which will be shown to characterize the market-clearing risk-premium process in the large population limit, by the idea proposed by Fujii & Takahashi [16].

Suppose that a risk-premium process $\theta \in \mathbb{H}^2_{\rm BMO}(\mathbb{P}, \mathbb{F}^0; \mathbb{R}^{d_0})$ is given. Recall that θ has values in $\text{Range}(\sigma^\top) = \text{Ker}(\sigma)^\perp$, i.e. $\theta_s^\top \in L_s$ for every $s \in [0,T]$. By repeating the analysis done in Section 3, one can show that the unique optimal strategy of agent-i, $1 \le i \le N$, is given by

$$p_t^{i,*} \ (= (\pi_t^{i,*})^\top \sigma_t) = Z_t^{i,0\|} + \frac{\theta_t^\top}{\gamma^i}, \quad t \in [0,T], \tag{4.1}$$

where $Z^{i,0}$ is associated to the solution $(Y^i, Z^{i,0}, Z^i)$ of the following BSDE:

$$Y_t^i = F^i + \int_t^T \Big(-Z_s^{i,0\|}\theta_s - \frac{|\theta_s|^2}{2\gamma^i} + \frac{\gamma^i}{2}(|Z_s^{i,0\perp}|^2 + |Z_s^i|^2)\Big) ds - \int_t^T Z_s^{i,0} dW_s^0 - \int_t^T Z_s^i dW_s^i, \ t \in [0,T]. \tag{4.2}$$

Under Assumptions 3.1 and 4.1, we already know, from Corollary 3.1, that there is a unique bounded solution $(Y^i, Z^{i,0}, Z^i) \in \mathbb{S}^\infty(\mathbb{P}, \mathbb{F}^{0,i}) \times \mathbb{H}^2_{\rm BMO}(\mathbb{P}, \mathbb{F}^{0,i}) \times \mathbb{H}^2_{\rm BMO}(\mathbb{P}, \mathbb{F}^{0,i})$ to (4.2), $1 \le i \le N$.

Definition 4.2. *We say that the market-clearing condition is satisfied if*

$$\frac{1}{N}\sum_{i=1}^N \pi_t^{i,*} = 0, \quad dt \otimes \mathbb{P} - a.e.$$

The market-clearing condition given by Definiton 4.2 requires that the risk-premium process θ to be

$$\theta_t = -\Big(\frac{1}{N}\sum_{j=1}^N \frac{1}{\gamma^j}\Big)^{-1} \frac{1}{N}\sum_{j=1}^N (Z_t^{j,0\|})^\top, \quad t \in [0,T]. \tag{4.3}$$

Unfortunately, the suggested θ by (4.3) is inconsistent with our information assumption that requires θ is $\mathbb{F}^0$-adapted i.e. being dependent only on the common market information. However, at this moment, let us formally consider the N-coupled system of quadratic-BSDEs obtained from (4.2) with θ replaced by the one given in (4.3); $1 \le i \le N$,

$$\begin{aligned} Y_t^i = F^i + \int_t^T \Big\{ & Z_s^{i,0\|}\Big(\frac{1}{N}\sum_{j=1}^N \frac{1}{\gamma^j}\Big)^{-1}\frac{1}{N}\sum_{j=1}^N (Z_s^{j,0\|})^\top - \frac{1}{2\gamma^i}\Big(\frac{1}{N}\sum_{j=1}^N\frac{1}{\gamma^j}\Big)^{-2}\Big|\frac{1}{N}\sum_{j=1}^N Z_s^{j,0\|}\Big|^2 \\ & + \frac{\gamma^i}{2}(|Z_s^{i,0\perp}|^2 + |Z_s^i|^2)\Big\} ds - \int_t^T Z_s^{i,0} dW_s^0 - \int_t^T Z_s^i dW_s^i. \end{aligned} \tag{4.4}$$

It is important to observe that the coupling among the agents through θ is symmetric. Thus, from Assumption 4.1, *if there is a solution* $\{(Y^i, Z^{i,0}, Z^i), 1 \le i \le N\}$ to the system (4.4), then they are exchangeable. In particular, $(Z_t^{i,0})_{i=1}^N$ (and hence $(Z_t^{i,0\|})_{i=1}^N$) are exchangeable random variables, i.e. their joint distribution is invariant under the permutation $\sigma(i)$ of their orders. Thus, in this case, De Finetti's theory of exchangeable sequence of random variables would imply that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} Z_t^{i,0\|}=\mathbb{E}\Big[Z_t^{1,0\|}|\bigcap_{k\geq 1}\sigma\{Z_t^{j,0\|},j\geq k\}\Big] \quad \text{a.s.}$$

See, for example, [6][Theorem 2.1]. We can also naturally expect that the tail σ-field converges to $\mathcal{F}_t^0$ since $(\mathcal{F}^i)_{i\geq 1}$ are independent. In this way, we can expect, at least heuristically, that the equilibrium risk-premium process θ in the large-N limit may be given by

$$\theta_t=-\widehat{\gamma}\mathbb{E}[Z_t^{1,0\|}|\mathcal{F}_t^0]^\top=-\widehat{\gamma}\mathbb{E}[Z_t^{1,0\|}|\mathcal{F}^0]^\top,$$

where $\widehat{\gamma}$ is defined by

$$\widehat{\gamma}:=\frac{1}{\mathbb{E}[1/\gamma^1]}.$$

The above heuristic discussions motivate us to study the following mean-field BSDE defined on the filtered probability space $(\Omega^{0,1},\mathcal{F}^{0,1},\mathbb{P}^{0,1};\mathbb{F}^{0,1})$ by choosing the agent-1 as the representative;

$$Y_t=F^1+\int_t^T\Big(\widehat{\gamma}Z_s^{0\|}\overline{\mathbb{E}}[Z_s^{0\|}]^\top-\frac{\widehat{\gamma}^2}{2\gamma^1}|\overline{\mathbb{E}}[Z_s^{0\|}]|^2+\frac{\gamma^1}{2}(|Z_s^{0\perp}|^2+|Z_s^1|^2)\Big)ds-\int_t^T Z_s^0dW_s^0-\int_t^T Z_s^1dW_s^1,\ t\in[0,T]. \tag{4.5}$$

Here, we have defined that, for any $X\in\mathbb{H}^2(\mathbb{P}^{0,1},\mathbb{F}^{0,1})$,

$$\overline{\mathbb{E}}[X_t](\omega^0):=\begin{cases}\mathbb{E}[X_t|\mathcal{F}^0](\omega^0)=\mathbb{E}^{\mathbb{P}^1}[X_t(\omega^0,\cdot)] & \text{if it exits}\\ 0 & \text{otherwise}\end{cases}.$$

Note that we have $\mathbb{E}[X_t|\mathcal{F}_t^0]=\overline{\mathbb{E}}[X_t]$ a.s., because $\sigma(\{W_s^0-W_t^0,s\geq t\})$ does not provide any additional information. As in [6][Section 4.3.1], we always choose $\mathbb{F}^0$-progressively measurable modification of $\overline{\mathbb{E}}[X]$. In the remainder of this section, we show that there is a solution to this mean-field BSDE under some conditions. In Section 5, we will show that the risk-premium process defined as

$$\theta_t^{\rm mfg}:=-\widehat{\gamma}\overline{\mathbb{E}}[Z_t^{0\|}]^\top,\quad t\in[0,T] \tag{4.6}$$

by the solution of the mean-field BSDE (4.5) actually clears the market in the large population limit.

4.2 Existence of a solution to the mean-field BSDE

We will work on the filtered probability space $(\Omega^{0,1},\mathcal{F}^{0,1},\mathbb{P}^{0,1};\mathbb{F}^{0,1})$. For notational ease, we simply write (F,γ) instead of (F^1,γ^1) and investigate the well-posedness of the mean-field BSDE:

$$Y_t=F+\int_t^T\Big(\widehat{\gamma}Z_s^{0\|}\overline{\mathbb{E}}[Z_s^{0\|}]^\top-\frac{\widehat{\gamma}^2}{2\gamma}|\overline{\mathbb{E}}[Z_s^{0\|}]|^2+\frac{\gamma}{2}(|Z_s^{0\perp}|^2+|Z_s^1|^2)\Big)ds-\int_t^T Z_s^0dW_s^0-\int_t^T Z_s^1dW_s^1,\quad t\in[0,T]. \tag{4.7}$$

Note that, by Assumption 3.2 (ii), $\widehat{\gamma}$ is a strictly positive constant.

Lemma 4.1. *If there exists a bounded solution $(Y,Z^0,Z^1)\in\mathbb{S}^\infty\times\mathbb{H}^2\times\mathbb{H}^2$ to the BSDE (4.7), then $Z:=(Z^0,Z^1)$ is in $\mathbb{H}^2_{\rm BMO}$.*

Proof. By applying Itô formula to $e^{2\gamma Y_t}$ and using $|Z_t^0|^2 = |Z_t^{0\|}|^2 + |Z_t^{0\perp}|^2$, we get

$$\begin{aligned}
d(e^{2\gamma Y_t}) &= e^{2\gamma Y_t}\big(2\gamma dY_t + 2\gamma^2(|Z_t^0|^2 + |Z_t^1|^2)dt\big)\\
&= e^{2\gamma Y_t}\big(-2\gamma\widehat{\gamma}Z_t^{0\|}\overline{\mathbb{E}}[Z_t^{0\|}]^\top + \widehat{\gamma}^2|\overline{\mathbb{E}}[Z_t^{0\|}]|^2 - \gamma^2(|Z_t^{0\perp}|^2 + |Z_t^1|^2) + 2\gamma^2(|Z_t^0|^2 + |Z_t^1|^2)\big)dt\\
&\quad + e^{2\gamma Y_t}2\gamma(Z_t^0 dW_t^0 + Z_t^1 dW_t^1)\\
&\geq e^{2\gamma Y_t}\gamma^2(|Z_t^0|^2 + |Z_t^1|^2)dt + e^{2\gamma Y_t}2\gamma(Z_t^0 dW_t^0 + Z_t^1 dW_t^1).
\end{aligned}$$

Hence, for any $\tau \in \mathcal{T}^{1,0}$,

$$\mathbb{E}\big[e^{2\gamma Y_T} - e^{2\gamma Y_\tau}|\mathcal{F}_\tau^{1,0}\big] \geq \mathbb{E}\Big[\int_\tau^T e^{2\gamma Y_s}\gamma^2(|Z_s^0|^2 + |Z_s^1|^2)ds|\mathcal{F}_\tau^{1,0}\Big],$$

which then gives $\|(Z^0, Z^1)\|^2_{\mathbb{H}^2_{\mathrm{BMO}}} \leq \dfrac{1}{\underline{\gamma}^2}\exp\big(4\overline{\gamma}\|Y\|_{\mathbb{S}^\infty}\big)$. □

Proving the well-posedness of the mean-field BSDE (4.7) is a very difficult task. First of all, due to the presence of conditional expectations, the comparison principle is not available. This makes many of the existing techniques for qg-BSDEs such as in [33, 2, 3] inapplicable. Due to the conditional McKean-Vlasov nature of the BSDE (4.7), the traditional approach of MFGs using Shauder's fixed point theorem does not work, either. Second, we did not succeed to obtain any *a priori* bound on $\|Y\|_{\mathbb{S}^\infty}$ nor any stability result such as [14][Lemma 3.3]. Thus, the strategy of constructing a compact set for the decoupling functions of the regularized BSDEs under the Markovian setup as in [15][Theorem 4.1] cannot be used.

Therefore, in the following, we will try the method proposed by Tevzadze [41][Proposition 1]. Among the existing literature on qg-BSDEs, a unique characteristic of his work is to solve a problem by a contraction mapping theorem without relying on the comparison principle. This method is also adopted by Hu et.al. [25] to deal with an anticipated BSDE with quadratic growth terms, which is related to the optimization problem with delay.

For notational simplicity, let us define the driver f of the mean-field BSDE by, for any $(z^0, z^1) \in \mathbb{H}^2(\mathbb{P}^{0,1}, \mathbb{F}^{0,1}; \mathbb{R}^{1\times d_0} \times \mathbb{R}^{1\times d})$,

$$f(z_s^0, z_s^1) := \widehat{\gamma}z_s^{0\|}\overline{\mathbb{E}}[z_s^{0\|}]^\top - \frac{\widehat{\gamma}^2}{2\gamma}|\overline{\mathbb{E}}[z_s^{0\|}]|^2 + \frac{\gamma}{2}(|z_s^{0\perp}|^2 + |z_s^1|^2), \quad s \in [0, T].$$

Since

$$\begin{aligned}
f(z^0, z^1) &= -\Big|\frac{\widehat{\gamma}}{\sqrt{2\gamma}}\overline{\mathbb{E}}[z^{0\|}] - \frac{\sqrt{\gamma}}{\sqrt{2}}z^{0\|}\Big|^2 + \frac{\gamma}{2}|z^{0\|}|^2 + \frac{\gamma}{2}(|z^{0\perp}|^2 + |z^1|^2)\\
&= -\Big|\frac{\widehat{\gamma}}{\sqrt{2\gamma}}\overline{\mathbb{E}}[z^{0\|}] - \frac{\sqrt{\gamma}}{\sqrt{2}}z^{0\|}\Big|^2 + \frac{\gamma}{2}(|z^0|^2 + |z^1|^2)
\end{aligned}$$

we have, for any (z^0, z^1),

$$f^+(z^0, z^1) \leq \frac{\overline{\gamma}}{2}(|z^0|^2 + |z^1|^2), \quad f^-(z^0, z^1) \leq \frac{\widehat{\gamma}^2}{\underline{\gamma}}|\overline{\mathbb{E}}[z^{0\|}]|^2 + \frac{\overline{\gamma}}{2}|z^{0\|}|^2.$$

Hence, in particular,

$$|f(z^0,z^1)| \le \frac{\overline{\gamma}}{2}(|z^0|^2+|z^1|^2) + \frac{\widehat{\gamma}^2}{\underline{\gamma}}|\overline{\mathbb{E}}[z^0]|^2. \tag{4.8}$$

Moreover, by Assumption 3.2 (ii), there is a positive constant C_γ, which depends only on $(\underline{\gamma},\overline{\gamma},\widehat{\gamma})$, such that, for any $(z^0,z^1),(\acute{z}^0,\acute{z}^1) \in \mathbb{H}^2$,

$$\begin{aligned}&|f(z^0,z^1)-f(\acute{z}^0,\acute{z}^1)| \\ &\quad \le C_\gamma\big(|z^0|+|\acute{z}^0|+|z^1|+|\acute{z}^1|+|\overline{\mathbb{E}}[z^0]|+|\overline{\mathbb{E}}[\acute{z}^0]|\big)\big(|z^0-\acute{z}^0|+|z^1-\acute{z}^1|+|\overline{\mathbb{E}}[z^0-\acute{z}^0]|\big).\end{aligned} \tag{4.9}$$

Let us observe the following simple fact.

Lemma 4.2. *For any input $(z^0,z^1) \in \mathbb{H}^2_{\mathrm{BMO}}(\mathbb{P}^{0,1},\mathbb{F}^{0,1})$, the next inequality holds;*

$$\sup_{\tau\in\mathcal{T}^{0,1}}\Big\|\mathbb{E}\Big[\int_\tau^T |f(z^0_s,z^1_s)|ds|\mathcal{F}^{0,1}_\tau\Big]\Big\|_\infty \le c_\gamma\|(z^0,z^1)\|^2_{\mathbb{H}^2_{\mathrm{BMO}}},$$

where c_γ is a positive constant given by $c_\gamma := \dfrac{\overline{\gamma}}{2}+\dfrac{\widehat{\gamma}^2}{\underline{\gamma}}$.

Proof. For any input $z^0 \in \mathbb{H}^2_{\mathrm{BMO}}$, $(\overline{\mathbb{E}}[z^0_t], t\in[0,T])$ is an $\mathbb{F}^0$-adapted process and hence independent from $\mathcal{F}^1$. This implies that

$$\begin{aligned}\sup_{\tau\in\mathcal{T}^{0,1}}\Big\|\mathbb{E}\Big[\int_\tau^T |\overline{\mathbb{E}}[z^0_s]|^2 ds|\mathcal{F}^{0,1}_\tau\Big]\Big\|_\infty &= \sup_{\tau\in\mathcal{T}^{0}}\Big\|\mathbb{E}\Big[\int_\tau^T |\overline{\mathbb{E}}[z^0_s]|^2 ds|\mathcal{F}^{0}_\tau\Big]\Big\|_\infty \\ \le \sup_{\tau\in\mathcal{T}^{0}}\Big\|\mathbb{E}\Big[\int_\tau^T |z^0_s|^2 ds|\mathcal{F}^{0}_\tau\Big]\Big\|_\infty &\le \sup_{\tau\in\mathcal{T}^{0,1}}\Big\|\mathbb{E}\Big[\int_\tau^T |z^0_s|^2 ds|\mathcal{F}^{0,1}_\tau\Big]\Big\|_\infty.\end{aligned} \tag{4.10}$$

Now the conclusion immediately follows from (4.8). □

We now define the map $\Gamma : \mathbb{H}^2_{\mathrm{BMO}}(\mathbb{P}^{0,1},\mathbb{F}^{0,1};\mathbb{R}^{1\times d_0}\times\mathbb{R}^{1\times d}) \ni (z^0,z^1) \mapsto \Gamma(z^0,z^1) = (Z^0,Z^1) \in \mathbb{H}^2_{\mathrm{BMO}}(\mathbb{P}^{0,1},\mathbb{F}^{0,1};\mathbb{R}^{1\times d_0}\times\mathbb{R}^{1\times d})$ by

$$Y_t = F + \int_t^T f(z^0_s,z^1_s)ds - \int_t^T Z^0_s dW^0_s - \int_t^T Z^1_s dW^1_s, \tag{4.11}$$

where $\Gamma(z^0,z^1)=(Z^0,Z^1)$ is the stochastic integrands associated to the BSDE (4.11).

Lemma 4.3. *Under Assumption 3.2, the map Γ is well-defined.*

Proof. For any $(z^0,z^1) \in \mathbb{H}^2_{\mathrm{BMO}}$, the existence of the unique solution (Y,Z^0,Z^1) to (4.11) is obvious. By taking a conditional expectation, we have from Lemma 4.2,

$$\|Y\|_{\mathbb{S}^\infty} \le \|F\|_\infty + c_\gamma\|(z^0,z^1)\|^2_{\mathbb{H}^2_{\mathrm{BMO}}} < \infty.$$

Moreover, by Itô formula applied to $|Y_t|^2$, we obtain for any $t\in[0,T]$,

$$\mathbb{E}\Big[\int_t^T (|Z^0_s|^2+|Z^1_s|^2)ds|\mathcal{F}^{0,1}_t\Big] \le \|F\|^2_\infty + 2\|Y\|_{\mathbb{S}^\infty}\mathbb{E}\Big[\int_t^T |f(z^0_s,z^1_s)|ds|\mathcal{F}^{0,1}_t\Big].$$

Thus, $\|(Z^0,Z^1)\|^2_{\mathbb{H}^2_{\mathrm{BMO}}} \le \|F\|^2_\infty + 2c_\gamma\|Y\|_{\mathbb{S}^\infty}\|(z^0,z^1)\|^2_{\mathbb{H}^2_{\mathrm{BMO}}} < \infty$. □

For each positive constant R, let $\mathcal{B}_R$ be a subset of $\mathbb{H}^2_{\mathrm{BMO}}$ defined by

$$\mathcal{B}_R := \Big\{(z^0, z^1) \in \mathbb{H}^2_{\mathrm{BMO}}(\mathbb{P}^{0,1}, \mathbb{F}^{0,1}; \mathbb{R}^{1\times d_0} \times \mathbb{R}^{1\times d})\Big| \|(z^0, z^1)\|^2_{\mathbb{H}^2_{\mathrm{BMO}}} \le R^2\Big\}.$$

Proposition 4.1. *Let Assumption 3.2 be in force. If $\|F\|_\infty \le 1/(2c_\gamma)$, then with $R := \sqrt{2}\|F\|_\infty$, the set $\mathcal{B}_R$ is stable under the map Γ, i.e. $(z^0, z^1) \in \mathcal{B}_R$ implies $\Gamma(z^0, z^1) \in \mathcal{B}_R$. Moreover, in this case, the y-component of the solution to* (4.11) *satisfies $\|Y\|_{\mathbb{S}^\infty} \le \sqrt{3}R$.*

Proof. By Itô formula, we have

$$|Y_t|^2 + \int_t^T (|Z^0_s|^2 + |Z^1_s|^2)ds = |F|^2 + \int_t^T 2Y_s f(z^0_s, z^1_s)ds - \int_t^T 2Y_s(Z^0_s dW^0_s + Z^1_s dW^1_s).$$

Taking conditional expectations and using Lemma 4.2, we obtain

$$\begin{aligned} &|Y_t|^2 + \mathbb{E}\Big[\int_t^T (|Z^0_s|^2 + |Z^1_s|^2)ds|\mathcal{F}^{0,1}_t\Big] \\ &\le \|F\|^2_\infty + 2\|Y\|_{\mathbb{S}^\infty}\mathbb{E}\Big[\int_t^T |f(z^0_s, z^1_s)|ds|\mathcal{F}^{0,1}_t\Big] \\ &\le \|F\|^2_\infty + \|Y\|^2_{\mathbb{S}^\infty} + c^2_\gamma\|(z^0, z^1)\|^4_{\mathbb{H}^2_{\mathrm{BMO}}}. \end{aligned} \tag{4.12}$$

Taking $\operatorname{ess\,sup}_{(t,\omega)\in[0,T]\times\Omega^{0,1}}$ in the left-hand side with $Z := (Z^0, Z^1)$ and $z := (z^0, z^1)$, we get

$$\|Z\|^2_{\mathbb{H}^2_{\mathrm{BMO}}} \le \|F\|^2_\infty + c^2_\gamma\|z\|^4_{\mathbb{H}^2_{\mathrm{BMO}}}.$$

We now try to find an $R > 0$ such that

$$\|F\|^2_\infty + c^2_\gamma R^4 \le R^2$$

holds. By completing the square, one sees that this is solvable if and only if $\|F\|_\infty \le 1/(2c_\gamma)$, and in this case, we can choose $R = \sqrt{2}\|F\|_\infty$. This proves the first statement. Moreover, by rearranging the first inequality in (4.12), we get

$$|Y_t|^2 \le \|F\|^2_\infty + \frac{1}{2}\|Y\|^2_{\mathbb{S}^\infty} + 2c^2_\gamma\|z\|^4_{\mathbb{H}^2_{\mathrm{BMO}}}$$

and hence

$$\|Y\|^2_{\mathbb{S}^\infty} \le -2\|F\|^2_\infty + 4\Big(\|F\|^2_\infty + c^2_\gamma\|z\|^4_{\mathbb{H}^2_{\mathrm{BMO}}}\Big).$$

This yields $\|Y\|^2_{\mathbb{S}^\infty} \le 3R^2$ if $z \in \mathcal{B}_R$. $\square$

Now, we provide the second main result of this paper.

Theorem 4.1. *Let Assumption 3.2 be in force. If the terminal function F is small enough in the sense that*

$$\|F\|_\infty < \frac{1}{12\sqrt{2}C_\gamma}, \tag{4.13}$$

where C_γ is a constant used in (4.9)*, then there exists a unique solution (Y, Z^0, Z^1) to the mean-field BSDE* (4.7) *in the domain*

$$(Z^0, Z^1) \in \mathcal{B}_R, \quad \|Y\|_{\mathbb{S}^\infty} \le \sqrt{3}R$$

with $R := \sqrt{2}\|F\|_\infty$.

Proof. Note that the requirement (4.13) is more stringent than the one used in Proposition 4.1. Thus it suffices to prove that the map Γ is a strict contraction.

To prove the contraction, let us consider two arbitrary inputs $z := (z^0, z^1)$, $\acute{z} := (\acute{z}^0, \acute{z}^1) \in \mathcal{B}_R$. We set

$$Z := (Z^0, Z^1) := \Gamma(z), \quad \acute{Z} := (\acute{Z}^0, \acute{Z}^1) := \Gamma(\acute{z}),$$

and $Y, \acute{Y}$ as the y-component of the solution of (4.11) with input z and $\acute{z}$, respectively. For notational simplicity, we put

$$\Delta z := z - \acute{z}, \quad \Delta Y := Y - \acute{Y}, \quad \Delta Z := Z - \acute{Z}.$$

This gives

$$\Delta Y_t = \int_t^T \big(f(z_s^0, z_s^1) - f(\acute{z}_s^0, \acute{z}_s^1)\big)ds - \int_t^T \Delta Z_s^0 dW_s^0 - \int_t^T \Delta Z_s^1 dW_s^1,$$

and hence by Itô formula,

$$|\Delta Y_t|^2 + \int_t^T (|\Delta Z_s^0|^2 + |\Delta Z_s^1|^2)ds = \int_t^T 2\Delta Y_s(f(z_s^0, z_s^1) - f(\acute{z}_s^0, \acute{z}_s^1))ds - \int_t^T 2\Delta Y_s(\Delta Z_s^0 dW_s^0 + \Delta Z_s^1 dW_s^1).$$

By taking the conditional expectation, we have, from Hölder inequality,

$$\begin{aligned}
&|\Delta Y_t|^2 + \mathbb{E}\Big[\int_t^T (|\Delta Z_s^0|^2 + |\Delta Z_s^1|^2)ds|\mathcal{F}_t^{0,1}\Big] \le 2\|\Delta Y\|_{\mathbb{S}^\infty}\mathbb{E}\Big[\int_t^T |f(z_s^0, z_s^1) - f(\acute{z}_s^0, \acute{z}_s^1)|ds|\mathcal{F}_t^{0,1}\Big] \\
&\quad \le \|\Delta Y\|_{\mathbb{S}^\infty}^2 + \Big(\mathbb{E}\Big[\int_t^T |f(z_s^0, z_s^1) - f(\acute{z}_s^0, \acute{z}_s^1)|ds|\mathcal{F}_t^{0,1}\Big]\Big)^2 \\
&\quad \le \|\Delta Y\|_{\mathbb{S}^\infty}^2 + C_\gamma^2\Big(\mathbb{E}\Big[\int_t^T (|z_s^0| + |\acute{z}_s^0| + |z_s^1| + |\acute{z}_s^1| + |\overline{\mathbb{E}}[z_s^0]| + |\overline{\mathbb{E}}[\acute{z}_s^0]|)(|\Delta z_s^0| + |\Delta z_s^1| + |\overline{\mathbb{E}}[\Delta z_s^0]|)ds|\mathcal{F}_t^{0,1}\Big]\Big)^2 \\
&\quad \le \|\Delta Y\|_{\mathbb{S}^\infty}^2 + C_\gamma^2\Big(\mathbb{E}\Big[\int_t^T (|z_s^0| + |\acute{z}_s^0| + |z_s^1| + |\acute{z}_s^1| + |\overline{\mathbb{E}}[z_s^0]| + |\overline{\mathbb{E}}[\acute{z}_s^0]|)^2 ds|\mathcal{F}_t^{0,1}\Big]\Big) \\
&\qquad\qquad \times \Big(\mathbb{E}\Big[\int_t^T (|\Delta z_s^0| + |\Delta z_s^1| + |\overline{\mathbb{E}}[\Delta z_s^0]|)^2 ds|\mathcal{F}_t^{0,1}\Big]\Big) \\
&\quad \le \|\Delta Y\|_{\mathbb{S}^\infty}^2 + C_\gamma^2\Big(6\mathbb{E}\Big[\int_t^T (|z_s^0|^2 + |\acute{z}_s^0|^2 + |z_s^1|^2 + |\acute{z}_s^1|^2 + \overline{\mathbb{E}}[|z_s^0|]^2 + \overline{\mathbb{E}}[|\acute{z}_s^0|]^2)ds|\mathcal{F}_t^{0,1}\Big]\Big) \\
&\qquad\qquad \times \Big(3\mathbb{E}\Big[\int_t^T (|\Delta z_s^0|^2 + |\Delta z_s^1|^2 + |\overline{\mathbb{E}}[\Delta z_s^0]|^2)ds|\mathcal{F}_t^{0,1}\Big]\Big).
\end{aligned}$$

Using (4.10), taking $\operatorname{ess\,sup}_{(\mathrm{t},\omega)\in[0,\mathrm{T}]\times\Omega^{0,1}}$ in the both hands of the above inequality, we get

$$\|\Delta Z\|_{\mathbb{H}^2_{\mathrm{BMO}}}^2 \le C_\gamma^2\Big(12 \sup_{\tau\in\mathcal{T}^{0,1}}\Big\|\mathbb{E}\Big[\int_\tau^T (|z_s|^2 + |\acute{z}_s|^2)ds|\mathcal{F}_\tau^{0,1}\Big]\Big\|_\infty\Big)\Big(6 \sup_{\tau\in\mathcal{T}^{0,1}}\Big\|\mathbb{E}\Big[\int_\tau^T |\Delta z_s|^2 ds|\mathcal{F}_\tau^{0,1}\Big]\Big\|_\infty\Big).$$

Since $z, \acute{z} \in \mathcal{B}_R$, we get

$$\|\Delta Z\|_{\mathbb{H}^2_{\mathrm{BMO}}}^2 \le 144 C_\gamma^2 R^2 \|\Delta z\|_{\mathbb{H}^2_{\mathrm{BMO}}}^2.$$

Hence the map Γ on $\mathcal{B}_R$ becomes contraction if $R < \frac{1}{12C_\gamma}$. Under the choice of $R = \sqrt{2}\|F\|_\infty$, this is equivalent to

$$\|F\|_\infty < \frac{1}{12\sqrt{2}C_\gamma}.$$

In this case $\Delta Z \to 0$ in $\mathbb{H}^2_{\rm BMO}$ under the repeated application of the map Γ, and it is also clear that $\Delta Y \to 0$ in $\mathbb{S}^\infty$. The unique fixed point $Z \in \mathcal{B}_R$ of the map Γ and the associated Y gives a unique solution to the mean-field BSDE (4.7) in the domain $Z \in \mathcal{B}_R$. $\square$

Remark 4.1. *Recall that there is an invariance of the optimal trading strategy under the transformation given by* (3.12)*. Therefore, for our purposes to obtain an equilibrium model with exponential utility, the constraint on the terminal function F in Theorem 4.1 is not a direct restriction on the absolute size of liability, but on the size of deviation from its mean:*

$$\big|F - \mathbb{E}[F|\mathcal{F}_0^{0,1}]\big| = \big|F - \mathbb{E}[F|\mathcal{F}_0^1]\big|.$$

4.3 Existence under the special situation

Let us provide one special example where the mean-field BSDE (4.7) has, at least, one solution $(Y, Z^0, Z^1) \in \mathbb{S}^\infty \times \mathbb{H}^2_{\rm BMO} \times \mathbb{H}^2_{\rm BMO}$ even when the terminal function F does not satisfy the constraint (4.13). Let us rewrite the mean-field BSDE (4.7) with the rescaled variables: [5]

$$(y, z^0, z^1) := (\gamma Y, \gamma Z^0, \gamma Z^1), \quad G = \gamma F,$$

which yields

$$y_t = G + \int_t^T \Big(\widehat{\gamma} z_s^{0\|} \mathbb{E}\Big[\frac{1}{\gamma} z_s^{0\|}\Big]^\top - \frac{\widehat{\gamma}^2}{2}\Big|\mathbb{E}\Big[\frac{1}{\gamma} z_s^{0\|}\Big]\Big|^2 + \frac{1}{2}(|z_s^{0\perp}|^2 + |z_s^1|^2)\Big) ds - \int_t^T z_s^0 dW_s^0 - \int_t^T z_s^1 dW_s^1, \ t \in [0, T]. \tag{4.14}$$

We put the following assumption.

Assumption 4.2. *The rescaled terminal function G has a separable form:*

$$G = G^0 + G^1,$$

where G^0 (resp. G^1) is a bounded $\mathcal{F}_T^0$ (resp. $\mathcal{F}_T^1$)-measurable random variable.

Remark 4.2. *In terms of the original liability F $(= F^1)$, the above condition is equivalent to assume that F has the following structure:*

$$F = \frac{1}{\gamma}\widetilde{F}^0 + \widetilde{F}^1,$$

where $\widetilde{F}^0$ (resp. $\widetilde{F}^1$) is a bounded $\mathcal{F}_T^0$ (resp. $\mathcal{F}_T^1$)-measurable random variable. In the financial market with distribution of agents as specified by Assumption 4.1, this implies that the part of liability dependent on the common noise are distributed as inversely proportional to the agents' risk-averseness parameters $(\gamma^i, i \in \mathbb{N})$.

Theorem 4.2. *Under Assumptions 3.2 and 4.2, there is, at least, one solution $(y, z^0, z^1) \in \mathbb{S}^\infty \times \mathbb{H}^2_{\rm BMO} \times \mathbb{H}^2_{\rm BMO}$ to* (4.14)*, or equivalently a solution $(Y, Z^0, Z^1) \in \mathbb{S}^\infty \times \mathbb{H}^2_{\rm BMO} \times \mathbb{H}^2_{\rm BMO}$ to* (4.7)*.*

Proof. Consider the following two BSDEs:

$$y_t^0 = G^0 + \int_t^T \frac{1}{2}|z_s^0|^2 ds - \int_t^T z_s^0 dW_s^0, \quad t \in [0, T],$$
$$y_t^1 = G^1 + \int_t^T \frac{1}{2}|z_s^1|^2 ds - \int_t^T z_s^1 dW_s^1, \quad t \in [0, T].$$

[5] Recall that, in this subsection, we are working on the convention $(F, \gamma) = (F^1, \gamma^1)$.

It is clear that there exists a unique solution $(y^i, z^i) \in \mathbb{S}^\infty \times \mathbb{H}^2_{\mathrm{BMO}}$ with $i = 0, 1$ by the standard results on qg-BSDEs in [33]. Since (y^0, z^0) is $\mathbb{F}^0$-adapted and so is $z^{0\|}$, we have

$$\overline{\mathbb{E}}\Big[\frac{1}{\gamma} z_s^{0\|}\Big] = \overline{\mathbb{E}}\Big[\frac{1}{\gamma}\Big] z_s^{0\|} = \frac{1}{\widehat{\gamma}} z_s^{0\|}.$$

Therefore, (y^0, z^0) also solves the BSDE

$$y_t^0 = G^0 + \int_t^T \Big(\widehat{\gamma} z_s^{0\|} \overline{\mathbb{E}}\Big[\frac{1}{\gamma} z_s^{0\|}\Big]^\top - \frac{\widehat{\gamma}^2}{2}\Big|\overline{\mathbb{E}}\Big[\frac{1}{\gamma} z_s^{0\|}\Big]\Big|^2 + \frac{1}{2}|z_s^{0\perp}|^2\Big) ds - \int_t^T z_s^0 dW_s^0.$$

It is now clear that $(y, z^0, z^1) := (y^0 + y^1, z^0, z^1)$ provides a solution to (4.14). $\square$

Remark 4.3. *Under the assumptions used in Theorem 4.2, we actually have a closed form solution,*

$$y_t^j = \ln\big(\mathbb{E}\big[\exp(G^j)|\mathcal{F}_t^j\big]\big), \;\; j = 0, 1.$$

This result can be easily confirmed by the Cole-Hopf transformation, $\exp(y_t^j)$.

5 Market clearing in the large population limit

Finally, in this section, we shall show that the process $(\theta_t^{\mathrm{mfg}}, t \in [0, T]) \in \mathbb{H}^2_{\mathrm{BMO}}$ defined by (4.6) in terms of the solution to the mean-field BSDE is actually a good approximate of the risk-premium process in the market-clearing equilibrium.

In order to treat the large population limit, we first enlarge the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ as follows.

- $\Omega := \Omega^0 \times \prod_{i=1}^\infty \Omega^i$ and $(\mathcal{F}, \mathbb{P})$ is the completion of $\big(\mathcal{F}^0 \otimes \bigotimes_{i=1}^\infty \mathcal{F}^i, \mathbb{P}^0 \otimes \bigotimes_{i=1}^\infty \mathbb{P}^i\big)$. $\mathbb{F}$ denotes the complete and the right-continuous augmentation of $(\mathcal{F}_t^0 \otimes \bigotimes_{i=1}^\infty \mathcal{F}_t^i)_{t \geq 0}$. $\mathbb{E}[\cdot]$ denotes the expectation with respect to $\mathbb{P}$.

Suppose that the financial market is defined as in Assumption 3.1 with the process μ given by $(\mu_t := \sigma_t \theta_t^{\mathrm{mfg}}, t \in [0, T])$. Notice that the process θ^{mfg} (and hence also μ) is $\mathbb{F}^0$-adapted and consistent with our assumption on the information structure. In particular, this means that each agent (agent-i) can implement her strategy based on the common and her own idiosyncratic informations $\mathbb{F}^0 \otimes \mathbb{F}^i$ without taking care of idiosyncratic noise of the other agents. Therefore, for each $i \in \mathbb{N}$, the optimal trading strategy of the agent-i is provided by, as in (4.1),

$$p_t^{i,*} \; (= (\pi_t^{i,*})^\top \sigma_t) = Z_t^{i,0\|} - \frac{\widehat{\gamma}}{\gamma^i} \overline{\mathbb{E}}[\mathcal{Z}_t^{0\|}], \quad t \in [0, T], \tag{5.1}$$

where $Z^{i,0}$ is associated to the solution of the BSDE (see, (4.2))

$$\begin{aligned} Y_t^i = F^i &+ \int_t^T \Big(\widehat{\gamma} Z_s^{i,0\|} \overline{\mathbb{E}}[\mathcal{Z}_s^{0\|}]^\top - \frac{\widehat{\gamma}^2}{2\gamma^i}|\overline{\mathbb{E}}[\mathcal{Z}_s^{0\|}]|^2 + \frac{\gamma^i}{2}(|Z_s^{i,0\perp}|^2 + |Z_s^i|^2)\Big) ds \\ &- \int_t^T Z_s^{i,0} dW_s^0 - \int_t^T Z_s^i dW_s^i, \; t \in [0, T], \end{aligned} \tag{5.2}$$

and $\mathcal{Z}^0$ is associated to the solution of the mean-field BSDE (4.5) [6]

$$\mathcal{Y}^1_t = F^1+\int_t^T\Big(\widehat{\gamma}\mathcal{Z}^{0\|}_s\overline{\mathbb{E}}[\mathcal{Z}^{0\|}_s]^\top-\frac{\widehat{\gamma}^2}{2\gamma^1}|\overline{\mathbb{E}}[\mathcal{Z}^{0\|}_s]|^2+\frac{\gamma^1}{2}(|\mathcal{Z}^{0\perp}_s|^2+|\mathcal{Z}^1_s|^2)\Big)ds-\int_t^T \mathcal{Z}^0_s dW^0_s-\int_t^T \mathcal{Z}^1_s dW^1_s,\ t\in[0,T].$$

From Theorem 4.1 and Theorem 4.2, we already know that there exists a bounded solution (possibly not unique) $(\mathcal{Y}^1,\mathcal{Z}^0,\mathcal{Z}^1)\in\mathbb{S}^\infty\times\mathbb{H}^2_{\text{BMO}}\times\mathbb{H}^2_{\text{BMO}}$ to the mean-field BSDE (4.5) under certain conditions.

Here is the last main result of this paper.

Theorem 5.1. *Let Assumptions 3.1 and 4.1 be in force. Assume in addition that there is a bounded solution $(\mathcal{Y}^1,\mathcal{Z}^0,\mathcal{Z}^1)\in\mathbb{S}^\infty\times\mathbb{H}^2_{\text{BMO}}\times\mathbb{H}^2_{\text{BMO}}$ to the mean-field BSDE* (4.5) *and that we select an arbitrary but fixed solution from it to define the risk-premium process $(\theta^{\text{mfg}}_t := -\widehat{\gamma}\overline{\mathbb{E}}[\mathcal{Z}^{0\|}_t]^\top,\ t\in[0,T])$. Then, θ^{mfg} clears the market in the large population limit in the sense that, the agents' optimal trading strategies $(\pi^{i,*})_{i\in\mathbb{N}}$ satisfy the estimate*

$$\mathbb{E}\int_0^T\Big|\frac{1}{N}\sum_{i=1}^N\pi^{i,*}_t\Big|^2dt\le\frac{C}{N}$$

with some constant $C>0$ uniformly in $N\in\mathbb{N}$.

Proof. For a given θ^{mfg} $(=-\widehat{\gamma}\overline{\mathbb{E}}[\mathcal{Z}^{0\|}]^\top)$, which is in $\mathbb{H}^2_{\text{BMO}}$ by (4.10), it follows from Corollary 3.1 that the BSDE (5.2) has a unique bounded solution $(Y^i,Z^{i,0},Z^i)\in\mathbb{S}^\infty\times\mathbb{H}^2_{\text{BMO}}\times\mathbb{H}^2_{\text{BMO}}$ for every $i\in\mathbb{N}$. In particular, this uniqueness of the solution implies $(Y^1,Z^{1,0},Z^1)=(\mathcal{Y}^1,\mathcal{Z}^0,\mathcal{Z}^1)$, the latter of which is the one used to define the process θ^{mfg}. Thus we have $\overline{\mathbb{E}}[\mathcal{Z}^{0\|}]=\overline{\mathbb{E}}[Z^{1,0\|}]$.

The uniqueness of the solution of (5.2) also implies, by Yamada-Watanabe Theorem (see, for example, [6][Theorem 1.33]), that there exists a some measurable function Φ such that

$$(Y^i,Z^{i,0},Z^i)=\Phi\Big(W^0,\xi^i,\gamma^i,W^i,(\theta^{\text{mfg}},F^i)\Big),\ \forall i\in\mathbb{N}$$

where Φ depends only on the joint distribution $\mathcal{L}\big(W^0,\xi^i,\gamma^i,W^i,(\theta^{\text{mfg}},F^i)\big)$. Since θ^{mfg} is $\mathbb{F}^0$-adapted and F^i is $\mathcal{F}^0$-conditionally i.i.d, this expression implies that the solutions $\{(Y^i,Z^{i,0},Z^i),\ i\in\mathbb{N}\}$ are $\mathcal{F}^0$-conditionally i.i.d.

Since $\pi^{i,*}_t=(\sigma_t\sigma_t^\top)^{-1}\sigma_t(p^{i,*}_t)^\top$, we have, from Assumption 3.1 (ii),

$$\mathbb{E}\int_0^T\Big|\frac{1}{N}\sum_{i=1}^N\pi^{i,*}_t\Big|^2dt\le C\mathbb{E}\int_0^T\Big|\frac{1}{N}\sum_{i=1}^N p^{i,*}_t\Big|^2dt$$

with some constant C. Since $\mathcal{Z}^0=Z^{1,0}$, we have

$$\frac{1}{N}\sum_{i=1}^N p^{i,*}_t=\frac{1}{N}\sum_{i=1}^N(Z^{i,0\|}_t-\overline{\mathbb{E}}[Z^{1,0\|}_t])+\frac{1}{N}\sum_{i=1}^N\Big(1-\frac{\widehat{\gamma}}{\gamma^i}\Big)\overline{\mathbb{E}}[Z^{1,0\|}_t],$$

[6]For a clear distinction, we have changed the symbols.

which then yields

$$\mathbb{E}\int_0^T\Big|\frac{1}{N}\sum_{i=1}^N p_t^{i,*}\Big|^2dt \le 2\mathbb{E}\int_0^T\Big|\frac{1}{N}\sum_{i=1}^N(Z_t^{i,0\|}-\overline{\mathbb{E}}[Z_t^{1,0\|}])\Big|^2dt+2\widehat{\gamma}^2\mathbb{E}\int_0^T\Big|\frac{1}{N}\sum_{i=1}^N\Big(\frac{1}{\widehat{\gamma}}-\frac{1}{\gamma^i}\Big)\overline{\mathbb{E}}[Z_t^{1,0\|}]\Big|^2dt$$
$$=2\mathbb{E}\int_0^T\Big|\frac{1}{N}\sum_{i=1}^N(Z_t^{i,0\|}-\overline{\mathbb{E}}[Z_t^{1,0\|}])\Big|^2dt+2\widehat{\gamma}^2\mathbb{E}\Big[\Big|\frac{1}{N}\sum_{i=1}^N\Big(\frac{1}{\widehat{\gamma}}-\frac{1}{\gamma^i}\Big)\Big|^2\Big]\mathbb{E}\int_0^T|\overline{\mathbb{E}}[Z_t^{1,0\|}]|^2dt,$$

where we used the independence of $(\gamma^i)_{i\in\mathbb{N}}$ and $\mathcal{F}^0$ in the second line. Since $(1/\gamma^i)_{i=1}^N$ are i.i.d. and $(Z_t^{i,0\|})_{i=1}^N$ are $\mathcal{F}^0$-conditionally i.i.d., the cross terms of the two expectations both vanish. Hence, we obtain

$$\mathbb{E}\int_0^T\Big|\frac{1}{N}\sum_{i=1}^N p_t^{i,*}\Big|^2dt \le \frac{2}{N^2}\sum_{i=1}^N\mathbb{E}\int_0^T|Z_t^{0,i}-\overline{\mathbb{E}}[Z_t^{0,1}]|^2dt+\frac{2\widehat{\gamma}^2}{N^2}\sum_{i=1}^N\mathbb{E}\Big[\Big|\frac{1}{\widehat{\gamma}}-\frac{1}{\gamma^i}\Big|^2\Big]\mathbb{E}\int_0^T|Z_t^{1,0}|^2dt$$
$$\le\frac{4}{N}\Big(1+\frac{\widehat{\gamma}^2}{\underline{\gamma}^2}\Big)\|Z^{0,1}\|^2_{\mathbb{H}^2_{\mathrm{BMO}}},$$

which gives the desired estimate. □

Remark 5.1. *Since the contraction mapping approach by Tevzadze [41][Proposition 1] used in our proof of Theorem 4.1 is applicable to multi-dimensional setups, it may be used to prove the existence of the equilibrium among the finite number of (say, N) agents, by directly solving the coupled system of qg-BSDEs* (4.4)*. However, for this purpose, we need to relax the assumption on the $\mathbb{F}^0$-adaptedness of the risk-premium process θ. Moreover, this approach requires the smallness of the* **total** *size of liabilities among the agents; $\|\sqrt{\sum_{i=1}^N|F^i|^2}\|_\infty$, and hence the constraint becomes more and more stringent as the population grows.*

6 Conclusion and discussions

In this paper, we studied a problem of equilibrium price formation among many investors with exponential utility. We allowed the agents to be heterogeneous in their initial wealth, risk-averseness parameter, as well as stochastic liability at the terminal time. We showed that the equilibrium risk-premium process of risky stocks is characterized by the solution to a novel mean-field BSDE, whose driver has quadratic growth both in the stochastic integrands and in their conditional expectations. We proved the existence of a solution to the mean-field BSDE under several conditions and showed that the resultant risk-premium process actually clears the market in the large population limit.

There are several directions for further research. We can likely to apply similar techniques to study mean-field equilibrium for other utilities. Applications to macroeconomic models in the presence of consumption in addition to the investment may also be possible. The novel mean-field BSDE (4.5) also deserves further study. The same type of BSDEs are likely to appear in similar and more general applications of the proposed method.

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