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Abstract

We consider a discrete-time robust utility maximisation with ***semistatic strategies***, and the associated indifference prices of exotic options. For this purpose, we introduce a robust form of convex integral functionals on the space of bounded continuous functions on a Polish space, and establish some key regularity and representation results, in the spirit of the classical Rockafellar theorem, in terms of the duality formed with the space of Borel measures. These results (together with the standard Fenchel duality and minimax theorems) yield a duality for the robust utility maximisation problem as well as a representation of associated indifference prices, where the presence of static positions in the primal problem appears in the dual problem as a ***marginal constraint*** on the martingale measures. Consequently, the resulting indifference prices are consistent with the observed prices of vanilla options.

Key Words: Integral functionals, semistatic strategies, robust utility, indifference valuation

1 Introduction

This paper consists of two parts. The first part is concerned with the following form of ***robust convex integral functionals***:

$$I_{\varphi,\mathcal{P}}(f) := \sup_{P\in\mathcal{P}} \int_{\Omega} \varphi(\omega, f(\omega))P(d\omega), \quad f \in C_b(\Omega), \tag{1.1}$$

where Ω is a Polish space, $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a normal convex integrand, and $\mathcal{P}$ is a convex set of Borel probability measures on Ω that is compact for the weak topology induced by $C_b(\Omega)$. This is a robust version of convex integral functionals. In the classical case with $\mathcal{P}$ being a singleton, say $\{\mathbb{P}\}$, Rockafellar [26] considered (among many others) an integral functional $I_{\varphi,\{\mathbb{P}\}} =: I_{\varphi,\mathbb{P}}$ on $L_\infty(\mathbb{P})$ and, under mild integrability assumptions on φ, found its conjugate on $L_\infty(\mathbb{P})'$ in the form:

$$I^*_{\varphi,\mathbb{P}}(\nu) = I_{\varphi^*,\mathbb{P}}(d\nu_r/d\mathbb{P}) + \sup_{\zeta\in L_\infty(\mathbb{P}), I_{\varphi,\mathbb{P}}(\zeta)<\infty} \nu_s(\zeta),$$

where $\nu = \nu_r + \nu_s$ is the (unique) Yosida-Hewitt decomposition of $\nu \in L_\infty(\mathbb{P})'$ into regular (σ-additive) part ν_r and singular (purely finitely additive) part ν_s, and

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$\varphi^*(\omega, y) = \sup_{x\in\mathbb{R}}(xy - \varphi(\omega, x))$. In particular, if $I_{\varphi,\mathbb{P}}$ is finite everywhere on $L_\infty(\mathbb{P})$, the second (singular) term is trivial, and $I_{\varphi,\mathbb{P}}$ is continuous for the Mackey topology $\tau(L_\infty(\mathbb{P}), L_1(\mathbb{P}))$. Similar representations on other decomposable spaces of measurable functions (including (σ-finite) Orlicz spaces; e.g. [19]) are found as well, and [26] and [24] obtained similar results in the non-decomposable space $C_0(X)$. For more information, see [28, Ch. 14] and [27].

In Section 2, we establish a result in the spirit of Rockafellar on the regularity and representation for the robust integral functionals of the form (1.1) in terms of the duality $\langle C_b(\Omega), \mathrm{ca}(\Omega)\rangle$, where $\mathrm{ca}(\Omega)$ is the Banach space of (finite signed) Borel measures on Ω. Specifically, under reasonable assumptions, we show that $I_{\varphi,\mathcal{P}}$ is continuous for the Mackey topology $\tau(C_b(\Omega), \mathrm{ca}(\Omega))$, and that the conjugate, on $\mathrm{ca}(\Omega)$, is a robust divergence functional associated to the conjugate of φ (Theorem 2.8). A similar robust integral functional is considered in [21], but there the set $\mathcal{P}$ is supposed to be dominated by a single probability measure $\mathbb{P}$ and the domain space is $L_\infty(\mathbb{P})$ (which is decomposable), while we do not suppose $\mathcal{P}$ is dominated, and we chose $C_b(\Omega)$ (which is not decomposable) for the domain space.

Our motivation for the robust integral functionals is to establish a duality between a robust partial hedging and valuation problem for exotic options ***consistent with the observed prices of vanilla options***. This is the content of the second part. In Section 3, we consider a discrete-time robust utility maximisation problem with semistatic strategies. Basic ingredients are the path-space $\mathbb{R}^N$, the coordinate process $(S_i)_{1\leq i\leq N} = \mathrm{id}_{\mathbb{R}^N}$ with $S_0 \equiv s_0$ (constant), and a sequence $(\mu_i)_{1\leq i\leq N}$ of distributions on $\mathbb{R}$ such that the set $\mathcal{M}_\mu$ of martingale measures Q for S with the marginal constraint

$$Q \circ S_i^{-1} = \mu_i, \quad i = 1, ..., N, \tag{1.2}$$

is non-empty. Each $Q \in \mathcal{M}_\mu$ is thought of as a calibrated pricing measure. By a ***semistatic*** strategy, we mean a pair $(H, (f_i)_{i\leq N})$ of predictable process $H = (H_i)_{i\leq N}$ and $(f_i)_{i\leq N} \in C_b(\mathbb{R})^N$, where each f_i is viewed as a vanilla option maturing at i with payoff $f_i(S_i)$, which is supposed to be priced in the market at $\mu_i(f_i) := \int_{\mathbb{R}} f_i d\mu_i$; so the gain from investing in $(H, (f_i)_{i\leq N})$ is

$$\sum\nolimits_{i\leq N} H_i(S_i - S_{i-1}) + \sum\nolimits_{i\leq N} (f_i(S_i) - \mu_i(f_i)) =: H \bullet S_N + \Gamma_{(f_i)_{i\leq N}}.$$

Then given a utility function $U : \mathbb{R} \to \mathbb{R}$, a possibly non-dominated set $\mathcal{P}$ of Borel probability measures on $\mathbb{R}^N$, and an exotic option specified by a real function Ψ on the path-space $\mathbb{R}^N$, the basic robust utility maximisation problem is:

$$u_\Psi(x) = \sup_\xi \inf_{P\in\mathcal{P}} \mathbb{E}_P\left[U\left(x + \xi - \Psi\right)\right], \quad x \in \mathbb{R}, \tag{1.3}$$

where ξ runs through (the gains from) suitable semi-static strategies. (The precise formulation will be given in Section 3.) This problem induces seller's and buyer's indifference prices of Ψ; namely

$$p_U^{\mathrm{sell}}(\Psi) := \inf\{x \in \mathbb{R} : u_\Psi(x) \geq u_0(0)\}; \quad p_U^{\mathrm{buy}}(\Psi) = -p_U^{\mathrm{sell}}(-\Psi) \leq p_U^{\mathrm{sell}}(\Psi).$$

As other indifference prices (see e.g. [17]); $p_U^{\mathrm{sell}}(\Psi)$ is to be understood as the minimal price p of Ψ such that selling Ψ at p yields a better utility than doing nothing.

By means of the regularity and representation results in Section 2, we provide, in Theorem 3.1, a dual representation of the value function $u_{\Psi}(x)$ where the dual problem is a minimisation over $\mathcal{M}_\mu$ of a certain robust divergence functional. This duality result yields a representation of the associated indifference prices of Ψ:

$$p_U^{\mathrm{sell}}(\Psi) = \sup_{Q\in\mathcal{M}_\mu} \left(\mathbb{E}_Q[\Psi] - \gamma_{V,\mathcal{P}}(Q)\right),$$

where $\gamma_{V,\mathcal{P}}$ is a certain positive convex function on the set of probability measures. In particular, $p_U^{\mathrm{buy}}(\Psi), p_U^{\mathrm{sell}}(\Psi)$ lie in the ***model-free pricing bound*** in [5]:

$$\Big[\inf_{Q\in\mathcal{M}_\mu} \mathbb{E}_Q[\Psi],\ \sup_{Q\in\mathcal{M}_\mu} \mathbb{E}_Q[\Psi]\Big], \tag{1.4}$$

so the indifference prices are, in a certain sense, fair prices consistent with calibration; see the last paragraph of this introduction.

Related literature. There is now a vast literature on robust utility maximisation (without static positions), either dominated or not; see [4] for a survey with extensive references. To the best of our knowledge, the duality for the semistatic robust utility maximisation problem (1.3) with general utility function U (on $\mathbb{R}$), and all options $f_i \in C_b(\mathbb{R})$, or (essentially) equivalently all call options available for static positions at each i, is new, where the last point appears in the dual problem as the ***full exact marginal constraint***. However, [3] (see also [13]) obtained a similar duality (with a different setup of $\mathcal{P}$) in the case of exponential utility with finitely many vanilla options, in that (in our notation) each f_i is restricted to the span of a finite number of fixed options, say $\mathrm{span}(f_{i,k}; k \le m_k)$, and accordingly, the constraint on martingale measures in the dual problem is of a weaker form $\mathbb{E}_Q[f_{i,k}] = \mu_i(f_{i,k})$ instead of (1.2). Some numerical results are also presented in [23] to non-robust semistatic exponential indifference valuation with finitely many options in illiquid market (without duality). Also, in [29], a robust exponential utility indifference valuation with a single marginal constraint at the maturity (of exotic option) is considered in a continuous time uncertain volatility framework. There a utility maximisation proof of Strassen's theorem (see (3.4) below) is also given.

Another related problem that originally inspired us is the (multi-marginal) ***martingale optimal transport (MOT)***, which is to minimise $\mathbb{E}_Q[\Psi]$ over the set $\mathcal{M}_\mu$. In this line, [5] proved that the infimum $\inf_{Q\in\mathcal{M}_\mu} \mathbb{E}_Q[\Psi]$ is attained and is, in financial terms, equal to the maximum sub-hedging cost for Ψ by the semistatic strategies (see [5, Th. 1.1] for the precise statement). A similar duality holds for $\sup_{Q\in\mathcal{M}_\mu} \mathbb{E}_Q[\Psi]$ as well with obvious changes. (See also [9, 12] for similar dualities in different setups, and [6, 7] for recent developments of (mainly 2-marginal) MOT.) These duality results give the interval in (1.4) a clear financial meaning as the ***model-free pricing bound consistent with calibration***. In particular, our indifference prices can indeed be viewed as fair prices consistent with calibration, and yield better (or not worse) bounds at the cost of small hedging error; this was the original point of view of this study though the quantitative evaluation as well as a good choice of the set $\mathcal{P}$ are left for further investigations. There are also some nonlinear generalisations of MOT ([22], [15]), typically of the form $\inf_{Q\in\mathcal{M}} \left(\mathbb{E}_Q[\Psi] + \sum_{k\le N} \rho_k(Q\circ S_k^{-1})\right)$ where ρ_k is a convex penalty function on the set of Borel probability measures on $\mathbb{R}$ and $\mathcal{M}$ is the set of all martingale measures for $(S_i)_{i\leq N}$ (without marginal constraint); the case with $\rho_k = \delta_{\{\mu_k\}}$ is the MOT.

2 Robust Convex Integral Functionals

2.1 Preliminaries

In this paper, all the vector spaces are real, and when X is a Banach space, $\mathbb{B}_X$ denotes its closed unit ball.

In the sequel, Ω is a ***Polish space*** with the Borel σ-field $\mathcal{B}(\Omega)$, and $C_b(\Omega)$ is the Banach space of bounded continuous function with $\|f\|_\infty = \sup_{\omega\in\Omega}|f(\omega)|$, while $\mathrm{ca}(\Omega)$ is the Banach space of (finite signed) Borel measures on Ω with $\|\nu\| = |\nu|(\Omega)$. Also, $L_p(\mu) := L_p(\Omega, \mathcal{B}(\Omega), \mu)$ for positive $\mu \in \mathrm{ca}(\Omega)$, and $\mathrm{Prob}(\Omega)$ denotes the closed convex subset of $\mathrm{ca}(\Omega)$ consisting of probability measures (i.e. those $\mu \in \mathrm{ca}(\Omega)$ with $\mu \geq 0$ and $\mu(\Omega) = 1$). For $\mu \in \mathrm{Prob}(\Omega)$, we write $\mathbb{E}_\mu[f]$ for $\int_\Omega f d\mu$. Next, if $\langle X, X'\rangle$ is a (separated) dual system, the weak topology $\sigma(X, X')$ (resp. the Mackey topology $\tau(X, X')$) is the weakest (resp. finest) locally convex topology on X consistent with the duality $\langle X, X'\rangle$, i.e. making X' the dual of X. More concretely, $\sigma(X, X')$ is the topology of pointwise convergence on X' while $\tau(X, X')$ is the topology of uniform convergence on $\sigma(X', X)$-compact absolutely convex subsets of X'; see [1, Sec.5.14-18] or [16, Ch. 2, Sec. 8-13] for details.

The duality $\langle C_b(\Omega), \mathrm{ca}(\Omega)\rangle$. The space $\mathrm{ca}(\Omega)$ is (isometrically isomorphic to) a closed subspace of the dual $C_b(\Omega)'$ (proper unless Ω is compact) via $\langle f, \mu\rangle = \int_\Omega f d\mu =: \mu(f)$, which makes $\langle C_b(\Omega), \mathrm{ca}(\Omega)\rangle$ a (separated) dual system. In the sequel, we are basically interested in the duality $\langle C_b(\Omega), \mathrm{ca}(\Omega)\rangle$, but the proof of Theorem 2.8 below also involves the duality $\langle C_b(\Omega), C_b(\Omega)'\rangle$. The dual $C_b(\Omega)'$ is identified as the space of (finite signed) regular Borel measures on the Stone-Čech compactification $\beta\Omega$, and Ω is (homeomorphic to) a dense G_δ, hence Borel, subset of $\beta\Omega$. Then $\mathrm{ca}(\Omega)$ is regarded as the subspace of $C_b(\Omega)'$ consisting of those measures supported by (the image in $\beta\Omega$ of) Ω. The next lemma describes $\mathrm{ca}(\Omega)$ in $C_b(\Omega)'$ sorely in terms of Ω (without passing to $\beta\Omega$).

Lemma 2.1 ([10], Prop. 5 on p. IX.59 or [8], Th. 7.10.6). *An $F \in C_b(\Omega)'$ lies in $\mathrm{ca}(\Omega)$, i.e. $F(f) = \int_\Omega f d\mu$ for some $\mu \in \mathrm{ca}(\Omega)$ iff for any $\varepsilon > 0$, there exists a compact set $K \subset \Omega$ such that $|F(g)| \leq \varepsilon$ whenever $g \in \mathbb{B}_{C_b(\Omega)}$ and $g = 0$ on K.*

The following version of Prokhorov's theorem characterises the $\sigma(\mathrm{ca}(\Omega), C_b(\Omega))$-compact sets; here the sufficiency follows from Lemma 2.1 and $\sigma(\mathrm{ca}(\Omega), C_b(\Omega)) = \sigma(C_b(\Omega)', C_b(\Omega))|_{\mathrm{ca}(\Omega)}$ (so $\sigma(\mathrm{ca}(\Omega), C_b(\Omega))$-bounded $\Leftrightarrow$ weak* bounded in $C_b(\Omega)'$ $\Leftrightarrow$ norm bounded), while the necessity is by "gliding hump".

Lemma 2.2 (Prokhorov's theorem; [8], Th. 8.6.7 and 8.6.8). *(If Ω is Polish,) a set $\Lambda \subset \mathrm{ca}(\Omega)$ is relatively $\sigma(\mathrm{ca}(\Omega), C_b(\Omega))$-compact iff it is bounded in (total variation) norm and uniformly tight, i.e. for any $\varepsilon > 0$, there exists a compact set $K \subset \Omega$ such that $\sup_{\nu\in\Lambda}|\nu|(\Omega \setminus K) < \varepsilon$.*

Remark 2.3. The restriction of $\sigma(\mathrm{ca}(\Omega), C_b(\Omega))$ to the set of probability measures is precisely what probabilists call the weak topology, and Prokhorov's theorem is usually stated for probability measures, but we will need it on the whole $\mathrm{ca}(\Omega)$. ♦

Generalities on convex functions. Given a duality $\langle X, X'\rangle$, a convex function $F : X \to \mathbb{R} \cup \{+\infty\}$ is called proper if $\mathrm{dom}(F) := \{x \in X : F(x) < \infty\} \neq \emptyset$. By the Hahn-Banach theorem, such an F is lower semicontinuous (lsc) for $\sigma(X, X')$, equivalently for $\tau(X, X')$, iff F has the Fenchel-Moreau dual representation by X':

$$F(x) = \sup_{x' \in X'} \left(\langle x, x' \rangle - F^*(x') \right), \quad x \in X,$$

where $F^*(x') := \sup_{x \in X}(\langle x, x'\rangle - F(x))$, $x' \in X'$, the conjugate of F. Also,

Lemma 2.4 (Moreau-Rockafellar theorem; e.g. [20]). *Let $\langle X, X'\rangle$ be a dual pair. A finite lsc convex function $F : X \to \mathbb{R}$ is $\tau(X, X')$-continuous at 0 (then on the whole X) iff F^* has $\sigma(X', X)$-compact sublevel sets, i.e. $\{x' \in X' : F^*(x') \leq c\}$, $c \in \mathbb{R}$, are $\sigma(X', X)$-compact. In this case, $F(x) = \max_{x' \in X'} (\langle x, x'\rangle - F^*(x'))$, $\forall x \in X$.*

For the last part, note that $x' \mapsto F^*(x') - \langle x, x'\rangle$ is the conjugate of $y \mapsto F(x + y)$ which is $\tau(X, X')$-continuous if F is, so $\{x' \in X' : \langle x, x'\rangle - F^*(x') \geq c\}$, $c \in \mathbb{R}$, are $\sigma(X', X)$-compact while $\langle x, \cdot\rangle - F^*$ is upper semicontinuous for the same topology.

2.2 *The functionals*

Let $\mathcal{P}$ be a set of Borel probability measures on Ω that we suppose

(2.1) $\qquad \mathcal{P}$ is convex and $\sigma(\mathrm{ca}(\Omega), C_b(\Omega))$-compact,

but we ***do not*** suppose that it is dominated, i.e. that there is a single probability measure $\mathbb{P}$ such that $P \ll \mathbb{P}$ for all $P \in \mathcal{P}$, so it does not embed in an L_1-space, and the common uniform integrability arguments do not work.

Let $\mathbb{L}_0(\mathcal{P})$ be the vector space of (equivalence classes modulo equality "P-a.s. for all $P \in \mathcal{P}$" of) real valued Borel functions on Ω, and define

$$\mathbb{L}_1(\mathcal{P}) := \left\{ \xi \in \mathbb{L}_0(\mathcal{P}) : \|\xi\|_{1,\mathcal{P}} := \sup_{P \in \mathcal{P}} \mathbb{E}_P[|\xi|] < \infty \right\},$$
$$\mathbb{L}_{1,b}(\mathcal{P}) := \left\{ \xi \in \mathbb{L}_1(\mathcal{P}) : \lim_n \|\xi \mathbb{1}_{\{|\xi|>n\}}\|_{1,\mathcal{P}} = 0 \right\}.$$

It is known (see [14]), and easily verified, that $\mathbb{L}_1(\mathcal{P})$ is a Banach space and $\mathbb{L}_{1,b}(\mathcal{P})$ is its closed subspace. In this paper, we just use these spaces to simplify the notation. Also, as usual, we do not differentiate a class $\xi \in \mathbb{L}_0(\mathcal{P})$ and its representatives $f \in \xi$; so we regard $C_b(\Omega)$ as a subspace of $\mathbb{L}_0(\mathcal{P})$ consisting of those ξ admitting a bounded continuous representative.

Lemma 2.5 ([14], Prop. 19). *If $\eta \in \mathbb{L}_{1,b}(\mathcal{P})$, then any $\varepsilon > 0$ admits a $\delta > 0$ such that for any $A \in \mathcal{B}(\Omega)$ with $\sup_{P \in \mathcal{P}}(A) \leq \delta$, one has $\sup_{P \in \mathcal{P}} \mathbb{E}_P[|\eta| \mathbb{1}_A] \leq \varepsilon$. In particular (under (2.1)), if $\eta \in \mathbb{L}_{1,b}(\mathcal{P})$, we have*

$$\forall \varepsilon > 0,\ \exists \textit{a compact } K \subset \Omega \textit{ such that } \sup_{P \in \mathcal{P}} \mathbb{E}_P[|\eta| \mathbb{1}_{K^c}] \leq \varepsilon.$$

The next ingredient is a random function $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ which we suppose:

Assumption 2.6. *$\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ is such that*

(2.2) $\quad \forall \omega \in \Omega,\ x \mapsto \varphi(\omega, x)$ *is an (everywhere finite) convex function*;

(2.3) $\quad \forall x \in \mathbb{R},\ \omega \mapsto \varphi(\omega, x)$ *is upper semicontinuous (usc)*;

(2.4) $\quad \forall x \in \mathbb{R},\ \varphi(\cdot, x)^+ \in \mathbb{L}_{1,b}(\mathcal{P})$ *i.e.* $\lim_n \sup_{P \in \mathcal{P}} \mathbb{E}_P[\varphi(\cdot, x)\mathbb{1}_{\{\varphi(\cdot,x)\geq n\}}] = 0.$

(2.5) $\quad \varphi(\cdot, 0)^- \in \mathbb{L}_1(\mathcal{P}).$

Several remarks on assumptions are in order. First, by (2.2), $x \mapsto \varphi(\omega, x)$, $\omega \in \Omega$, are continuous (being a finite valued convex function on $\mathbb{R}$), while by (2.3), $\omega \mapsto \varphi(\omega, x)$, $x \in \mathbb{R}$, are Borel; hence in the terminology of convex analysis (see e.g. [28]), φ is a ***convex Carathéodory integrand***, a fortiori it is a (finite-valued) ***normal convex integrand***, i.e. the *epigraphical mapping*

$$\omega \mapsto \{(x, \alpha) \in \mathbb{R} \times \mathbb{R} : \varphi(\omega, x) \leq \alpha\}$$

is a closed-valued measurable multifunction. Then the (partial) conjugate

$$\varphi^*(\omega, y) := \sup_{x \in \mathbb{R}}(xy - f(\omega, x)), \quad \forall y \in \mathbb{R},$$

is also a proper normal convex integrand, and Young's inequality holds:

$$xy \leq \varphi(\omega, x) + \varphi^*(\omega, y),\ \forall \omega \in \Omega,\ x, y \in \mathbb{R}. \tag{2.6}$$

The normality of φ implies that it is jointly measurable, so $\omega \mapsto \varphi(\omega, f(\omega))$ is measurable whenever $f : \Omega \to \mathbb{R}$ is. Then (2.3) implies (hence is equivalent to):

$$\forall f \in C_b(\Omega),\ \omega \mapsto \varphi(\omega, f(\omega)) \text{ is usc.} \tag{2.3$'$}$$

For if $f \in C_b(\Omega)$ and $\omega_n \to \omega$ in Ω, then $K = \{\omega, \omega_n; n \geq 1\}$ is compact, so by the usc, $(\varphi(\omega', \cdot))_{\omega' \in K}$ is a pointwise bounded family of convex functions; hence [25, Th. 10.6] gives us a constant $c > 0$ such that $\sup_{\omega \in K} |\varphi(\omega, x) - \varphi(\omega, y)| \leq c|x - y|$ whenever $x, y \in [-\|f\|_\infty, \|f\|_\infty]$. Consequently,

$$\begin{aligned}&\limsup_n \{\varphi(\omega_n, f(\omega_n)) - \varphi(\omega, f(\omega))\} \\ &\qquad \leq c \limsup_n |f(\omega_n) - f(\omega)| + \limsup_n \varphi(\omega_n, f(\omega)) - \varphi(\omega, f(\omega)) \leq 0.\end{aligned}$$

Similarly, (2.4) already implies

$$\forall f \in C_b(\Omega),\ \varphi(\cdot, f)^+ \in \mathbb{L}_{1,b}(\mathcal{P}). \tag{2.4$'$}$$

Indeed, by convexity, $\varphi(\cdot, f) \leq \varphi(\cdot, -\|f\|_\infty)^+ + \varphi(\cdot, \|f\|_\infty)^+ \in \mathbb{L}_{1,b}(\mathcal{P})$.

Finally, given (2.2) and (2.4), (2.5) is equivalent to:

$$\exists \eta \in \mathbb{L}_1(\mathcal{P}) \text{ such that } \varphi^*(\cdot, \eta)^+ \in \mathbb{L}_1(\mathcal{P}). \tag{2.7}$$

Indeed, by the normality of φ^* and $\varphi(\cdot,0) = \sup_y(-\varphi^*(\cdot,y))$, $\Xi(\omega) := \{y \in \mathbb{R} : \varphi(\omega,0) \leq 1 - \varphi^*(\omega,y)\}$ is a nonempty closed valued measurable multifunction. Thus Kuratowski-Ryll-Nardzewski's measurable selection theorem yields a measurable function $f : \Omega \to \mathbb{R}$ such that $f(\omega) \in \Xi(\omega)$, i.e. $\varphi(\omega,0) \leq 1 - \varphi^*(\omega, f(\omega))$. Thus $\varphi^*(\cdot, f) \leq 1 - \varphi(\cdot,0)^- \in \mathbb{L}_1(\mathcal{P})$, and $|f| = f\mathrm{sgn}(f) \leq \varphi(\cdot,-1)^+ + \varphi(\cdot,1)^+ + \varphi^*(\cdot,f)^+ \in \mathbb{L}_1(\mathcal{P})$.

Now we define

$$I_{\varphi,\mathcal{P}}(f) := \sup_{P\in\mathcal{P}} \mathbb{E}_P[\varphi(\cdot,f)] = \sup_{P\in\mathcal{P}} \int_\Omega \varphi(\omega, f(\omega))P(d\omega), \quad f \in C_b(\Omega). \tag{2.8}$$

We check that this is well-defined under (2.1) and Assumption 2.6.

Lemma 2.7. *Under (2.1) and Assumption 2.6, $I_{\varphi,\mathcal{P}}$ is well-defined as a finite-valued convex function on $C_b(\Omega)$, and it is lower semicontinuous for the topology of pointwise convergence on bounded sets:*

$$\sup_n \|f_n\|_\infty < \infty,\ f_n \to f \text{ pointwise} \ \Rightarrow\ I_{\varphi,\mathcal{P}}(f) \leq \liminf_n I_{\varphi,\mathcal{P}}(f_n). \tag{2.9}$$

In particular, $I_{\varphi,\mathcal{P}}$ is norm-lsc, hence norm continuous (being finite-valued) on $C_b(\Omega)$.

Proof. Picking an $\eta \in \mathbb{L}_1(\mathcal{P})$ as in (2.7),

$$\forall f \in C_b(\Omega),\ \varphi(\cdot,f) \geq -\|f\|_\infty|\eta| - \varphi^*(\cdot,\eta)^+ \in \mathbb{L}_1 \subset \textstyle\bigcap_{P\in\mathcal{P}} L_1(P).$$

Thus for each $P \in \mathcal{P}$, $f \mapsto \mathbb{E}_P[\varphi(\cdot,f)]$ is well-defined as a convex function on $C_b(\Omega)$, and it is lsc on bounded sets for the pointwise convergence by Fatou's lemma, hence so is their pointwise supremum $I_{\varphi,\mathcal{P}}(f) = \sup_{P\in\mathcal{P}} \mathbb{E}_P[\varphi(\cdot,f)]$. Then (2.4′) guarantees that $I_{\varphi,\mathcal{P}}(f) < \infty$ for all $f \in C_b(\Omega)$. $\square$

We next define the φ^*-divergence functional by

$$J_{\varphi^*}(\nu|P) := \begin{cases} \mathbb{E}_P[\varphi^*(\cdot, d\nu/dP)] & \text{if } \nu \ll P, \\ +\infty & \text{otherwise,} \end{cases} \qquad \forall \nu \in \mathrm{ca}(\Omega),\ P \in \mathcal{P}. \tag{2.10}$$

This is a jointly convex function with values in $(-\infty,\infty]$. To see this, let

$$\tilde{\varphi}^*(\cdot,y,z) := \sup_x (xy - z\varphi(\cdot,x)) = z\varphi^*\left(\cdot, \frac{y}{z}\right)\mathbb{1}_{\{z>0\}} + \infty\mathbb{1}_{\{y\neq 0, z=0\}} \geq -z\varphi(\cdot,0)^+.$$

This $\tilde{\varphi}^*$ is convex in $(y,z) \in \mathbb{R}\times\mathbb{R}^+$ and $J_V(\nu|P) = \mathbb{E}_{\mathbb{P}}[\tilde{\varphi}^*(\cdot, \frac{d\nu}{d\mathbb{P}}, \frac{dP}{d\mathbb{P}})] > -\infty$ (by (2.4)) whenever $\nu, P \ll \mathbb{P} \in \mathrm{Prob}(\Omega)$. Since any finite number of finite signed measures are dominated by a single probability, J_V is jointly convex on $\mathrm{ca}(\Omega)\times\mathcal{P}$, and $J_V(\nu|P) < \infty$ if $d\nu/dP = \eta \in \mathbb{L}_1(\mathcal{P})$ as in (2.7). Since $\mathcal{P}$ is convex,

$$J_{\varphi^*,\mathcal{P}}(\nu) := \inf_{P\in\mathcal{P}} J_{\varphi^*}(\nu|P), \quad \forall \nu \in \mathrm{ca}, \tag{2.11}$$

called ***the robust φ^*-divergence functional***, is convex on $\mathrm{ca}(\Omega)$ and not identically $+\infty$. Further, (2.6) yields $\nu(f) \leq \mathbb{E}_P[\varphi(\cdot,f)] + J_{\varphi^*}(\nu|P) \leq I_{\varphi,\mathcal{P}}(f) + J_{\varphi^*}(\nu|P)$ for any $f \in C_b(\Omega)$, $\nu \in \mathrm{ca}(\Omega)$, $P \in \mathcal{P}$, and taking the infimum over $P \in \mathcal{P}$,

$$\nu(f) \leq I_{\varphi,\mathcal{P}}(f) + J_{\varphi^*,\mathcal{P}}(\nu), \quad \forall f \in C_b(\Omega),\ \nu \in \mathrm{ca}(\Omega). \tag{2.12}$$

In particular, $J_{\varphi^*,\mathcal{P}}(\nu) \geq -I_{\varphi,\mathcal{P}}(0) > -\infty$, so $J_{\varphi^*,\mathcal{P}}$ is proper.

2.3 A Duality Result

Now the main result of this section is the following:

Theorem 2.8 (Rockafellar-Type Duality). *Suppose (2.1), and Assumption 2.6. Then* $I_{\varphi,\mathcal{P}} : C_b(\Omega) \to \mathbb{R}$ *is* $\tau(C_b(\Omega), \mathrm{ca}(\Omega))$*-continuous, and its conjugate is given by*

$$I^*_{\varphi,\mathcal{P}}(\nu) := \sup_{f \in C_b(\Omega)} \Big(\nu(f) - I_{\varphi,\mathcal{P}}(f)\Big) = J_{\varphi^*,\mathcal{P}}(\nu), \quad \nu \in \mathrm{ca}(\Omega). \tag{2.13}$$

In particular, it holds that

$$I_{\varphi,\mathcal{P}}(f) = \max_{\nu \in \mathrm{ca}(\Omega)} \Big(\nu(f) - J_{\varphi^*,\mathcal{P}}(\nu)\Big), \quad f \in C_b(\Omega). \tag{2.14}$$

In view of the Moreau-Rockafellar theorem (Lemma 2.4), the $\tau(C_b(\Omega), \mathrm{ca}(\Omega))$-continuity is equivalent to (1) $I_{\varphi,\mathcal{P}}$ is $\sigma(C_b(\Omega), \mathrm{ca}(\Omega))$-lsc, and (2) the conjugate $I^*_{\varphi,\mathcal{P}}$ has $\sigma(\mathrm{ca}(\Omega), C_b(\Omega))$-compact sublevels, i.e.

$$\forall c \in \mathbb{R},\ \Lambda_c := \{\nu \in \mathrm{ca}(\Omega) : I^*_{\varphi,\mathcal{P}}(\nu) \le c\} \text{ is } \sigma(\mathrm{ca}(\Omega), C_b(\Omega))\text{-compact}. \tag{2.15}$$

Regarding (1), we already know from Lemma 2.7 that $I_{\varphi,\mathcal{P}}$ is ***sequentially*** lsc for $\sigma(C_b(\Omega), \mathrm{ca}(\Omega))$ (the $\sigma(C_b(\Omega), \mathrm{ca}(\Omega))$-convergence implies the pointwise convergence as $\mathrm{ca}(\Omega)$ contains the point masses), but it need not imply the full lower semicontinuity (i.e. for nets, not only for sequences). Indeed, unlike $L_\infty(\mathbb{P})$ which is a common choice of the domain space in the dominated case, $C_b(\Omega)$ is neither a dual (unless $\beta\Omega$ is extremally disconnected) nor a predual space (unless Ω is compact). Thus the common techniques using Krein-Šmulian theorem as well as a probabilistic description of the Mackey topology on bounded sets (due to Grothendieck) are not available.

Remark 2.9 (Infinite φ?). Many of existing results on (classical) convex integral functionals are stated for possibly infinite proper integrand φ, where an additional singular term appears in the the conjugate as (1.1). In the robust case, however, our previous work [21] suggests that this type of "exact" representation does not generally hold when φ is infinite (even if $\mathcal{P}$ is dominated), and the current situation is even more complicated due to the non-decomposability of $C_b(\Omega)$. Anyway, with infinite φ, we can no longer hope for the $\tau(C_b(\Omega), \mathrm{ca}(\Omega))$-continuity of the integral functional, and we are forced to work with the duality $\langle C_b(\Omega), C_b(\Omega)'\rangle$ which is not what we want in view of financial application. We thus do not seek this direction in this paper. ♦

Given the $\tau(C_b(\Omega), \mathrm{ca}(\Omega))$-continuity (on the whole $C_b(\Omega)$), the Fenchel duality theorem (see e.g. [31], Th.7.15 with $g = \delta_{\mathcal{C}}$) yields that

Corollary 2.10 (Meta duality). *Under the assumptions of Theorem 2.8, it holds for any nonempty convex set* $\mathcal{C} \subset C_b(\Omega)$ *that*

$$\inf_{f \in \mathcal{C}} I_{\varphi,\mathcal{P}}(f) = -\min_{\nu \in \mathrm{ca}(\Omega)} \Big(J_{\varphi^*,\mathcal{P}}(-\nu) + \sup_{g \in \mathcal{C}} \nu(g)\Big). \tag{2.16}$$

If in addition $\mathcal{C}$ *is a convex cone, the RHS is equal to* $-\min_{\nu \in \mathcal{C}^\circ} J_{\varphi^*,\mathcal{P}}(-\nu)$*, where* $\mathcal{C}^\circ$ *is the one-sided polar of* $\mathcal{C}$ *in* $\langle C_b(\Omega), \mathrm{ca}(\Omega)\rangle$*, i.e.*

$$\mathcal{C}^\circ = \{\nu \in \mathrm{ca}(\Omega) : \nu(g) \le 1,\ \forall g \in \mathcal{C}\}.$$

A typical and motivating example of normal integrand φ is of the following type:

Proposition 2.11. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a (deterministic finite-valued) convex function, and $B : \Omega \to \mathbb{R}$ be a usc function such that*

$$\varphi((1+\varepsilon)B)^+ \in \mathbb{L}_{1,b}(\mathcal{P}) \text{ and } \varphi(-\varepsilon B)^+ \in \mathbb{L}_1(\mathcal{P}) \text{ for some } \varepsilon > 0. \tag{2.17}$$

Then $\varphi_B(\omega, x) := \varphi(x + B(\omega))$ satisfies Assumption 2.6 with the conjugate

$$\varphi_B^*(\cdot, y) = \sup_x (xy - \varphi(x+B)) = \varphi^*(y) - yB.$$

Thus under (2.1), the functional $I_{\varphi_B,\mathcal{P}}$ is $\tau(C_b(\Omega), \mathrm{ca}(\Omega))$-continuous on $C_b(\Omega)$ with conjugate $J_{\varphi_B^*,\mathcal{P}}$. Further, $J_{\varphi^*,\mathcal{P}}(\nu) < \infty \Leftrightarrow J_{\varphi_B^*,\mathcal{P}}(\nu) < \infty$, and $B \in L_1(\nu)$ whenever $\in_{\lambda>0} J_{V,\mathcal{P}}(\lambda\nu) < \infty$. In particular,

$$J_{\varphi_B^*,\mathcal{P}}(\nu) = \begin{cases} J_{\varphi^*,\mathcal{P}}(\nu) - \nu(B) & \text{if } \inf_{\lambda>0} J_{\varphi^*,\mathcal{P}}(\lambda\nu) < \infty, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.18}$$

Proof. The deterministic convex function φ clearly satisfies Assumption 2.6, and φ_B is usc in ω since B is. Thus φ_B is a Carathéodory integrand. Denoting $\rho_\alpha(x) = \frac{1}{\alpha}\varphi(\alpha x)^+$, $\alpha > 0$, the convexity of φ yields

$$\frac{1+\varepsilon}{\varepsilon}\varphi\left(\frac{\varepsilon}{1+\varepsilon}x\right) - \rho_\varepsilon(-B) \le \varphi_B(x) \le \frac{\varepsilon}{1+\varepsilon}\varphi\left(\frac{1+\varepsilon}{\varepsilon}x\right) + \rho_{1+\varepsilon}(B). \tag{2.19}$$

Thus (2.17) shows that φ_B satisfies (2.4) and (2.5). Further, taking the conjugate, $\varphi_B^*(\cdot, y) = \sup_x (xy - \varphi(x+B)) = \varphi^*(y) - yB$, and

$$\frac{\varepsilon}{1+\varepsilon}\varphi^*(y) - \rho_{1+\varepsilon}(B) \le \varphi_B^*(y) \le \frac{1+\varepsilon}{\varepsilon}\varphi^*(y) + \rho_\varepsilon(-B). \tag{2.20}$$

This shows that $J_{\varphi_B^*,\mathcal{P}}(\nu) < \infty \Leftrightarrow J_{\varphi^*,\mathcal{P}}(\nu) < \infty \Rightarrow B \in L_1(\nu)$. Then noting that $B \in L_1(\nu)$ iff $B \in L_1(\lambda\nu)$ for some (then any) $\lambda > 0$, we see that $B \in L_1(\nu)$ whenever $\inf_{\lambda>0} J_{V,\mathcal{P}}(\lambda\nu) < \infty$; in particular, (2.18) holds. □

2.4 *Proof of Theorem 2.8*

Though Theorem 2.8 is stated entirely in terms of the duality $\langle C_b(\Omega), \mathrm{ca}(\Omega)\rangle$ (with $\mathrm{ca}(\Omega)$ rather than $C_b(\Omega)'$), the proof relies on the duality $\langle C_b(\Omega), C_b(\Omega)'\rangle$. By Lemma 2.7, $I_{\varphi,\mathcal{P}}$ is norm continuous. Since the norm topology is the Mackey topology $\tau(C_b(\Omega), C_b(\Omega)')$, Moreau-Rockafellar's theorem (Lemma 2.4) tells us that

$$I_{\varphi,\mathcal{P}}(f) = \sup_{\nu \in C_b(\Omega)'} \left(\nu(f) - I_{\varphi,\mathcal{P}}^*(\nu)\right), \quad f \in C_b(\Omega), \tag{2.21}$$

where the conjugate $I_{\varphi,\mathcal{P}}^*(\nu) = \sup_{f \in C_b(\Omega)}(\nu(f) - I_{\varphi,\mathcal{P}}(f))$ is now considered on $C_b(\Omega)'$, and $\{\nu \in C_b(\Omega)' : I_{\varphi,\mathcal{P}}(\nu) \le c\}$, $c \in \mathbb{R}$, are $\sigma(C_b(\Omega)', C_b(\Omega))$-compact ($\Leftrightarrow$ closed and bounded in norm). All we need to get the $\tau(C_b(\Omega), \mathrm{ca}(\Omega))$-continuity of $I_{\varphi,\mathcal{P}}$ is to replace the dual $C_b(\Omega)'$ by $\mathrm{ca}(\Omega)$.

Proof of Theorem 2.8: Mackey continuity. We first claim that

$$I^*_{\varphi,\mathcal{P}}(\nu) = \infty \text{ if } \nu \in C_b(\Omega)' \setminus \mathrm{ca}(\Omega). \tag{2.22}$$

To see this, note first that for any $\nu \in C_b(\Omega)'$,

$$\sup_{f\in C_b(\Omega)} \Big(\nu(f) - I_{\varphi,\mathcal{P}}(f)\Big) \geq \sup_n \sup_{g\in \mathbb{B}_{C_b(\Omega)}} \Big(\nu(ng) - I_{\varphi,\mathcal{P}}(ng)\Big), \tag{2.23}$$

while for any $g \in \mathbb{B}_{C_b(\Omega)}$, $\varphi(\cdot, ng)^+ \leq \varphi(\cdot, -n)^+ + \varphi(\cdot, n)^+ =: \beta_n \in \mathbb{L}_{1,b}(\mathcal{P})$ by convexity and (2.4). Thus Lemma 2.5 yields compact sets $K_n \subset \Omega$, $n \geq 1$, with

$$\sup_{g\in\mathbb{B}_{C_b(\Omega)}} \sup_{P\in\mathcal{P}} \mathbb{E}_P[\varphi(\cdot, ng)^+ \mathbb{1}_{K_n^c}] \leq \sup_{P\in\mathcal{P}} \mathbb{E}_P[\beta_n \mathbb{1}_{K_n^c}] \leq 1.$$

On the other hand, if $\nu \in C_b(\Omega)' \setminus C_b(\Omega)$, Lemma 2.1 gives an $\varepsilon > 0$ and a sequence $g_n \in \mathbb{B}_{C_b(\Omega)}$ such that $g_n \mathbb{1}_{K_n} = 0$ and $\nu(g_n) > \varepsilon$ (since $|\nu(g_n)| = \nu(g_n) \vee \nu(-g_n)$); hence

$$\nu(ng_n) - I_{\varphi,\mathcal{P}}(ng_n) \geq n\varepsilon - \underbrace{\sup_{P\in\mathcal{P}} \mathbb{E}_P[\varphi(\cdot, ng_n)^+ \mathbb{1}_{K_n^c}]}_{\leq 1} - \underbrace{\sup_{P\in\mathcal{P}} \mathbb{E}_P[\varphi(\cdot, 0)^+]}_{<\infty}.$$

Combined with (2.23), we deduce $\sup_{f\in C_b(\Omega)}(\nu(f) - I_{\varphi,\mathcal{P}}(f)) = \infty$.

Now by (2.22) and (2.21), we have

$$I_{\varphi,\mathcal{P}}(f) \overset{(2.21)}{=} \sup_{\nu\in C_b(\Omega)'} \Big(\nu(f) - I^*_{\varphi,\mathcal{P}}(\nu)\Big) \overset{(2.22)}{=} \sup_{\nu\in\mathrm{ca}(\Omega)} \Big(\nu(f) - I^*_{\varphi,\mathcal{P}}(\nu)\Big).$$

Thus $I_{\varphi,\mathcal{P}}$ is $\sigma(C_b(\Omega), \mathrm{ca}(\Omega))$-lsc. (2.22) shows also that the sublevel set Λ_c in $\mathrm{ca}(\Omega)$ coincides with that considered in $C_b(\Omega)'$, i.e.

$$\Lambda_c = \Big\{\nu \in \mathrm{ca}(\Omega) : I^*_{\varphi,\mathcal{P}}(\nu) \leq c\Big\} = \Big\{\nu \in C_b(\Omega)' : I^*_{\varphi,\mathcal{P}}(\nu) \leq c\Big\}.$$

As noted above (see the comment following (2.21)), the last set is $\sigma(C_b(\Omega)', C_b(\Omega))$-compact. Consequently, Λ_c is a $\sigma(C_b(\Omega)', C_b(\Omega))$-compact subset of $C_b(\Omega)'$ lying in $\mathrm{ca}(\Omega)$, so it is $\sigma(C_b(\Omega)', C_b(\Omega))|_{\mathrm{ca}(\Omega)} = \sigma(\mathrm{ca}(\Omega), C_b(\Omega))$-compact. Now being $\sigma(C_b(\Omega), \mathrm{ca}(\Omega))$-lsc with the conjugate having $\sigma(\mathrm{ca}(\Omega), C_b(\Omega))$-compact sublevels, Lemma 2.4 shows that $I_{\varphi,\mathcal{P}}$ is $\tau(C_b(\Omega), \mathrm{ca}(\Omega))$-continuous. □

We proceed to the conjugate formula (2.13). We derive it from the classical Rockafellar theorem on $L_\infty(\mathbb{P})$ and a minimax argument. The latter needs the following simple lemma.

Lemma 2.12. *Suppose (2.1)–(2.5). Then for each $f \in C_b(\Omega)$, $P \mapsto \mathbb{E}_P[\varphi(\cdot, f)]$ is (affine, hence concave and) $\sigma(\mathrm{ca}(\Omega), C_b(\Omega))$-usc on $\mathcal{P}$.*

Proof. Let $g := \varphi(\cdot, f)$ which is usc with $g^+ \in \bigcap_{P\in\mathcal{P}} L_1(P)$, and $g_m := g \vee (-m)$. Since $\mathbb{E}_P[g] = \inf_m \mathbb{E}_P[g_m]$, it suffices that $P \mapsto \mathbb{E}_P[g_m]$, $m \geq 1$, are usc. For each n, $g_m \wedge n$ is a bounded usc function, so $P \mapsto \mathbb{E}_P[g_m \wedge n]$ is $\sigma(\mathrm{ca}(\Omega), C_b(\Omega))$-usc on $\mathcal{P} \subset \mathrm{Prob}(\Omega)$ (see e.g. [1, Th. 15.5]). Then note that $g_m - g_m \wedge n \leq g_m \mathbb{1}_{\{g_m>n\}} =$

$g^+ \mathbb{1}_{\{g>n\}}$, so $\lim_n \sup_{P\in\mathcal{P}} \mathbb{E}_P\left[g_m - g_m \wedge n\right] = 0$ by (2.4′) ($\Leftarrow$ (2.4)). Thus if $P_k \to P$ in $(\mathcal{P}, \sigma(\mathrm{ca}(\Omega), C_b(\Omega))$, one has

$$\limsup_k \mathbb{E}_{P_k}[g_m] \le \sup_{P\in\mathcal{P}} \mathbb{E}_P[g_m - g_m \wedge n] + \underbrace{\limsup_k \mathbb{E}_{P_k}[g_m \wedge n]}_{\le \mathbb{E}_P[g_m\wedge n]\le \mathbb{E}_P[g_m]}.$$

Letting $n \to \infty$, we get $\limsup_k \mathbb{E}_{P_k}[g_m] \le \mathbb{E}_P[g_m]$. Since $\sigma(\mathrm{ca}(\Omega), C_b(\Omega))$ is metrisable on $\mathcal{P} \subset \mathrm{Prob}(\Omega)$, this proves the claim. □

Proof of Theorem 2.8: the conjugate formula (2.13). Given $\nu \in \mathrm{ca}(\Omega)$, the function $(f, P) \mapsto \nu(f) - \mathbb{E}_P[\varphi(\cdot, f)]$ on $C_b(\Omega)\times\mathcal{P}$, is concave in $f \in C_b(\Omega)$, and $\sigma(\mathrm{ca}(\Omega), C_b(\Omega))$-lsc and convex in $P \in \mathcal{P}$ by Lemma 2.12. Since the set $\mathcal{P}$ is $\sigma(\mathrm{ca}(\Omega), C_b(\Omega))$-compact, the (usual) minimax theorem yields that

$$\begin{aligned}\sup_{f\in C_b(\Omega)} \left(\nu(f) - I_{\varphi,\mathcal{P}}(f)\right) &= \sup_{f\in C_b(\Omega)} \inf_{P\in\mathcal{P}} (\nu(f) - \mathbb{E}_P[\varphi(\cdot, f)]) \\ &= \inf_{P\in\mathcal{P}} \sup_{f\in C_b(\Omega)} (\nu(f) - \mathbb{E}_P[\varphi(\cdot, f)]).\end{aligned}$$

Therefore it suffices to show that

$$\forall \nu \in \mathrm{ca}(\Omega),\ \forall P \in \mathcal{P},\ \sup_{f\in C_b(\Omega)} (\nu(f) - \mathbb{E}_P[\varphi(\cdot, f)]) = J_\varphi(\nu|P). \tag{2.24}$$

So fix $\nu \in \mathrm{ca}(\Omega)$, $P \in \mathcal{P}$, and pick a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{B}(\Omega))$ with $\nu, P \ll \mathbb{P}$ (e.g. $\mathbb{P} = \frac{1}{2}(|\nu|/\|\nu\| + P)$). Then consider

$$\varphi_P(\omega, x) := \frac{dP}{d\mathbb{P}}\varphi(\omega, x).$$

This is a finite-valued normal convex integrand with the conjugate

$$\varphi_P^*(\cdot, y) = \frac{dP}{d\mathbb{P}}\varphi^*(\cdot, y/(dP/d\mathbb{P}))\mathbb{1}_{\{dP/d\mathbb{P}>0\}} + \infty\mathbb{1}_{\{dP/d\mathbb{P}=0, y\neq 0\}}.$$

Note that $\varphi_P(\cdot, x)^+ \in L_1(\mathbb{P})$ for all $x \in \mathbb{R}$ by (2.4), and by (2.7), $\varphi_P^*(\cdot, \zeta)^+ \in L_1(\mathbb{P})$ for some $\zeta \in L_1(\mathbb{P})$ (with a slight abuse of notation, $\zeta = \eta\frac{dP}{d\mathbb{P}}$, with the η in (2.7) does the job). Thus the classical Rockafellar theorem ([26, Th.1]) shows that

$$\sup_{\xi\in L_\infty(\mathbb{P})} (\nu(\xi) - \mathbb{E}_\mathbb{P}[\varphi_P(\cdot, \xi)]) = \mathbb{E}_\mathbb{P}\left[\varphi_P^*(\cdot, d\nu/d\mathbb{P})\right] = J_\varphi(\nu|P),$$

where note that $\frac{d\nu}{d\mathbb{P}}/\frac{dP}{d\mathbb{P}} = \frac{d\nu}{d\mathbb{P}}$ if $\nu \ll P$ etc. Thus it remains to show that

$$\sup_{f\in C_b(\Omega)} (\nu(f) - \mathbb{E}_P[\varphi(\cdot, f)]) = \sup_{\xi\in L_\infty(P)} (\nu(\xi) - \mathbb{E}_\mathbb{P}[\varphi_P(\cdot, \xi)]). \tag{2.25}$$

Of course, "$\le$" is clear. For "$\ge$," let $\xi \in L_\infty(\mathbb{P})$ and pick a bounded representative $f \in \xi$ (relative to $L_\infty(\mathbb{P})$). Now for each $\varepsilon > 0$, Lusin's theorem yields a compact set $K_\varepsilon \subset \Omega$ such that $\mathbb{P}(K_\varepsilon^c) < \varepsilon$ and $f|_{K_\varepsilon}$ is continuous, then Tietze's theorem gives us its continuous extension $f_\varepsilon \in C(\Omega)$ with $\|f_\varepsilon\|_\infty = \|f|_{K_\varepsilon}\|_\infty \le \|\xi\|_\infty$. Then noting that

$\|g\|_\infty \leq c \Rightarrow |\varphi(\cdot, g)| \leq \varphi(\cdot, c)^+ + \varphi(\cdot, -c)^+ + c + \varphi^*(\cdot, \eta)^+ =: \kappa_c \in L_1(P)$ where η is as in (2.7),

$$\begin{aligned}
&\nu(\xi) - \mathbb{E}_{\mathbb{P}}[\varphi_P(\cdot, \xi)] \\
&\quad = \nu(f_\varepsilon) - \mathbb{E}_P[\varphi(\cdot, f_\varepsilon)] + \nu((f - f_\varepsilon)\mathbb{1}_{K_\varepsilon^c}) + \mathbb{E}_P[\{\varphi(\cdot, f_\varepsilon) - \varphi(\cdot, f)\}\mathbb{1}_{K_\varepsilon^c}] \\
&\quad \leq \sup_{g \in C_b(\Omega)} (\nu(g) - \mathbb{E}_P[\varphi(\cdot, g)]) + 2\|\xi\|_\infty |\nu|(K_\varepsilon^c) + 2\mathbb{E}_P[\kappa_{\|\xi\|_\infty} \mathbb{1}_{K_\varepsilon^c}].
\end{aligned}$$

The last two terms tend to 0 as $\varepsilon \to 0$ since $P, |\nu| \ll \mathbb{P}$. □

3 Semistatic Robust Utility Indifference Valuation

We proceed to the robust utility maximisation problem. Let $\Omega = \mathbb{R}^N$ ($N \in \mathbb{N}$), which we think of as the N-period discrete time path-space, and $S = (S_i)_{1 \leq i \leq N}$ the coordinate process, i.e. $(S_i(\omega))_{1 \leq i \leq N} = \mathrm{id}_{\mathbb{R}^N}(\omega)$, $\omega \in \mathbb{R}^N$, with $S_0(\omega) = s_0$ (constant) as the (discounted) underlying assets. Also, we are given a set $\mathcal{P}$ of Borel probability measures on $\mathbb{R}^N$, viewed as the set of possible models for S. As in Section 2, we suppose

(3.1) $\qquad \mathcal{P}$ is a convex and $\sigma(\mathrm{ca}(\mathbb{R}^N), C_b(\mathbb{R}^N))$-compact.

Let $\mathcal{H}$ denote the vector space of processes $H = (H_i)_{1 \leq i \leq N}$ such that

(3.2) $\qquad H_1$ is constant; $H_t = h_t(S_1, \ldots, S_{t-1})$ for some $h_t \in C_b(\mathbb{R}^{t-1})$, $\forall i \geq 2$.

Each $H \in \mathcal{H}$ is predictable (for the filtration generated by S), and is thought of as a self-financing dynamic strategy with gain $H \bullet S_t := \sum_{i \leq t} H_i(S_i - S_{i-1})$, the discrete stochastic integral. Note also that for a probability measure $Q \in \mathrm{Prob}(\mathbb{R}^N)$,

(3.3) $\qquad S$ is a Q-martingale $\Leftrightarrow$ $S_t \in L_1(Q)$, $\forall t$ and $\mathbb{E}_Q[H \bullet S_N] = 0$, $\forall H \in \mathcal{H}$.

The next ingredient is a family $\mu = (\mu_i)_{1 \leq i \leq N}$ of distributions on $\mathbb{R}$ such that

$$\mathcal{M}_\mu := \left\{ Q \in \mathrm{Prob}(\mathbb{R}^N) : S \text{ is a } Q\text{-martingale, } (Q \circ S_i^{-1})_{i \leq N} = (\mu_i)_{i \leq N} \right\} \neq \emptyset,$$

for which it is necessary and sufficient that:

(3.4) $\qquad \int |x| d\mu_i < \infty$, $\int x d\mu_i = s_0$ and $i \mapsto \int f d\mu_i$ is increasing for every convex function $f : \mathbb{R} \to \mathbb{R}$ (i.e. increasing in convex order).

This is ***Strassen's theorem*** ([30], Th. 8). Then $\mathcal{M}_\mu$ is a $\sigma(\mathrm{ca}(\mathbb{R}^N), C_b(\mathbb{R}^N))$-compact convex set ([5], Prop. 2.4). In (idealised) reality, such a family $(\mu_i)_{i \leq N}$ is calculated from the prices of call options via the relation (due to [11]):

$$Q(\xi \leq K) = 1 + \lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}_Q[(\xi - K - \varepsilon)^+] - \mathbb{E}_Q[(\xi - K)^+]}{\varepsilon},$$

(if call options of all the strikes are available). In this sense, each $Q \in \mathcal{M}_\mu$ is a pricing measure ***calibrated to the call prices in the market*** (see e.g. [18] for more detailed exposition). Then every vanilla option with payoff function $f \in C_b(\mathbb{R})$ and maturity $i \le N$ is priced at $\mathbb{E}_Q[f(S_i)] = \mu_i(f)$ for all $Q \in \mathcal{M}_\mu$; thus the final gain from investing in (f, i) is $f(S_i) - \mu_i(f)$. A static position is any $(f_i)_{i \le N} \in C_b(\mathbb{R})^N$ where each f_i is a vanilla option maturing at i, and any pair $(H, (f_i)_{i \le N}) \in \mathcal{H} \times C_b(\mathbb{R})^N$ is called a ***semistatic strategy***, whose gain is

$$H \bullet S_N + \sum_{i \le N} (f_i(S_i) - \mu_i(f_i)) =: H \bullet S_N + \Gamma_{(f_i)_{i \le N}}.$$

Finally, let $U : \mathbb{R} \to \mathbb{R}$ be a utility function (finite on the whole $\mathbb{R}$; e.g. exponential) that is strictly concave, differentiable and satisfies the ***Inada condition***

$$\lim_{x \downarrow -\infty} U'(x) = +\infty \quad \text{and} \quad \lim_{x \uparrow \infty} U'(x) = 0. \tag{3.5}$$

Then its conjugate $V(y) := \sup_{x \in \mathbb{R}}(U(x) - xy)$ is a proper convex function such that $\operatorname{int}\operatorname{dom}(V) = (0, \infty)$ on which it is strictly convex, differentiable, and

$$V'(0) := \lim_{y \downarrow 0} V'(y) = -\infty, \; V'(\infty) := \lim_{y \uparrow \infty} V'(y) = +\infty. \tag{3.6}$$

Now for each initial cost $x \in \mathbb{R}$ and (the payoff function of) an exotic option $\Psi : \mathbb{R}^N \to \mathbb{R}$, we consider the robust utility maximisation with ***semistatic strategies***:

$$u_\Psi(x) := \sup_{H \in \mathcal{H}, f \in C_b(\mathbb{R})^N} \inf_{P \in \mathcal{P}} \mathbb{E}_P \left[U\left(x + H \bullet S_N + \Gamma_f - \Psi \right) \right]. \tag{3.7}$$

The main result of this Section is the following.

Theorem 3.1 (Duality). *Suppose (3.1), (3.4), (3.5) as well as*

$$\inf_{\lambda > 0, Q \in \mathcal{M}_\mu} J_{V,\mathcal{P}}(\lambda Q) < \infty; \tag{3.8}$$

$$\lim_n \sup_{P \in \mathcal{P}} \mathbb{E}_P \left[U(\alpha |S_i|)^- \mathbb{1}_{\{|S_i| > n\}} \right] = 0, \quad \forall i \in \{1, \ldots, N\}, \forall \alpha > 0. \tag{3.9}$$

Then for any upper semicontinuous function $\Psi : \mathbb{R}^N \to \mathbb{R}$ with linear growth (i.e. $|\Psi(\omega)| \le c(1 + |\omega_1| + \cdots + |\omega_N|)$ for some $c > 0$), it holds that

$$u_\Psi(x) = \min_{\lambda > 0, Q \in \mathcal{M}_\mu} \left(J_{V,\mathcal{P}}(\lambda Q) - \lambda \mathbb{E}_Q[\Psi] + \lambda x \right). \tag{3.10}$$

The duality easily gives us representations of associated robust utility indifference prices as risk measures. Note that in view of (3.8),

$$\gamma_{V,\mathcal{P}}(Q) := \inf_{\lambda > 0} \frac{1}{\lambda} \left(J_{V,\mathcal{P}}(\lambda Q) - u_0(0) \right), \quad Q \in \operatorname{Prob}\left(\mathbb{R}^N\right),$$

defines a positive proper convex function with $\inf_{Q \in \operatorname{Prob}(\mathbb{R}^N)} \gamma_{V,\mathcal{P}}(Q) = 0$.

Corollary 3.2 (Indifference prices). *Under the assumptions of Theorem 3.1,*

(1) For any upper semicontinuous $\Psi : \mathbb{R}^N \to \mathbb{R}$ with linear growth,

$$p_U^{\mathrm{sell}}(\Psi) := \inf\{x : u_\Psi(x) \ge u_0(0)\} = \sup_{Q \in \mathcal{M}_\mu} \left(\mathbb{E}_Q[\Psi] - \gamma_{V,\mathcal{P}}(Q) \right). \tag{3.11}$$

(2) For any lower semicontinuous $\Psi : \mathbb{R}^N \to \mathbb{R}$ with linear growth,

$$p_U^{\mathrm{buy}}(\Psi) := -p_U^{\mathrm{sell}}(-\Psi) = \inf_{Q\in\mathcal{M}_\mu} \left(\mathbb{E}_Q[\Psi] + \gamma_{V,\mathcal{P}}(Q)\right). \tag{3.12}$$

(3) In particular, for any continuous $\Psi : \mathbb{R}^N \to \mathbb{R}$ with linear growth,

$$\inf_{Q\in\mathcal{M}_\mu} \mathbb{E}_Q[\Psi] \leq p_U^{\mathrm{buy}}(\Psi) \leq p_U^{\mathrm{sell}}(\Psi) \leq \sup_{Q\in\mathcal{M}_\mu} \mathbb{E}_Q[\Psi]. \tag{3.13}$$

Proof. (2) follows from (1), and (3) is a combination of (1) and (2). The derivation of (1) from (3.10) is also standard: by (3.10), $u_\Psi(x) \geq u_0(0)$ iff for any $\lambda > 0$ and $Q \in \mathcal{M}_\mu$, $J_{V,\mathcal{P}}(\lambda Q) - \lambda\mathbb{E}_Q[\Psi] + \lambda x \geq u_0(0)$; then rearrange the terms and take the infimum over $\lambda > 0$ and $Q \in \mathcal{M}_\mu$. □

The estimate (3.13) says that the indifference prices lie in the ***model-free pricing bound*** in the sense of [5]:

$$[p_U^{\mathrm{buy}}(\Psi), p_U^{\mathrm{sell}}(\Psi)] \subset \Big[\inf_{Q\in\mathcal{M}_\mu} \mathbb{E}_Q[\Psi], \sup_{Q\in\mathcal{M}_\mu} \mathbb{E}_Q[\Psi]\Big], \tag{3.14}$$

for any continuous $\Psi : \mathbb{R}^N \to \mathbb{R}$ with linear growth (then $\Psi \in \bigcap_{Q\in\mathcal{M}_\mu} L_1(Q)$). In particular, the indifference prices are consistent with the observed vanilla prices, i.e. if $\Psi(\omega) = g(\omega_i) = g(S_i(\omega))$ (i.e. a vanilla option with maturity i), then

$$p_U^{\mathrm{sell}}(\Psi) = p_U^{\mathrm{buy}}(\Psi) = \mu_i(g).$$

Remark 3.3 (A trivial case). In the situation of the paper (with full marginal constraint), $\mathcal{M}_\mu$ itself is (convex and) $\sigma(\mathrm{ca}(\mathbb{R}^N), C_b(\mathbb{R}^N))$-compact. If we take $\mathcal{P} = \mathcal{M}_\mu$, then $J_{V,\mathcal{M}_\mu}(Q) \leq J_V(Q|Q) = V(1) < \infty$ ($\forall Q \in \mathcal{M}_\mu$), and $\gamma_{V,\mathcal{M}_\mu}(Q) = 0$ on $\mathcal{M}_\mu$; thus in this case, the buyer's/seller's indifference prices coincide, respectively, with sub/super-hedging prices, i.e. the two intervals in (3.14) coincide. The choice of a "nice" $\mathcal{P}$ as well as a quantitative analysis are left for future topics. ♦

Example 3.4 (Exponential case; cf. [3], [29]). As one might expect, the situation is much simpler if the utility function is exponential, i.e.

$$U(x) = -e^{-x}, \quad x \in \mathbb{R}.$$

Letting $\mathcal{E}(Q|P) = \mathbb{E}_P[\frac{dQ}{dP}\log\frac{dQ}{dP}]$ if $Q \ll P$ and otherwise $+\infty$ (the relative entropy) and $\mathcal{E}_\mathcal{P}(Q) = \inf_{P\in\mathcal{P}} \mathcal{E}(Q|P)$, a straightforward calculation shows that

$$J_V(\lambda Q|P) = \lambda\mathcal{E}(Q|P) + \lambda\log\lambda - \lambda.$$

Thus for any continuous $\Psi : \mathbb{R}^N \to \mathbb{R}$ with linear growth and a $\sigma(\mathrm{ca}(\mathbb{R}^N), C_b(\mathbb{R}^N))$-compact convex set $\mathcal{P}$ with $\inf_{Q\in\mathcal{M}_\mu} \mathcal{E}_\mathcal{P}(Q) < \infty$ and

$$\lim_k \sup_{P\in\mathcal{P}} \mathbb{E}_P[\exp(\alpha|S_i|)\mathbb{1}_{\{|S_i|>k\}}] = 0,\ \forall\alpha > 0,\ i \leq N, \tag{3.15}$$

one has

$$\sup_{H\in\mathcal{H}, f\in C_b(\mathbb{R})^N} \inf_{P\in\mathcal{P}} \mathbb{E}_P\left[-e^{-(x+H\bullet S_N+\Gamma_f-\Psi)}\right] = -e^{-x-\min_{Q\in\mathcal{M}_\mu}\left(\mathcal{E}_\mathcal{P}(Q)-\mathbb{E}_Q[\Psi]\right)}.$$

In particular,

$$p_{\exp}^{\mathrm{sell}}(\Psi) = \max_{Q\in\mathcal{M}_\mu}\Big\{\mathbb{E}_Q[\Psi] - \Big(\mathcal{E}_\mathcal{P}(Q) - \inf_{Q'\in\mathcal{M}_\mu}\mathcal{E}(Q')\Big)\Big\}. \quad \diamond$$

3.1 Ramifications

From the financial motivation, it is important to note that the duality (3.10) is somehow stable for the choice of the admissible sets. In Theorem 3.1, we chose $C_b(\mathbb{R})^N$ for static positions and $\mathcal{H}$ (given by (3.2)) for the dynamic ones. We first examine the largest choice. Let $\mathcal{H}_s$ be the set of predictable processes H (for the filtration generated by S) such that $H \bullet S$ is a supermartingale under all $Q \in \mathcal{M}_\mu$, and consider $\prod_{i\le N} L_1(\mu_i)$ for static positions. Then for any $f = (f_i)_{i\le N} \in \prod_{i\le N} L_1(\mu_i)$, $H \in \mathcal{H}_s$ and $Q \in \mathcal{M}_\mu$,

$$\begin{aligned}&\mathbb{E}_P[U(x + H \bullet S_N + \varGamma_f - \varPsi)] \le J_V(\lambda Q|P) + \lambda\mathbb{E}_Q[x + H \bullet S_N + \varGamma_f - \varPsi]\\ &\quad\le J_V(\lambda Q|P) + \lambda\mathbb{E}_Q[x - \varPsi],\end{aligned}$$

where $\mathbb{E}_Q[\varGamma_f] = \sum_{i\le N}\{\mathbb{E}_Q[f_i(S_i)] - \mu_i(f_i)\} = 0$ if $f \in \prod_{i\le N} L_1(\mu_i)$; hence

$$\begin{aligned}&\sup_{H\in\mathcal{H}_s, f\in\prod_{i\le N} L_1(\mu_i)} \inf_{P\in\mathcal{P}} \mathbb{E}_P[U(x + H \bullet S_N + \varGamma_f - \varPsi)]\\ &\qquad\le \inf_{\lambda>0, Q\in\mathcal{M}_\mu} (J_{V,\mathcal{P}}(\lambda Q) + \lambda x - \lambda\mathbb{E}_Q[\varPsi]) = u_\varPsi(x).\end{aligned}$$

Since $C_b(\mathbb{R}) \subset L_1(\mu_i)$ and $\mathcal{H} \subset \mathcal{H}_s$, we deduce that

Corollary 3.5. *Under the assumptions of Theorem 3.1, it holds that*

$$\begin{aligned}&\sup_{H\in\mathcal{H}_s, f\in\prod_{i\le N} L_1(\mu_i)} \inf_{P\in\mathcal{P}} \mathbb{E}_P[U(x + H \bullet S_N + \varGamma_f - \varPsi)]\\ &\qquad= \min_{\lambda>0, Q\in\mathcal{M}_\mu} (J_{V,\mathcal{P}}(\lambda Q) + \lambda x - \lambda\mathbb{E}_Q[\varPsi])\,.\end{aligned}$$

A bit more general formulation is to choose, for each $i \le N$, a subset $\mathcal{S}_i \subset L_1(\mu_i)$ for static positions maturing at i. For instance, [3] considered the case where each $\mathcal{S}_i$ is spanned by a finite number (possibly 0) of fixed options. Here we consider the one spanned by call options of all the strikes:

$$\mathcal{S}_{\mathrm{call}} := \mathrm{span}\,((\cdot - K)^+ : K \in \mathbb{R}) \subset L_1(\mu_i).$$

Note that every element of $\mathcal{S}_{\mathrm{call}}$ is piecewise linear, while any bounded piecewise linear function lies in $\mathcal{S}_{\mathrm{call}} + \mathbb{R} = \{g + a : g \in \mathcal{S}_{\mathrm{call}}, a \in \mathbb{R}\}$.

Corollary 3.6 (Duality with calls only). *Under the assumptions of Theorem 3.1,*

$$\begin{aligned}&\sup_{H\in\mathcal{H}, f\in\mathcal{S}^N_{\mathrm{call}}} \inf_{P\in\mathcal{P}} \mathbb{E}_P[U(x + H \bullet S_N + \varGamma_f - \varPsi)]\\ &\qquad= \min_{\lambda>0, Q\in\mathcal{M}_\mu} (J_{V,\mathcal{P}}(\lambda Q) + \lambda x - \lambda\mathbb{E}_Q[\varPsi])\,.\end{aligned} \tag{3.16}$$

Proof. Will be given in Section 3.2. □

Remark 3.7 (Finitely many options). Another possible (and rather more realistic) formulation is to set each $\mathcal{S}_i$ to be the span of a finite number (possibly 0) of fixed options, say $\mathcal{S}_i = \mathrm{span}(f_{i,1}, \ldots, f_{i,m_i})$ as considered in [3] and [13] (and [9] in superhedging). In this case, the duality (3.16) with ***exact marginal constraint*** is no longer true. But a similar duality with martingale measures Q with constraints $\mathbb{E}_Q[f_{i,k}(S_i)] = \mu_i(f_{i,k})$ holds; see [3], Th. 2.2. ♦

3.2 Proofs of Theorem 3.1 and Corollary 3.6

We shall apply the results of Section 2 to the normal integrands

$$\varphi_{H,\Psi}(\omega, x) = -U(-x + H \bullet S_N(\omega) - \Psi(\omega)), \quad H \in \mathcal{H}.$$

Note first that for each $H \in \mathcal{H}$, $\omega \mapsto H \bullet S_N(\omega)$ is continuous with linear growth, i.e. $|H \bullet S_N(\omega)| \leq c(1 + \sum_{i\leq N} |\omega_i|)$ for some $c > 0$ (say $c = 2(|S_0| \vee 1) \max_{i\leq N} \|H_i\|_\infty$). Thus under the assumptions of Theorem 3.1, $\omega \mapsto B_{H,\Psi}(\omega) := -H \bullet S_N(\omega) + \Psi(\omega)$ is usc with linear growth since Ψ is. If $c > 0$ is a linear growth constant for $B_{H,\Psi}$, then letting $\varphi_U(x) = -U(-x)$, which is convex,

$$\begin{aligned}\varphi_U(\pm\alpha|B_{H,\Psi}|) &\leq \varphi_U\Big(\alpha c\Big(1 + \sum\nolimits_{i\leq N} |S_i|\Big)\Big)\\ &\leq \frac{\varphi_U(2\alpha c)^+}{2} + \sum\nolimits_{i\leq N} \frac{\varphi_U(2\alpha cN|S_i|)^+}{2N} \overset{(3.9)}{\in} \mathbb{L}_{1,b}(\mathcal{P}),\end{aligned}$$

for any $\alpha > 0$. Hence Proposition 2.11 applied to φ_U and $B_{H,\Psi}$ yields that $\varphi_{H,\Psi}(\cdot, x) = (\varphi_U)_{B_H}(\cdot, x)$ satisfies Assumption 2.6, and we have

Lemma 3.8. *Under the assumptions of Theorem 3.1, it holds that for any $H \in \mathcal{H}$,*

$$I_{\varphi_{H,\Psi},\mathcal{P}}(f) = -\inf_{P\in\mathcal{P}} \mathbb{E}_P\left[U(-f + H \bullet S_N - \Psi)\right]$$

is continuous on $C_b(\mathbb{R}^N)$ for $\tau(C_b(\mathbb{R}^N), \mathrm{ca}(\mathbb{R}^N))$; $-H \bullet S_N + \Psi \in L_1(\nu)$ whenever $\inf_{\lambda>0} J_{V,\mathcal{P}}(\lambda\nu) < \infty$; and the conjugate of $I_{\varphi_{H,\Psi},\mathcal{P}}$ is given on $\mathrm{ca}(\mathbb{R}^N)$ as

$$I^*_{\varphi_{H,\Psi},\mathcal{P}}(\nu) = \begin{cases} J_{V,\mathcal{P}}(\nu) + \nu(H \bullet S_N - \Psi) & \text{if } \inf_{\alpha>0} J_{V,\mathcal{P}}(\alpha\nu) < \infty,\\ +\infty & \text{otherwise.}\end{cases} \tag{3.17}$$

In particular, $\mathrm{dom}(I^*_{\varphi_{H,\Psi},\mathcal{P}}) = \mathrm{dom}(J_{V,\mathcal{P}}) \subset \mathrm{ca}(\mathbb{R}^N)^+$ (since $\mathrm{dom}(V) \subset \mathbb{R}^+$), and $\Psi, S_i \in L_1(\nu)$, $i \leq N$, whenever $\inf_{\lambda>0} J_{V,\mathcal{P}}(\lambda\nu) < \infty$.

Recall that $\Gamma_f = \sum_{i\leq N}(f_i(S_i) - \mu_i(f_i))$ for $f = (f_i)_{i\leq N} \in C_b(\mathbb{R})^N$, and let

$$\mathcal{D} := \left\{\Gamma_f : f \in C_b(\mathbb{R})^N\right\}. \tag{3.18}$$

This is a vector subspace of $C_b(\mathbb{R}^N)$, so its (one-sided) polar in $\mathrm{ca}(\mathbb{R}^N)$ is $\mathcal{D}^\circ = \{\nu \in \mathrm{ca}(\mathbb{R}^N) : \nu(\psi) = 0, \psi \in \mathcal{D}\}$ (which is linear), and a probability measure Q lies in $\mathcal{D}^\circ$ iff $\mathbb{E}_Q[f(S_i) - \mu_i(f)] = 0$ for $i \leq N$ and $f \in C_b(\mathbb{R})$ iff $Q \circ S_i^{-1} = \mu_i$, $i \leq N$. In other words,

$$\mathcal{D}^\circ \cap \mathrm{ca}(\mathbb{R}^N)^+ = \left\{\lambda Q : \lambda \geq 0, Q \in \mathrm{Prob}(\mathbb{R}^N), Q \circ S_i^{-1} = \mu_i, i \leq N\right\}. \tag{3.19}$$

In particular, by (3.8) and $\mathrm{dom}(J_{V,\mathcal{P}}) \subset \mathrm{ca}(\mathbb{R}^N)$ (Lemma 3.8),

$$\emptyset \neq \mathcal{M}_\mu \cap \mathrm{dom}(J_{V,\mathcal{P}}) \subset \mathcal{D}^\circ \cap \mathrm{dom}(J_{V,\mathcal{P}}) \subset \mathrm{ca}(\mathbb{R}^N)^+, \tag{3.20}$$

Note also that in view of (3.4) and the linear growth assumption on Ψ,

$$\nu \in \mathcal{D}^\circ \cap \mathrm{ca}(\mathbb{R}^N)^+ \Rightarrow S_i \in L_1(\nu), \text{ so } \Psi, H \bullet S_N \in L_1(\nu), \forall H \in \mathcal{H}. \tag{3.21}$$

Consequently, on $\mathcal{D}^\circ\, \mathrm{ca}(\mathbb{R}^N)^+$, (3.17) simplifies to

$$I^*_{\varphi_{H,\Psi},\mathcal{P}}(\nu) = J_{V,\mathcal{P}}(\nu) + \nu(H \bullet S_N - \Psi), \quad \nu \in \mathcal{D}^\circ \cap \mathrm{ca}(\mathbb{R}^N)^+. \tag{3.22}$$

Proof of Theorem 3.1. Replacing Ψ by $\Psi - x$, which does not affect the assumptions on Ψ, it suffices to prove the case of $x = 0$. In this case, Corollary 2.10 and Lemma 3.8 show that

$$\sup_{f\in\mathcal{D}} \inf_{P\in\mathcal{P}} \mathbb{E}_P[U(f + H \bullet S_N - \Psi)] = -\inf_{f\in -\mathcal{D}} I_{\varphi_{H,\Psi},\mathcal{P}}(f) = \min_{\nu\in\mathcal{D}^\circ} I^*_{\varphi_{H,\Psi},\mathcal{P}}(\nu).$$

Since $\mathrm{dom}(I_{\varphi_{H,\Psi},\mathcal{P}}) = \mathrm{dom}(J_{V,\mathcal{P}})$, we deduce that

$$\min_{\nu\in\mathcal{D}^\circ} I^*_{\varphi_{H,\Psi},\mathcal{P}}(\nu) = \min_{\nu\in\mathcal{D}^\circ\cap\mathrm{dom}(J_{V,\mathcal{P}})} I^*_{\varphi_{H,\Psi},\mathcal{P}}(\nu) \overset{(3.20)}{<} \infty.$$

Then note that $\{\nu \in \mathcal{D}^\circ \cap \mathrm{dom}(J_{V,\mathcal{P}}) : I^*_{\varphi_{H,\Psi},\mathcal{P}}(\nu) \le c\} = \mathcal{D}^\circ \cap \{\nu \in \mathrm{ca}(\mathbb{R}^N) : I^*_{\varphi_{H,\Psi},\mathcal{P}}(\nu) \le c\}$, $c \in \mathbb{R}$, are convex and $\sigma(\mathrm{ca}(\mathbb{R}^N), C_b(\mathbb{R}^N))$-compact since $I_{\varphi_{H,\Psi},\mathcal{P}}$ is $\tau(C_b(\mathbb{R}^N), \mathrm{ca}(\mathbb{R}^N))$-continuous, and $H \mapsto I^*_{\varphi_{H,\Psi},\mathcal{P}}(\nu)$ is concave for each $\nu \in \mathcal{D}^\circ \cap \mathrm{dom}(J_{V,\mathcal{P}})$. Thus the lop-sided minimax theorem ([2, Th.6.2.7 on p.319]) shows

$$\begin{aligned}\sup_{H\in\mathcal{H}} \min_{\nu\in\mathcal{D}^\circ\cap\mathrm{dom}(J_{V,\mathcal{P}})} I^*_{\varphi_{H,\Psi},\mathcal{P}}(\nu) &= \min_{\nu\in\mathcal{D}^\circ\cap\mathrm{dom}(J_{V,\mathcal{P}})} \sup_{H\in\mathcal{H}} I^*_{\varphi_{H,\Psi},\mathcal{P}}(\nu)\\ &\overset{(3.22)}{=} \min_{\nu\in\mathcal{D}^\circ\cap\mathrm{dom}(J_{V,\mathcal{P}})} \left(J_{V,\mathcal{P}}(\nu) + \sup_{H\in\mathcal{H}} \nu(H \bullet S_N) - \nu(\Psi)\right) =: (*).\end{aligned}$$

Then note that for $\nu \in \mathcal{D}^\circ \cap \mathrm{dom}(J_{V,\mathcal{P}})$, $\sup_{H\in\mathcal{H}} \nu(H \bullet S_N) \in \{0, \infty\}$ since $\mathcal{H}$ is linear (and $S_i \in L_1(\nu)$ by (3.21)), and by (3.3), it is 0 iff $\nu = \alpha Q$ for a martingale measure Q for S and $\alpha \ge 0$. Summing up with (3.19) (and $\mathrm{dom}(J_{V,\mathcal{P}}) \subset \mathrm{ca}(\mathbb{R}^N)^+$),

$$(*) = \min_{\lambda\ge 0, Q\in\mathcal{M}_\mu} \left(J_{V,\mathcal{P}}(\lambda Q) - \lambda\mathbb{E}_Q[\Psi]\right).$$

We complete the proof by showing that the minimum on the RHS is attained by a non-zero λQ. To see this, note first that $J_{V,\mathcal{P}}(0) - 0(\Psi) = V(0)$. Pick, by (3.8), a $Q \in \mathcal{M}_\mu$ with $J_V(\lambda Q|P) < \infty$ for some $\lambda > 0$ and $P \in \mathcal{P}$; then $Q \ll P$, so $P(dQ/dP > 0) > 0$. Putting $\eta := \lambda dQ/dP$, $\alpha \mapsto G(\alpha) := V(\alpha\eta) - \alpha\eta\Psi$ is (finite-valued and) convex. Since $V'(0) = \lim_{\downarrow 0} V'(\alpha) = -\infty$, $\frac{G(\alpha)-G(0)}{\alpha} \downarrow -\infty\mathbb{1}_{\{\eta>0\}}$, and since $G(1) = V(\eta) - \eta\Psi \in L_1(P)$, we deduce that

$$\frac{J_V(\alpha\nu|P) - \alpha\nu(\Psi) - V(0)}{\alpha} = \mathbb{E}_P\left[\frac{G(\alpha) - G(0)}{\alpha}\right] \downarrow -\infty.$$

Thus $J_V(\alpha\nu|P) - \alpha\nu(\Psi) < V(0)$ for some $\alpha > 0$. □

For the proof of Corollary 3.6, we need a simple lemma. Let

$$\mathcal{D}_{\mathrm{call}} = \left\{\sum\nolimits_{i\le N} (f_i(S_i) - \mu(f_i)) : f_i \in \mathcal{S}_{\mathrm{call}}\right\}.$$

Lemma 3.9. *Any $\psi \in \mathcal{D}$ admits a sequence $(\psi_n)_n$ in $\mathcal{D}_{\mathrm{call}} \cap C_b(\mathbb{R})$ with $\psi_n \to \psi$ in $\tau(C_b(\mathbb{R}^N), \mathrm{ca}(\mathbb{R}^N))$.*

Proof. Since $\mathcal{D}$ consists of functions $\sum_{i\le N}(f_i \circ S_i - \mu_i(f_i))$ with $f_i \in C_b(\mathbb{R})$, and $(f_i + a) \circ S_i - \mu_i(f_i + a) = f_i \circ S_i - \mu_i(f_i)$ if a is a constant, it suffices to show that each $g \in C_b(\mathbb{R})$ admits a sequence $(g_n)_n$ of bounded piecewise linear functions on $\mathbb{R}$ such that $g_n \circ S_i - \mu_i(g_n) \to g \circ S_i - \mu_i(g)$ in $\tau(C_b(\mathbb{R}^N), \mathrm{ca}(\mathbb{R}^N))$. So fix $g \in C_b(\mathbb{R})$. For each n, g is uniformly continuous on $[-n, n]$, so one can find a piecewise linear function $g_n : [-n, n] \to \mathbb{R}$ with $|g - g_n| \le 1/n$ on $[-n, n]$. Extend g_n to the entire $\mathbb{R}$ by setting $g(x) = g(-n)$ (resp. $g(n)$) if $x < -n$ (resp. $> n$), which is piecewise linear and $|g_n| \le 2\|g\|_\infty$ on $\mathbb{R} \setminus [-n, n]$. Now for each $i \le N$,

$$|\mu_i(g - g_n)| \le \frac{1}{n} + 2\|g\|_\infty \mu_i\left(\mathbb{R} \setminus [-n, n]\right) \to 0.$$

Also, for any $\sigma(\mathrm{ca}(\mathbb{R}^N), C_b(\mathbb{R}^N))$-compact ($\Leftrightarrow$ bounded uniformly tight) $\Lambda \subset \mathrm{ca}(\mathbb{R}^N)$,

$$\sup_{\nu\in\Lambda} |\nu(g \circ S_i - g_n \circ S_i)| \le \frac{1}{n} \sup_{\nu\in\Lambda} \underbrace{|\nu|\left([-n, n]^N\right)}_{\le\|\nu\|} + 2\|g\|_\infty \sup_{\nu\in\Lambda} |\nu|\left(\mathbb{R}^N \setminus [-n, n]^N\right).$$

By the uniform tightness, the RHS tends to 0 as $n \to \infty$. This shows that $g_n \circ S_i \to g \circ S_i$ in $\tau(C_b(\mathbb{R}^N), \mathrm{ca}(\mathbb{R}^N))$. Summing up, $g_n \circ S_i - \mu_i(g_n) \to g \circ S_i - \mu_i(g)$ in $\tau(C_b(\mathbb{R}^N), \mathrm{ca}(\mathbb{R}^N))$. □

Proof of Corollary 3.6. Since $f \mapsto \inf_{P\in\mathcal{P}} \mathbb{E}_P[U(f + H \bullet S_N - \Psi)] = -I_{\varphi_{H,\Psi},\mathcal{P}}(-f)$ is $\tau(C_b(\mathbb{R}^N), \mathrm{ca}(\mathbb{R}^N))$-continuous on $C_b(\mathbb{R}^N)$ for each $H \in \mathcal{H}$, Lemma 3.9 yields

$$\begin{aligned} u_\Psi(0) = \sup_{H\in\mathcal{H}} \sup_{\psi\in\mathcal{D}} -I_{\varphi_{H,\Psi},\mathcal{P}}(-\psi) &\le \sup_{H\in\mathcal{H}} \sup_{\psi\in\mathcal{D}_{\mathrm{call}}} -I_{\varphi_{H,\Psi},\mathcal{P}}(-\psi) \\ &\le \min_{\lambda>0, Q\in\mathcal{M}_\mu} \left(J_{V,\mathcal{P}}(\lambda Q) - \lambda\mathbb{E}_Q[\Psi]\right) = u_\Psi(0), \end{aligned}$$

where in the second line, note that any $g \in \mathcal{S}_{\mathrm{call}}$ (is Lipschitz, hence) has a linear growth, so $g \in L_1(\mu_i)$, thus Corollary 3.5 proves the second "≤." □

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