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Multi-agent Equilibrium Model with Heterogeneous Views on Fundamental Risks in Incomplete Market

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Abstract

This paper considers a multi-agent optimal investment problem with conservative, aggressive, or neutral sentiments in an incomplete market by a BSDE approach. Particularly, we formulate the conservative, aggressive, or neutral sentiments of the agents by a sup-inf/inf-sup, sup-sup, or sup problem where we take infimum or supremum on a choice of a probability measure depending on the view types and supremum on trading strategies. To the best of our knowledge, this is the first attempt to investigate a multi-agent equilibrium model in an incomplete setting with heterogeneous views on Brownian motions. Moreover, we show a square-root case where a group of agents has either conservative, aggressive, or neutral sentiments on the fundamental risks and a general case where the Sharpe ratio process of the risky asset and the optimal trading strategies in equilibrium are explicitly solved by a BSDE approach. Finally, we present numerical examples of the trading strategies and the expected return process in equilibrium under heterogeneous sentiments, which explain how the conservative, aggressive, or neutral sentiments affect the Sharpe ratio process of the risky asset and the trading strategies of the agents in equilibrium.

1 Introduction

This paper investigates a multi-agent optimal investment problem under an incomplete market setting with heterogeneous views on fundamental risks represented by Brownian motions. Specifically, we consider an exponential utility case, where the degrees of risk aversion and the view types on the fundamental risks differ among the agents. Particularly, we obtain the Sharpe ratio process of the risky asset and the optimal trading strategies of the agents in equilibrium. To the best of our knowledge, this is the first attempt to solve for a general equilibrium in a multi-agent model under an incomplete market setting with heterogeneous views of the agents.

Sentiments of the market participants affect asset prices in financial markets, such as bond prices and stock prices, which has been particularly observed after the global

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financial crisis (e.g., Nakatani et al. [21] and Nishimura et al. [20]). In the financial market, the number of fundamental risks driving the market is generally considered to exceed the number of risky assets as in stochastic volatility models. Thus, considering heterogeneous views of market participants in an incomplete market setting has theoretical importance.

Moreover, this study is useful since the way the expected return on the risky asset changes when the sentiment of the market participants varies is essential in constructing a profitable trading strategy. For instance, when the major market participants in the stock market have different views and when their views change, it affects the stock prices through the trading of the market participants. If the sentiment changes are expected, we can construct a trading strategy by predicting how the expected return on the risky asset shifts. Furthermore, policymakers, such as central banks, can estimate the effect of their announcement on the stock market through the change in the bandwidth of the sentiments of market participants in the model.

In this study, we observe how the heterogeneous sentiments affect the Sharpe ratio process, or equivalently the expected return process when the volatility process is given, and investment strategies in an incomplete market setting, where the number of Brownian motions is greater than the number of risky assets and the state-price density process that defines the risk-neutral probability measure used for securities pricing is not uniquely determined.

For related literature, Petersen et al. [26] introduce relative entropy constraints to a stochastic system where the worst case is considered as a choice of a probability measure. Hansen and Sargent [9] apply the robust control theory to finance. They deal with a single agent's utility maximization problem with a choice of a probability measure on the conservative side. Also, as a problem of a choice of a probability measure, Chen and Epstein [5] investigate an optimal consumption problem in finance with ambiguity on risks.

As for optimal portfolio problems with ambiguity and general equilibrium under uncertainty, Beissner et al. [2] investigate the alpha max-min expected utility with ambiguity on risks of a single agent, which indicates a view lying between the most aggressive and the most conservative side. Beissner and Riedel [3] also examine equilibrium under economy with Knightian uncertainty. Beissner [4] investigates equilibrium with Knightian uncertainty in a complete market. Kizaki et al. [14] consider a multi-agent optimal portfolio problem with conservative and aggressive sentiments in a complete market setting.

Moreover, Choi and Larsen [6] derive incomplete market equilibrium under exponential utility without a choice of a probability measure. Kizaki et al. [15] investigate equilibrium under an incomplete market setting where agents have different income or payout profiles and risk aversion parameters and apply the results to life-cycle investment and reinsurance pricing.

Our work is different in that we incorporate heterogeneous views on the fundamental risks of multiple agents to obtain the expected return process in equilibrium in an incomplete market setting. Although in the complete market setting in Kizaki et al. [14], utilizing the fact that the state-price density process exists uniquely and the optimal portfolios of agents are expressed with the state-price density process, the process in equilibrium is obtained imposing a market clearing condition, and as a result, the interest rate process and the market price of risk are obtained. On the other hand, in the incomplete market setting, the state-price density process is not uniquely determined, and the same technique cannot be applied. Instead, we need to solve a system of BSDEs, which is the difference.

Also, for sentiments in the markets, Nishimura et al. [20] and Nakatani et al. [21] estimate sentiment factors in the interest rate models by using a text mining approach for the Japanese government bond markets. Saito and Takahashi [29] investigate a sup-inf problem on aggressive and conservative sentiments for a given state variable process. Saito and Takahashi [30] solve a sup-sup-inf problem for a single agent, where the agent works on an optimal investment problem under aggressive and conservative sentiments by a Malliavin calculus approach. Our work investigates a multi-agent model with sup-inf/inf-sup, sup-sup, or sup problems for individual optimization problems in an incomplete setting, where we solve for an equilibrium expected return process of the risky asset and the subjective probability measure of the agents, useful in pricing assets with heterogeneous views on fundamental risks.

Specifically, for optimal portfolio problems on multi-agent systems, Yang et al. [35] investigate principal-agent problems for a contract design with multiple agents, where a principal solves a utility maximization problem. Leung et al. [19] consider a decentralized robust portfolio optimization problem with a cooperative-competitive multi-agent system. (For other studies on multi-agent systems, see e.g., Kumar & Bhattacharya[16], Lee et al.[18], Park et al.[25], Pinto et al.[28], Gharesifard et al. [8] and Yang et al. [36],[37].

For applications of stochastic control to optimal portfolio problems in financial risk management, see, e.g., Cui et al. [7], Kasbekar et al.[13], He et al.[10], Ni et al.[22],[23], [24], Ye and Zhou [38], Lamperski and Cowan [17], Sen [32], Jiang and Fu [11], Wu et al.[34], Aybat et al. [1]). Our study differs in that we investigate market clearing on assets among the agents, who have different views on Brownian motions and risk-aversion parameters, to obtain the expected return process in equilibrium.

The contributions of this study are as follows. To the best of our knowledge, this study is the first attempt to investigate the multi-agent equilibrium under an incomplete market setting where the agents have heterogeneous views on fundamental risks. Kizaki et al. [14] obtained the market equilibrium where the agents have heterogeneous views on fundamental risks but in a complete market setting. This study extends the case to an incomplete market setting, where the number of Brownian motions that drive the market exceeds the number of risky tradable assets. Specifically, with a square-root case, where the standard results for the existence and uniqueness of a solution for the BSDE with stochastic Lipschitz driver do not apply since the terminal condition is unbounded, we first solve the sup-inf/inf-sup, sup-sup, or sup problem for the portfolio (the sup part) and the conservative or aggressive view (the inf or sup part) Then, we provide a general case in Section 5 and a Gaussian case in Appendix B, in which the existence and uniqueness of a solution and the comparison principle for standard BSDEs hold, is also included.

The organization of this paper is as follows. After Section 2 introduces the equilibrium multi-agent model in an incomplete market, Section 3 shows a square-root case, where the excess return process and the optimal strategies in equilibrium are explicitly solved. Section 4 presents numerical examples. Section 5 provides a theorem for a general case that also includes a Gaussian case in Appendix B. Finally, Section 6 concludes. Appendix A provides the proof of the theorem for the general case in Section 5. Appendix B presents a Gaussian state process case, an example of the general case in Section 5 where the system of BSDEs is reduced to separate BSDEs.

2 Setting

In this section, we explain the multi-agent model with heterogeneous views on fundamental risks in an incomplete market to obtain the Sharpe ratio process and the optimal trading strategies in equilibrium. Firstly, we describe the setting of the financial market and then introduce the individual optimization problem of each agent. We consider the following financial market where there are $\bar{I}$ agents, where $\bar{I} \geq 2$, trading one risky asset and a money market account and $\bar{I}$ agents consists of three types of agents, that is, I agents with conservative views on fundamental risks, I' agents with aggressive views, and I'' agents with neutral views, where $I, I', I'' \geq 0$, and $\bar{I} = I + I' + I''$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Also, we let (W_Y, W_S) be a two-dimensional Brownian motion defined on the probability space, where W_Y represents the Brownian motion driving the common factor process Y and W_S the one inherent to a risky asset price. $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ be the augmented filtration generated by the two-dimensional Brownian motion (W_Y, W_S), and $\mu, \mu_Y, \sigma, \sigma_Y, \rho_S, \hat{\rho}_S := \sqrt{1-\rho_S^2}$ be $\mathcal{R}$-valued $\{\mathcal{F}_t\}$-progressively measurable processes defined on $[0, T]$, where μ, μ_Y represent the expected return of the risky asset price and the factor process, σ, σ_Y the volatility, and ρ_S the correlation between the Brownian motions of the risky asset process and the factor process. Particularly, we assume $\sigma_t > 0, |\rho_{S,t}| \leq 1,\ 0 \leq t \leq T$, and hence $|\hat{\rho}_{S,t}| \leq 1$. Let $\lambda = (\lambda_Y, \lambda_S)^\top$ be a $\mathcal{R}^2$-valued $\{\mathcal{F}_t\}$-progressively measurable process, which represents the views on the fundamental risks, lying in some intervals. Namely, let $\Lambda_i = \{\lambda = (\lambda_Y, \lambda_S)^\top | |\lambda_{Y,t}| \leq \bar{\lambda}_{Y,i,t}, |\lambda_{S,t}| \leq \bar{\lambda}_{S,i,t}\}$ with exogenously given $\bar{\lambda}_{Y,i,t}, \bar{\lambda}_{S,i,t} > 0$ be the set of views on the fundamental risks of agent i, where $\bar{\lambda}_{Y,i}, \bar{\lambda}_{S,i},\ i = 1, \ldots, I + I'$, are $\mathcal{R}$-valued $\{\mathcal{F}_t\}$-progressively measurable processes.

Let S_0, S_1 be the price process of the money market account and the risky asset satisfying SDEs

$$\begin{aligned} dS_{0,t} &= rS_{0,t}dt,\ S_{0,0} = 1, \\ dS_{1,t} &= \mu_t S_{1,t}dt + \sigma_t S_{1,t}(\rho_{S,t}dW_{Y,t} + \hat{\rho}_{S,t}dW_{S,t}), \\ S_{1,0} &= p > 0, \end{aligned} \tag{1}$$

where the initial value of the risky asset price p is exogenously given. Specifically, we assume $r \equiv 0$ throughout this study and obtain the expected return process μ in equilibrium, which satisfies market clearing conditions.

Suppose that there exist $\bar{I} = I + I' + I''$ agents in the market who trade the money market account and the risky asset aiming to maximize their expected utility on the sum of the wealth and the wealth shock represented by the state process at the terminal time T in the following. Let π_i be the portfolio process satisfying $\int_0^T |\pi_{i,s}\mu_s| ds < \infty$, $\int_0^T \pi_{i,s}^2 \sigma_s^2 ds < \infty,\ P-a.s.$, which describes the allocation of the agent i's portfolio on the risky asset on value basis. Then X^{π_i}, the wealth process of agent i, satisfies an SDE

$$\begin{aligned} dX_t^{\pi_i} &= \pi_{i,t}\frac{dS_{1,t}}{S_{1,t}} \\ &= \pi_{i,t}\theta_t\sigma_t dt + \pi_{i,t}\sigma_t(\rho_{S,t}dW_{Y,t} + \hat{\rho}_{S,t}dW_{S,t}), \\ X_0^{\pi_i} &= x_{i,0}, \end{aligned} \tag{2}$$

where $\theta_t = \frac{\mu_t}{\sigma_t}$ and $x_{i,0} \in \mathcal{R},\ i = 1, \ldots, \bar{I}$.

Let $\mathcal{A}_i$ be a set of admissible strategies which will be specified depending on the respective cases in Section 3 and Appendix B so that arbitrage opportunities, where the agent makes a profit with a positive probability without losing money, are excluded for agent i.

Next, let Y be an endowment process which satisfies an SDE

$$dY_t = \mu_{Y,t}dt + \sigma_{Y,t}dW_{Y,t},\ Y_0 = y_0, \tag{3}$$

where $y_0 \in \mathcal{R}$. This state process Y is a source of incompleteness, which cannot be traded in the market and could affect μ, σ, ρ_S, $\hat{\rho}_S$ of the risky asset price process S_1 in (1). We assume that there is a one-time wealth shock Y_T at the terminal T, which may be understood as the economic state at the terminal, and the wealth shock is common among the $\bar{I}$ agents.

Agent i, $i = 1, \ldots, \bar{I}$ has an exponential utility function u_i on the sum of the terminal wealth $X_T^{\pi_i}$ and the one-time wealth shock Y_T, where $u_i(x) = -\exp(-\gamma_i x)$, $\gamma_i > 0$.

Remark 1 *We remark that although we assume the one-time wealth shock is Y_T for all $i = 1, \ldots, \bar{I}$, which is common among the agents, for simplicity, we can also handle the case where the wealth shock at the terminal T for agent i is a linear functional of Y_T such as $\alpha_i Y_T + \beta_i$, where $\alpha_i, \beta_i \in \mathcal{R}$ are constants, in the same way. This model corresponds to the case where $\alpha_i = 1$ and $\beta_i = 0$, which indicates a positive wealth shock if $Y_T > 0$. When $\alpha_i < 0$ with $Y_T > 0$, this implies a negative wealth shock.*

2.1 Individual optimization problems of three Types of agents

Also, agent i has either conservative, aggressive, or neutral views on the fundamental risks related to the risky asset price and the state process. The agent aims to maximize its expected utility by choosing the trading strategy π_i while minimizing when the agent is conservative or maximizing when the agent is aggressive with respect to the views $\lambda_{Y,i}$ and $\lambda_{S,i}$ on the fundamental risks W_Y and W_S, respectively. Thus, we consider the following sup-inf/inf-sup, sup-sup, or sup problem as the individual optimization problem.

(i) *When agent i is conservative* $(i = 1, \ldots, I)$,

$$\sup_{\pi_i \in \mathcal{A}_i} \inf_{\lambda_i \in \Lambda_i} E^{P^{\lambda_i}}[-\exp(-\gamma_i(X_T^{\pi_i} + Y_T))], \tag{4}$$

$$\inf_{\lambda_i \in \Lambda_i} \sup_{\pi_i \in \mathcal{A}_i} E^{P^{\lambda_i}}[-\exp(-\gamma_i(X_T^{\pi_i} + Y_T))], \tag{5}$$

(ii) *when agent i is aggressive* $(i = I + 1, \ldots, I + I')$,

$$\sup_{\lambda_i \in \Lambda_i} \sup_{\pi_i \in \mathcal{A}_i(\lambda_i)} E^{P^{\lambda_i}}[-\exp(-\gamma_i(X_T^{\pi_i} + Y_T))], \tag{6}$$

where $\mathcal{A}_i(\lambda_i)$ is defined appropriately in the following specific settings to exclude arbitrage opportunities are excluded for each fixed λ_i, and

(iii) *when agent i is neutral* $(i = I + I' + 1, \ldots, \bar{I})$,

$$\sup_{\pi_i \in \mathcal{A}_i} E^{P}[-\exp(-\gamma_i(X_T^{\pi_i} + Y_T))], \tag{7}$$

where P^{λ_i} in (i) and (ii) is defined as

$$\frac{dP^{\lambda_i}}{dP} = \exp\left(-\frac{1}{2}\int_0^T (\lambda_{S,i,t}^2 + \lambda_{Y,i,t}^2)dt + \int_0^T \lambda_{S,i,t}dW_{S,t} + \int_0^T \lambda_{Y,i,t}dW_{Y,t}\right).$$

As in Section 3 and Appendix B, when the weak version of Novikov's condition (e.g., Corollary 3.5.14 in Karatzas and Shreve [12]) holds, by Girsanov's theorem, $dW_{S,t} = dW_{S,t}^{\lambda_i} + \lambda_{S,i,t}dt$, $dW_{Y,t} = dW_{Y,t}^{\lambda_i} + \lambda_{Y,i,t}dt$, where $(W_Y^{\lambda_i}, W_S^{\lambda_i})$ is a Brownian motion under P^{λ_i}. Thus, the terms $\lambda_{S,i,t}dt$ and $\lambda_{Y,i,t}dt$ with the views $\lambda_{S,i,t}$ and $\lambda_{Y,i,t}$ indicate the bias of agent i on the instantaneous increment of the fundamental risks $dW_{S,t}$ and $dW_{Y,t}$.

2.2 Market equilibrium

Then, given the optimal trading strategies $\pi_i^*, i = 1, \ldots, \bar{I}$ obtained by the individual optimization problems (4) and (5), (6), or (7), we call that the market is in equilibrium if the following market-clearing conditions are satisfied.

$$\sum_{i=1}^{\bar{I}} \pi_{i,t}^* = \pi_t^s, \ \forall t \in [0, T], \tag{8}$$

and

$$\sum_{i=1}^{\bar{I}} (X_t^{\pi_i^*} - \pi_{i,t}^*) = 0, \ \forall t \in [0, T], \tag{9}$$

where π^s is a $\mathcal{R}$-valued $\{\mathcal{F}_t\}$-progressively measurable process, which represents the net supply of the risky asset in value basis.

Here, we also assume

$$\pi_0^s = \sum_{i=1}^{\bar{I}} x_{i,0}, \tag{10}$$

which indicates that the initial net supply is equal to the total initial wealth.

(8) is the market clearing condition for the risky asset position where the total demand of the risky asset is equal to the net supply of the risky asset π^s, and (9) is the market clearing condition for the money market account, where we assume that the net position of the money market is zero. In the following specific settings, we consider the zero-net supply $\pi^s = 0$ as a base case, and in addition to this, we investigate a positive supply case $\pi^s > 0$.

Under this setting, in the following sections, we aim to find the Sharpe ratio process θ of S_1, or equivalently the expected return process μ when the volatility process σ is given, in (1) in equilibrium, which is obtained by the following procedures.

Concretely, first presupposing the form of the views on the fundamental risks $\lambda_i^* = (\lambda_{Y,i}^*, \lambda_{S,i}^*)^\top, i = 1, \ldots, \bar{I}$, we solve the individual optimization problems (4) and (5), (6), or (7). Then, imposing the market clearing conditions (8) and (9), we obtain the candidate of the Sharpe ratio process in equilibrium.

Then, in theorems, given the candidate of the Sharpe ratio process θ and the volatility process σ, we solve the individual optimization problems (4) and (5), (6), or (7) and confirm that the market is in equilibrium.

3 Conservative, aggressive, and neutral sentiments in a square-root model

In this section, we present a square-root case where the equilibrium is concretely obtained by solving individual optimization problems for the three types of agents and imposing the market clearing conditions. As we will observe in the general case in Section 5, obtaining an equilibrium reduces to solve a system of BSDEs in general. In this square-root case, the BSDEs reduce to a system of Riccati ODEs, which can be solved numerically.

For clarity, we restate the setting as follows. Let S_0, S_1 be the price process of the money market account and the risky asset satisfying SDEs

$$dS_{0,t} = rS_{0,t}dt,\ \ S_{0,0} = 1,$$
$$dS_{1,t} = \mu_t S_{1,t}dt + \sigma_t S_{1,t}(\rho_{S,t}dW_{Y,t} + \hat{\rho}_{S,t}dW_{S,t}); S_{1,0} = p > 0, \tag{11}$$

and the following square-root process for Y instead of (3)

$$dY_t = (\mu_{Y,1,t}Y_t + \mu_{Y,2,t})dt + \sigma_{Y,t}\sqrt{Y_t}dW_{Y,t}; Y_0 = y_0 > 0, \tag{12}$$

where $\mu_{Y,1}, \mu_{Y,2}, \sigma_Y$ are nonrandom processes with $\mu_{Y,1,t} \leq 0, \mu_{Y,2,t} \geq 0, \sigma_{Y,t} > 0,\ 0 \leq t \leq T$, the initial value of the risky asset price p is exogenously given. We also assume that Λ_i, the set of the views on the fundamental risks λ, has the following square-root form $\Lambda_i = \{\lambda_i = (l_{Y,i,t}\sqrt{Y_t}, l_{S,i,t}\sqrt{Y_t})^\top | -\lambda^\star_{Y,i,t} \leq l_{Y,i,t} \leq \lambda^\star_{Y,i,t}, -\lambda^\star_{S,i,t} \leq l_{S,i,t} \leq \lambda^\star_{S,i,t}\}$, where $\lambda^\star_{Y,i}$ and $\lambda^\star_{S,i}$ are nonnegative random processes. Then, we consider the individual optimization problems for the conservative, the aggressive, and the neutral agents as follows.

First, we note that X^{π_i}, the wealth process of agent i, satisfies an SDE below. That is, $dX^{\pi_i}_t = \frac{\pi_{i,t}}{S_{1,t}}dS_{1,t} + r_t\frac{X^{\pi_i}_t - \pi_{i,t}}{S_{0,t}}dS_{0,t}$ with $r_t = 0$ and a given $x_{i,0}$ such that $\sum_{i=1}^{\bar{I}} x_{i,0} = \pi^s_0$:

$$dX^{\pi_i}_t = \pi_{i,t}\sigma_t\theta_t dt + \pi_{i,t}\sigma_t(\rho_{S,t}dW_{Y,t} + \hat{\rho}_{S,t}dW_{S,t});\ \ X^{\pi_i}_0 = x_{i,0}, \tag{13}$$

where $\theta_t = \frac{\mu_t}{\sigma_t}$.

[**Individual Optimization for the three types of agents**]

- **(Conservative agents)**

  First, we consider the sup-inf/inf-sup problem of the conservative agents i, $i = 1, \ldots, I$,

$$\sup_{\pi_i \in \mathcal{A}_i} \inf_{\lambda_i \in \Lambda_i} E^{P^{\lambda_i}}[-\exp(-\gamma_i(X^{\pi_i}_T + Y_T))],$$
$$\inf_{\lambda_i \in \Lambda_i} \sup_{\pi_i \in \mathcal{A}_i} E^{P^{\lambda_i}}[-\exp(-\gamma_i(X^{\pi_i}_T + Y_T))]. \tag{14}$$

  We consider the set of admissible strategies $\mathcal{A}_i$ as $\mathcal{A}_i = \{\pi_i | X^{\pi_i} \text{is a } Q_i^{\lambda^*_i}\text{-supermartingale}\}$ so that arbitrage opportunities are excluded. In detail, if $\pi_i \in \mathcal{A}_i$, π_i is not an arbitrage strategy for the following reason. Suppose that π_i is an arbitrage opportunity, i.e., $X^{\pi_i}_0 = x_{i,0}$, $X^{\pi_i}_T \geq x_{i,0}$, $P(X^{\pi_i}_T > x_{i,0}) > 0$, then $Q_i^{\lambda^*_i}(X^{\pi_i}_T > x_{i,0}) > 0$ and since X^{π_i} is a supermartingale, $E^{Q_i^{\lambda^*_i}}[X^{\pi_i}_T] \leq X^{\pi_i}_0 = x_{i,0}$, which is a contradiction.

Here, $Q_i^{\lambda_i^*}$, which represents agent i's risk-neutral probability measure, is defined as

$$\frac{dQ_i^{\lambda_i^*}}{dP} = \frac{u_i'(X_T^{\pi_i^*} + Y_T)}{E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*} + Y_T)]} \frac{dP^{\lambda_i^*}}{dP}, \tag{15}$$

where we set $\lambda_{i,t}^* = (+\lambda_{Y,i,t}^\dagger \sqrt{Y_t}, +\lambda_{S,i,t}^\dagger \sqrt{Y_t})^\top$ with $\lambda_{Y,i,t}^\dagger := -\lambda_{Y,i,t}^\star, \lambda_{S,i,t}^\dagger := -\lambda_{S,i,t}^\star$ for $i = 1, \ldots, I$ (conservative agents), $\lambda_{Y,i,t}^\dagger := +\lambda_{Y,i,t}^\star, \lambda_{S,i,t}^\dagger := +\lambda_{S,i,t}^\star$ for $i = I + 1, \ldots, I + I'$ (aggressive agents), $\lambda_{Y,i,t}^\dagger = \lambda_{S,i,t}^\dagger = 0$ for $i = I + I' + 1, \ldots, \bar{I}$ (neutral agents),

$$\pi_{i,t}^* = \frac{1}{\gamma_i \sigma_t}(\theta_t + \rho_{S,t}\lambda_{Y,i,t}^\dagger \sqrt{Y_t} + \hat{\rho}_{S,t}\lambda_{S,i,t}^\dagger \sqrt{Y_t} - \gamma_i \rho_{S,t}\sigma_{Y,t} a_{i,t}^* \sqrt{Y_t}), \tag{16}$$

and

$$\theta_t = \sum_{j=1}^{\bar{I}} \frac{1}{\gamma_j} \Gamma(-\rho_{S,t}\lambda_{Y,j,t}^\dagger \sqrt{Y_t} - \hat{\rho}_{S,t}\lambda_{S,j,t}^\dagger \sqrt{Y_t} + \gamma_j \rho_{S,t} a_{j,t}^* \sigma_{Y,t} \sqrt{Y_t}). \tag{17}$$

Here, $\Gamma = \frac{1}{\sum_{k=1}^{I} \frac{1}{\gamma_k}}$ and $(a_1^*, \ldots, a_I^*)$ is a unique solution of Riccati equations (27) defined in Theorem 1 in the following.

- **(Aggressive agents)**

  In addition to the agents with conservative sentiments, one group of agents has aggressive views on the fundamental risks, and there is another group of agents who do not have any views on the risks. Specifically, suppose that in addition to the I agents who have pessimistic sentiments, there are I' agents, agents $I+1, \ldots, I+I'$, ($I \geq 0$) who have aggressive sentiments on the risks, and I'' agents ($I'' \geq 0$) without views on the risks. That is, instead of the individual optimization problem in (14), the problem for agent i, $i = I + 1, \ldots, I + I'$, is described as

$$\sup_{\lambda_i \in \Lambda_i} \sup_{\pi_i \in \mathcal{A}_i(\lambda_i)} E^{P^{\lambda_i}}[-\exp(-\gamma_i(X_T^{\pi_i} + Y_T))]. \tag{18}$$

Let π_i be a $\{\mathcal{F}_t\}$-progressively measurable process with $\int_0^T \pi_{i,s}^2 \sigma_s^2 ds < \infty, P - a.s.$

For agents i, $i = I+1, \ldots, I+I'$, we consider the set of admissible strategies $\mathcal{A}_i(\lambda_i)$ so that arbitrage opportunities are excluded for each view λ_i as follows. In detail, $\mathcal{A}_i(\lambda_i) = \{\pi_i | X^{\pi_i} \text{is a } Q_i^\lambda\text{-supermartingale}\}$, where Q_i^λ is defined as

$$\frac{dQ_i^\lambda}{dP} = \frac{u_i'(X_T^{\pi_i^{l,*}} + Y_T)}{E^{P^\lambda}[u_i'(X_T^{\pi_i^{l,*}} + Y_T)]} \frac{dP^{\lambda_i}}{dP},$$

where

$$\pi_{i,t}^{l,*} = \frac{1}{\gamma_i \sigma_t}(\theta_t + \rho_{S,t} l_{Y,i,t} \sqrt{Y_t} + \hat{\rho}_{S,t} l_{S,i,t} \sqrt{Y_t} - \gamma_i \rho_{S,t}\sigma_{Y,t} a_{i,t}^l \sqrt{Y_t}), \tag{19}$$

$$\theta_t = \sum_{j=1}^{\bar{I}} \frac{1}{\gamma_j} \Gamma(-\rho_{S,t}\lambda^{\dagger}_{Y,j,t}\sqrt{Y_t} - \hat{\rho}_{S,t}\lambda^{\dagger}_{S,j,t}\sqrt{Y_t} + \gamma_j \rho_{S,t} a^*_{j,t}\sigma_{Y,t}\sqrt{Y_t}). \tag{20}$$

Here, $\Gamma = \frac{1}{\sum_{k=1}^{\bar{I}} \frac{1}{\gamma_k}}$, and $(a^*_1, \ldots, a^*_{\bar{I}})$ and $(a^l_{I+1}, \ldots, a^l_{I+I'})$ are a unique solution of Riccati equations (27) and (30) defined in Theorem 1 in the following.

- **(Neutral agents)**

  Moreover, there is a group of I'' agents, agents i, $i = I + I' + 1, \ldots, \bar{I}$, who are neutral about the views on the risks with individual optimization problems

$$\sup_{\pi_i \in \mathcal{A}_i} E^P[-\exp(-\gamma_i(X_T^{\pi_i} + Y_T))]. \tag{21}$$

  Also, for the neutral agents i, $i = I + I' + 1, \ldots, \bar{I}$, we set the set of admissible strategies as $\mathcal{A}_i$, which is the same as the one for the conservative agents i, $i = 1, \ldots, I$ except for $\lambda^*_i = (0, 0)^\top$.

3.1 Market clearing conditions for zero-net supply case

Next, we define the riskless asset's and the risky asset's market clearing condition as

$$\sum_{i=1}^{\bar{I}} (X_t^{\pi^*_i} - \pi^*_{i,t}) = 0, \tag{22}$$

$$\sum_{i=1}^{\bar{I}} \pi^*_{i,t} = 0, \ \forall t \in [0, T], \tag{23}$$

where we set the supply of the riskless and the risky asset to be zero.

Specifically, assuming $r \equiv 0$, we obtain the Sharpe ratio process θ and the volatility process σ in equilibrium, which satisfy the market clearing conditions as follows.

Theorem 1 *Suppose that for the systems of Riccati equations (i) and (ii) in the following, each system has a unique global solution in* $[0, T]$.

Then, the Sharpe ratio process θ *in equilibrium is given by*

$$\theta_t = \sqrt{Y_t}\Gamma \left[-\sum_{i=1}^{\bar{I}} \frac{1}{\gamma_i} (\rho_{S,t}\lambda^{\dagger}_{Y,i,t} + \hat{\rho}_{S,t}\lambda^{\dagger}_{S,i,t} - \gamma_i \rho_{S,t}\sigma_{Y,t} a^*_{i,t}) \right]. \tag{24}$$

As a result, we obtain the optimal portfolio and the sentiment (π^*_i, λ^*_i) *that attains the individual optimization problems, the sup-inf/inf-sup, the sup-sup, and the sup problem (14),(18),(21), and satisfy the market clearing condition as*

$$\pi^*_{i,t} = \frac{1}{\gamma_i \sigma_t}(\theta_t + \rho_{S,t}\lambda^{\dagger}_{Y,i,t}\sqrt{Y_t} + \hat{\rho}_{S,t}\lambda^{\dagger}_{S,i,t}\sqrt{Y_t} - \gamma_i \rho_{S,t} a^*_{i,t}\sigma_{Y,t}\sqrt{Y_t}). \tag{25}$$

Moreover, the expected return process is given as

$$\mu_t = \sigma_t\sqrt{Y_t}\Gamma\left[-\sum_{i=1}^{\bar{I}}\frac{1}{\gamma_i}(\rho_{S,t}\lambda^{\dagger}_{Y,i,t} + \hat{\rho}_{S,t}\lambda^{\dagger}_{S,i,t} - \gamma_i\rho_{S,t}\sigma_{Y,t}a^*_{i,t})\right]. \tag{26}$$

Here, (i) the system of Riccati equations for $i = 1, \ldots, \bar{I}$ (conservative, aggressive, and neutral agents) is given as

$$\begin{aligned} &-\dot{a}^*_{i,t} \\ &= \frac{1}{2\gamma_i}\left(\sum_{j=1}^{\bar{I}}\frac{1}{\gamma_j}\Gamma(-\rho_{S,t}\lambda^{\dagger}_{Y,j,t} - \hat{\rho}_{S,t}\lambda^{\dagger}_{S,j,t} + \gamma_j\rho_{S,t}a^*_{j,t}\sigma_{Y,t}) + \rho_{S,t}\lambda^{\dagger}_{Y,i,t} + \hat{\rho}_{S,t}\lambda^{\dagger}_{S,i,t} - \gamma_i\rho_{S,t}a^*_{i,t}\sigma_{Y,t}\right)^2 \\ &\quad + a^*_{i,t}(\mu_{Y,1,t} + \lambda^{\dagger}_{Y,i,t}\sigma_{Y,t}) - \frac{1}{2}\gamma_i(a^*_{i,t})^2\sigma^2_{Y,t}, \\ &a^*_{i,T} = 1;\ \Gamma = \frac{1}{\sum_{k=1}^{\bar{I}}\frac{1}{\gamma_k}}, \end{aligned} \tag{27}$$

subject to for $i = 1, \ldots I + I'$ (conservative and aggressive agents),

$$\begin{aligned} &\frac{\rho_{S,t}}{\gamma_i}\left(\sum_{j=1}^{\bar{I}}\frac{1}{\gamma_j}\Gamma(-\rho_{S,t}\lambda^{\dagger}_{Y,j,t} - \hat{\rho}_{S,t}\lambda^{\dagger}_{S,j,t} + \gamma_j\rho_{S,t}\sigma_{Y,t}a^*_{j,t}) - \rho_{S,t}\lambda^{\star}_{Y,i,t} - \hat{\rho}_{S,t}\lambda^{\star}_{S,i,t} - \gamma_i\rho_{S,t}\sigma_{Y,t}a^*_{i,t}\right) \\ &\quad + \sigma_{Y,t}a^*_{i,t} \geq 0, \end{aligned} \tag{28}$$

$$\frac{\hat{\rho}_{S,t}}{\gamma_i}\left(\sum_{j=1}^{\bar{I}}\frac{1}{\gamma_j}\Gamma(-\rho_{S,t}\lambda^{\dagger}_{Y,j,t} - \hat{\rho}_{S,t}\lambda^{\dagger}_{S,j,t} + \gamma_j\rho_{S,t}\sigma_{Y,t}a^*_{j,t}) - \rho_{S,t}\lambda^{\star}_{Y,i,t} - \hat{\rho}_{S,t}\lambda^{\star}_{S,i,t} - \gamma_i\rho_{S,t}\sigma_{Y,t}a^*_{i,t}\right) \geq 0, \tag{29}$$

and (ii) for arbitrary $-\lambda^{\star}_{Y,i,t} \leq l_{Y,i,t} < \lambda^{\star}_{Y,i,t}, -\lambda^{\star}_{S,i,t} \leq l_{S,i,t} < \lambda^{\star}_{S,i,t},\ i = I+1, \ldots, I+I'$ (aggressive agents), the system of Riccati equations is given as

$$\begin{aligned} &-\dot{a}^l_{i,t} \\ &= \frac{1}{2\gamma_i}(\sum_{j=1}^{\bar{I}}\frac{1}{\gamma_j}\Gamma(-\rho_{S,t}\lambda^{\dagger}_{Y,j,t} - \hat{\rho}_{S,t}\lambda^{\dagger}_{S,j,t} + \gamma_j\rho_{S,t}a^*_{j,t}\sigma_{Y,t}) + \rho_{S,t}l_{Y,i,t} + \hat{\rho}_{S,t}l_{S,i,t} - \gamma_i\rho_{S,t}a^l_{i,t}\sigma_{Y,t})^2 \\ &\quad + a^l_{i,t}(\mu_{Y,1,t} + l_{Y,i,t}\sigma_{Y,t}) - \frac{1}{2}\gamma_i(a^l_{i,t})^2\sigma^2_{Y,t}, \\ &a^l_{i,T} = 1,\ i = I+1, \ldots, I+I'. \end{aligned} \tag{30}$$

Remark 2 *Regarding the solutions of the systems of Riccati equations (27) to obtain θ, π^*_i for all types of agents, and (30) to provide $\pi^{l,*}_i$ used in the proof for the individual optimization problem in the aggressive case in Section 3.1.2, by Picard-Lindelöf theorem of ODEs (e.g., Theorems 2.2 and 2.1.3 in Teschle[33]), each system of Riccati equations has a unique solution up to some blow-up time $T_{blow-up}$ with respect to time to maturity $\tau = T - t$, i.e., the unique solution exists for $\tau \in [0, T_{blow-up})$, since the coefficients are locally Lipschitz continuous. Thus, in the Theorem 1, we suppose that $T < T_{blow-up}$. In the numerical example in Section 4, we will show a case where the Riccati equations are numerically solved, and the solutions satisfy the conditions (28) and (29).*

Remark 3 *Since*

$$\begin{aligned} dS_{1,t} &= \mu_t S_{1,t}dt + \sigma_t S_{1,t}(\rho_{S,t}dW_{Y,t} + \hat{\rho}_{S,t}dW_{S,t}) \\ &= (\mu_t + \sigma_t(\rho_{S,t}\lambda^{\dagger}_{Y,i,t}\sqrt{Y_t} + \hat{\rho}_{S,t}\lambda^{\dagger}_{S,i,t}\sqrt{Y_t}))S_{1,t}dt + \sigma_t S_{1,t}(\rho_{S,t}dW^{\lambda^*_i}_{Y,t} + \hat{\rho}_{S,t}dW^{\lambda^*_i}_{S,t}), \end{aligned} \tag{31}$$

$\lambda^{\dagger}_{S,i,t} = -\lambda^{\star}_{S,i,t}$ and $\lambda^{\dagger}_{Y,i,t} = -\lambda^{\star}_{Y,i,t}$ indicate that agent i sees less expected return on the risky asset process under the subjective probability measure $P^{\lambda^*_i}$ than the return under the physical probability measure P assuming that $\sigma_t, \rho_{S,t}, \hat{\rho}_{S,t} > 0$. As in the proof below, the conditions (28) and (29) make the individual optimization for a conservative agent attained at the low ends for the views ($\lambda^{\dagger}_{S,i,t} = -\lambda^{\star}_{S,i,t}$, $\lambda^{\dagger}_{Y,i,t} = -\lambda^{\star}_{Y,i,t}$), and the problem for an aggressive agent attained at the high ends ($\lambda^{\dagger}_{S,i,t} = +\lambda^{\star}_{S,i,t}$, $\lambda^{\dagger}_{Y,i,t} = +\lambda^{\star}_{Y,i,t}$). We also note that conditions (28) and (29) imply the following with the expression of $\pi^*_{i,t}$ as

$$\begin{aligned} &\pi^*_{i,t}\frac{\sigma_t\rho_{S,t}}{\sqrt{Y_t}} + \sigma_{Y,t}a^*_{i,t} \\ &\geq \frac{\rho_{S,t}}{\gamma_i}\left(\sum_{j=1}^{\bar{I}}\frac{1}{\gamma_j}\Gamma(-\rho_{S,t}\lambda^{\dagger}_{Y,j,t} - \hat{\rho}_{S,t}\lambda^{\dagger}_{S,j,t} + \gamma_j\rho_{S,t}\sigma_{Y,t}a^*_{j,t}) - \rho_{S,t}\lambda^{\star}_{Y,i,t} - \hat{\rho}_{S,t}\lambda^{\star}_{S,i,t} - \gamma_i\rho_{S,t}\sigma_{Y,t}a^*_{i,t}\right) \\ &\quad + \sigma_{Y,t}a^*_{i,t} \geq 0, \end{aligned} \tag{32}$$

and

$$\begin{aligned} &\pi^*_{i,t}\frac{\sigma_t\hat{\rho}_{S,t}}{\sqrt{Y_t}} \\ &\geq \frac{\hat{\rho}_{S,t}}{\gamma_i}\left(\sum_{j=1}^{\bar{I}}\frac{1}{\gamma_j}\Gamma(-\rho_{S,t}\lambda^{\dagger}_{Y,j,t} - \hat{\rho}_{S,t}\lambda^{\dagger}_{S,j,t} + \gamma_j\rho_{S,t}\sigma_{Y,t}a^*_{j,t}) - \rho_{S,t}\lambda^{\star}_{Y,i,t} - \hat{\rho}_{S,t}\lambda^{\star}_{S,i,t} - \gamma_i\rho_{S,t}\sigma_{Y,t}a^*_{i,t}\right) \\ &\geq 0, \end{aligned} \tag{33}$$

respectively. Particularly, (33) indicates the agent takes a long position under the current parameter settings $\sigma_t, \hat{\rho}_{S,t} > 0$. In addition, (32) automatically follows when (33) is satisfied under the assumption that $\sigma_{Y,t}$, $\sigma_t, \rho_{S,t}, \hat{\rho}_{S,t} > 0$, $a^*_{i,T} = 1$ implying that $a^*_{i,t}$ is most likely to be nonnegative. Moreover, we may consider the opposite case where some agents are short on the risky asset, which indicates that the agents see a higher expected return on the short risky asset positions under $P^{\lambda^*_i}$ than the drift under P and gives the opposite results, i.e., the views are attained at the high ends for the conservative agents and the low ends for the aggressive agents. For instance, while the conservative agents take long positions, we may consider the case without neutral agents, where the aggressive agents take short positions by replacing inequalities (28) and (29) with the following: For $i = I+1, \ldots, I+I'$,

$$\begin{aligned} &\pi^*_{i,t}\frac{\sigma_t\rho_{S,t}}{\sqrt{Y_t}} + \sigma_{Y,t}a^*_{i,t} = \\ &\frac{\rho_{S,t}}{\gamma_i}\left(\sum_{j=1}^{\bar{I}}\frac{1}{\gamma_j}\Gamma(-\rho_{S,t}\lambda^{\dagger}_{Y,j,t} - \hat{\rho}_{S,t}\lambda^{\dagger}_{S,j,t} + \gamma_j\rho_{S,t}\sigma_{Y,t}a^*_{j,t}) + \rho_{S,t}\lambda^{\star}_{Y,i,t} + \hat{\rho}_{S,t}\lambda^{\star}_{S,i,t} - \gamma_i\rho_{S,t}\sigma_{Y,t}a^*_{i,t}\right) \\ &\quad + \sigma_{Y,t}a^*_{i,t} \leq 0, \end{aligned} \tag{34}$$

and

$$\pi^*_{i,t}\frac{\sigma_t\hat{\rho}_{S,t}}{\sqrt{Y_t}} =$$

$$\frac{\hat{\rho}_{S,t}}{\gamma_i}\left(\sum_{j=1}^{\bar{I}}\frac{1}{\gamma_j}\Gamma(-\rho_{S,t}\lambda^{\dagger}_{Y,j,t}-\hat{\rho}_{S,t}\lambda^{\dagger}_{S,j,t}+\gamma_j\rho_{S,t}\sigma_{Y,t}a^*_{j,t})+\rho_{S,t}\lambda^{\star}_{Y,i,t}+\hat{\rho}_{S,t}\lambda^{\star}_{S,i,t}-\gamma_i\rho_{S,t}\sigma_{Y,t}a^*_{i,t}\right)$$

$$\leq 0. \tag{35}$$

Proof of Theorem 1 The following shows that the individual optimization problem is attained with (π^*_i, λ^*_i) in the conservative agents' case in Section 3.1.1. We provide the proof for agents with aggressive or neutral views in Section 3.1.2. Then, we prove that the market is in equilibrium, i.e., the market clearing conditions are satisfied.

We confirm that the market is in equilibrium as follows.

Since

$$\pi^*_{i,t} = \frac{1}{\gamma_i\sigma_t}(\theta_t + \rho_{S,t}\lambda^{\dagger}_{Y,i,t}\sqrt{Y_t} + \hat{\rho}_{S,t}\lambda^{\dagger}_{S,i,t}\sqrt{Y_t} - \gamma_i\rho_{S,t}a^*_{i,t}\sigma_{Y,t}\sqrt{Y_t}), \tag{36}$$

we have

$$\sigma_t\sum_{i=1}^{\bar{I}}\pi^*_{i,t}$$

$$= \sum_{i=1}^{\bar{I}}\frac{1}{\gamma_i}(\theta_t + \rho_{S,t}\lambda^{\dagger}_{Y,i,t}\sqrt{Y_t} + \hat{\rho}_{S,t}\lambda^{\dagger}_{S,i,t}\sqrt{Y_t} - \gamma_i\rho_{S,t}a^*_{i,t}\sigma_{Y,t}\sqrt{Y_t})$$

$$= (\sum_{i=1}^{\bar{I}}\frac{1}{\gamma_i})\theta_t - \sum_{i=1}^{\bar{I}}\frac{1}{\gamma_i}(-\rho_{S,t}\lambda^{\dagger}_{Y,i,t}\sqrt{Y_t} - \hat{\rho}_{S,t}\lambda^{\dagger}_{S,i,t}\sqrt{Y_t} + \gamma_i\rho_{S,t}a^*_{i,t}\sigma_{Y,t}\sqrt{Y_t})$$

$$= 0.$$

Thus, $\sum_{i=1}^{\bar{I}}\pi^*_{i,t} = 0$, which indicates (23).

Also, (22) follows from (13) and (23). ■

3.1.1 Proof for (π^*_i, λ^*_i) attaining the individual optimization in the conservative case

First, we show that (π^*_i, λ^*_i) attains the individual optimization problem for the agents with conservative sentiments.

Let

$$J_i(\pi_i, \lambda_i) = E^{P^{\lambda_i}}[-\exp(-\gamma_i(X^{\pi_i}_T + Y_T))].$$

First, an inequality $\sup_{\pi_i\in\mathcal{A}_i}\inf_{\lambda_i\in\Lambda_i}J_i(\lambda_i,\pi_i) \leq \inf_{\lambda_i\in\Lambda_i}\sup_{\pi_i\in\mathcal{A}_i}J_i(\lambda_i,\pi_i)$ naturally holds since the admissible set $\mathcal{A}_i$ is independent of λ_i, where $\mathcal{A}_i$ is a set of strategies π_i such that X^{π_i} is a supermartingale under $Q^{\lambda^*_i}_i$, where $Q^{\lambda^*_i}_i$ is defined by (15).

The opposite side of the inequality $\sup_{\pi_i\in\mathcal{A}_i}\inf_{\lambda_i\in\Lambda_i}J_i(\lambda_i,\pi_i) \geq \inf_{\lambda_i\in\Lambda_i}\sup_{\pi_i\in\mathcal{A}_i}J_i(\lambda_i,\pi_i)$ also holds, which can be proved by showing (λ^*_i, π^*_i) is the saddle point, i.e.,

$$J_i(\lambda^*_i, \pi_i) \leq J_i(\lambda^*_i, \pi^*_i) \leq J_i(\lambda_i, \pi^*_i),\ \forall\lambda_i\in\Lambda_i, \pi_i\in\mathcal{A}_i. \tag{37}$$

Thus, in this square-root case, the inf-sup and the sup-inf case are solved and proved to coincide.

In the following, we will show that the first part of (37)

$$J_i(\lambda_i^*, \pi_i) \le J_i(\lambda_i^*, \pi_i^*), \tag{38}$$

follows from the convex duality argument and the second part of (37)

$$J_i(\lambda_i^*, \pi_i^*) \le J_i(\lambda_i, \pi_i^*), \tag{39}$$

follows from the martingale representation of R_i under $P^{\lambda_i^*}$, where we define $R_{i,t} = -\exp(-\gamma_i(X_t^{\pi_i^*} + V_{i,t}))$ with V_i satisfying following BSDE.

For $\lambda_{i,t}^* = (+\lambda_{Y,i,t}^\dagger \sqrt{Y_t}, +\lambda_{S,i,t}^\dagger \sqrt{Y_t})^\top$, we consider a BSDE under $P^{\lambda_i^*}$

$$\begin{cases} dV_{i,t} = -f_i(Z_{i,t})dt + Z_{i,t}dW_{Y,t}^{\lambda_i^*}, \\ V_{i,T} = Y_T, \end{cases} \tag{40}$$

with

$$f_i(Z_{i,t}) = \frac{1}{2\gamma_i}(\theta_t^{\lambda_i^*} - \gamma_i \rho_{S,t} Z_{i,t})^2 - \frac{1}{2}\gamma_i Z_{i,t}^2, \tag{41}$$

$$\theta_t^{\lambda_i^*} = \theta_t + \rho_{S,t}\lambda_{Y,i,t}^\dagger \sqrt{Y_t} + \hat{\rho}_{S,t}\lambda_{S,i,t}^\dagger \sqrt{Y_t},$$

$$\theta_t = \sum_{j=1}^{\bar{I}} \Gamma \frac{1}{\gamma_j}(-\rho_{S,t}\lambda_{Y,j,t}^\dagger \sqrt{Y_t} - \hat{\rho}_{S,t}\lambda_{S,j,t}^\dagger \sqrt{Y_t} + \gamma_j \rho_{S,t} a_{j,t}^* \sigma_{Y,t}\sqrt{Y_t}),$$

$$Y_T = y_0 + \int_0^T ((\mu_{Y,1,t} + \lambda_{Y,i,t}^\dagger \sigma_{Y,t})Y_t + \mu_{Y,2,t})dt + \int_0^T \sigma_{Y,t}\sqrt{Y_t}dW_{Y,t}^{\lambda_i^*},$$

which can be solved as follows.

We show that V_i expressed as

$$V_{i,t} = a_{i,t}^* Y_t + b_{i,t}^*, \;\; a_{i,T}^* = 1, b_{i,T}^* = 0,$$

satisfies the BSDE (40), where $a_{i,t}^*, b_{i,t}^*$ are nonrandom processes differentiable with respect to t satisfying Riccati equations (27) and $-\dot{b}_{i,t}^* = a_{i,t}^* \mu_{Y,2,t}$.

Calculating $dV_{i,t}$ and comparing it with BSDE (40),

$$\begin{aligned} dV_{i,t} &= a_{i,t}^* dY_t + Y_t \dot{a}_{i,t}^* dt + \dot{b}_{i,t}^* dt \\ &= \{(a_{i,t}^*(\mu_{Y,1,t} + \lambda_{Y,i,t}^\dagger \sigma_{Y,t}) + \dot{a}_{i,t}^*)Y_t + a_{i,t}^* \mu_{Y,2,t} + \dot{b}_{i,t}^*\}dt + a_{i,t}^* \sigma_{Y,t}\sqrt{Y_t}dW_{Y,t}^{\lambda_i^*}, \end{aligned}$$

we have

$$Z_{i,t} = a_{i,t}^* \sigma_{Y,t}\sqrt{Y_t}. \tag{42}$$

Since f_i in (41) becomes

$$\frac{1}{2\gamma_i}(\sum_{j=1}^{\bar{I}} \frac{1}{\gamma_j}\Gamma(-\rho_{S,t}\lambda_{Y,j,t}^\dagger \sqrt{Y_t} - \hat{\rho}_{S,t}\lambda_{S,j,t}^\dagger \sqrt{Y_t} + \gamma_j \rho_{S,t} a_{j,t}^* \sigma_{Y,t}\sqrt{Y_t})$$
$$+\rho_{S,t}\lambda_{Y,i,t}^\dagger \sqrt{Y_t} + \hat{\rho}_{S,t}\lambda_{S,i,t}^\dagger \sqrt{Y_t} - \gamma_i \rho_{S,t} Z_{i,t})^2 - \frac{1}{2}\gamma_i Z_{i,t}^2,$$

substituting the expression of Z_i in (42), we have

$$\left(\frac{1}{2\gamma_i}(\sum_{j=1}^{\bar{I}}\frac{\Gamma}{\gamma_j}(-\rho_{S,t}\lambda^\dagger_{Y,j,t}-\hat{\rho}_{S,t}\lambda^\dagger_{S,j,t}+\gamma_j\rho_{S,t}a^*_{j,t}\sigma_{Y,t})+\rho_{S,t}\lambda^\dagger_{Y,i,t}+\hat{\rho}_{S,t}\lambda^\dagger_{S,i,t}-\gamma_i\rho_{S,t}a^*_{i,t}\sigma_{Y,t})^2\right.$$
$$\left.-\frac{\gamma_i}{2}(a^*_{i,t})^2\sigma^2_{Y,t}\right)Y_t,$$

which is equivalent to

$$-\{(a^*_{i,t}(\mu_{Y,t}+\lambda^\dagger_{Y,i,t}\sigma_{Y,t})+\dot{a}^*_{i,t})Y_t+a^*_{i,t}\mu_{Y,2,t}+\dot{b}^*_{i,t}\}.$$

Hence, we obtain the system of Riccati equations in (27),

$$-\dot{b}^*_{i,t}=a^*_{i,t}\mu_{Y,2,t},$$
$$-\dot{a}^*_{i,t}=$$
$$\left(\frac{1}{2\gamma_i}(\sum_{j=1}^{\bar{I}}\frac{\Gamma}{\gamma_j}(-\rho_{S,t}\lambda^\dagger_{Y,j,t}-\hat{\rho}_{S,t}\lambda^\dagger_{S,j,t}+\gamma_j\rho_{S,t}a^*_{j,t}\sigma_{Y,t})\right.$$
$$\left.+\rho_{S,t}\lambda^\dagger_{Y,i,t}+\hat{\rho}_{S,t}\lambda^\dagger_{S,i,t}-\gamma_i\rho_{S,t}a^*_{i,t}\sigma_{Y,t})^2-\frac{\gamma_i}{2}(a^*_{i,t})^2\sigma^2_{Y,t}\right)+a^*_{i,t}(\mu_{Y,t}+\lambda^\dagger_{Y,i,t}\sigma_{Y,t}). \tag{43}$$

By the assumption, the system of Riccati equations has a unique solution $(a^*_1,\dots,a^*_I)$ in $[0,T]$ that satisfies the conditions (28) and (29), and thus (V_i,Z_i) is a solution of BSDE (40).

Step 1. First, for $\lambda^*_{i,t}=(+\lambda^\dagger_{Y,i,t}\sqrt{Y_t},+\lambda^\dagger_{S,i,t}\sqrt{Y_t})^\top$, we show that π^*_i where $\pi^*_{i,t}=\frac{1}{\gamma_i\sigma_t}(\theta_t+\rho_{S,t}\lambda^\dagger_{Y,i,t}\sqrt{Y_t}+\hat{\rho}_{S,t}\lambda^\dagger_{S,i,t}\sqrt{Y_t}-\gamma_i\rho_{S,t}a^*_{i,t}\sigma_{Y,t}\sqrt{Y_t})$ attains the sup. That is, (38) holds.

Concretely, we consider

$$\sup_{\pi_i\in\mathcal{A}_i}E^{P^{\lambda^*_i}}[-\exp(-\gamma_i(X^{\pi_i}_T+Y_T))],$$

where

$$dY_t=[(\mu_{Y,1,t}+\sigma_{Y,t}\lambda^\dagger_{Y,i,t})Y_t+\mu_{Y,2,t}]dt+\sigma_{Y,t}\sqrt{Y_t}dW^{\lambda^*_i}_{Y,t},$$
$$dX^{\pi_i}_t=\pi_{i,t}\sigma_t(\theta_t+\rho_{S,t}\lambda^\dagger_{Y,i,t}\sqrt{Y_t}+\hat{\rho}_{S,t}\lambda^\dagger_{S,i,t}\sqrt{Y_t})dt$$
$$+\pi_{i,t}\sigma_t(\rho_{S,t}dW^{\lambda^*_i}_{Y,t}+\hat{\rho}_{S,t}dW^{\lambda^*_i}_{S,t}),$$

$$dW^{\lambda^*_i}_{Y,t}=dW_{Y,t}-\lambda^\dagger_{Y,i,t}\sqrt{Y_t}dt,$$
$$dW^{\lambda^*_i}_{S,t}=dW_{S,t}-\lambda^\dagger_{S,i,t}\sqrt{Y_t}dt.$$

First, we note the following martingale property for R_i, where $R_{i,t}=-\exp(-\gamma_i(X^{\pi^*_i}_t+V_{i,t}))$, and express the risk-neutral probability measure of agent i, $Q^{\lambda^*_i}_i$ in (15), by R_i.

Lemma 1 *For R_i defined as $R_{i,t}=-\exp(-\gamma_i(X^{\pi^*_i}_t+V_{i,t}))$, R_i is a $P^{\lambda^*_i}$-martingale.*

Proof.

$$
\begin{aligned}
dR_{i,t} &= -\gamma_i R_{i,t} d(X_t^{\pi_i^*} + V_{i,t}) + \frac{1}{2}\gamma_i^2 R_{i,t} d\langle X^{\pi_i^*} + V_i\rangle_t \\
&= -\gamma_i R_{i,t}\bigg((\pi_{i,t}^* \sigma_t \theta_t^{\lambda_i^*} - \frac{1}{2}\gamma_i((\pi_{i,t}^*\sigma_t\rho_{S,t} + Z_{i,t})^2 + (\pi_{i,t}^*\sigma_t\hat{\rho}_{S,t})^2) - f_i(Z_{i,t}))dt \\
&\quad + (\pi_{i,t}^*\sigma_t\rho_{S,t} + Z_{i,t})dW_{Y,t}^{\lambda_i^*} + \pi_{i,t}^*\sigma_t\hat{\rho}_{S,t}dW_{S,t}^{\lambda_i^*}\bigg) \\
&= -\gamma_i R_{i,t}\bigg((\pi_{i,t}^*\sigma_t\rho_{S,t} + Z_{i,t})dW_{Y,t}^{\lambda_i^*} + \pi_{i,t}^*\sigma_t\hat{\rho}_{S,t}dW_{S,t}^{\lambda_i^*}\bigg),
\end{aligned}
\tag{44}
$$

where the drift part is calculated as

$$
\begin{aligned}
&(\pi_{i,t}^*\sigma_t\theta_t^{\lambda^*} - \frac{1}{2}\gamma_i((\pi_{i,t}^*\sigma_t\rho_{S,t} + Z_{i,t})^2 + (\pi_{i,t}^*\sigma_t\hat{\rho}_{S,t})^2) - f_i(Z_{i,t})) \\
&= -\frac{1}{2}\gamma_i\sigma_t^2(\pi_{i,t}^* - \frac{1}{\gamma_i\sigma_t}(\theta_t^{\lambda_i^*} - \gamma_i\rho_{S,t}Z_{i,t}))^2 = 0,
\end{aligned}
$$

where $Z_{i,t} = a_{i,t}^*\sigma_{Y,t}\sqrt{Y_t}$, and by Theorem 3.2 in Shirakawa [31], the weak version of Novikov condition holds and R_i is a $P^{\lambda_i^*}$-martingale. ■

Next, $Q_i^{\lambda_i^*}$ in (15), the risk-neutral probability measure of agent i, is expressed as

$$
\frac{dQ_i^{\lambda_i^*}}{dP^{\lambda_i^*}} = \frac{u_i'(X_T^{\pi_i^*} + Y_T)}{E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*} + Y_T)]} = \frac{R_{i,T}}{E^{P^{\lambda_i^*}}[R_{i,T}]},
\tag{45}
$$

where

$$
u_i'(x) = \gamma_i \exp(-\gamma_i x).
$$

Since

$$
\begin{aligned}
u_i'(X_t^{\pi_i^*} + V_{i,t}) &= \gamma_i \exp(-\gamma_i(X_t^{\pi_i^*} + V_{i,t})) \\
&= -\gamma_i R_{i,t},
\end{aligned}
$$

and by (44)

$$
d(-\gamma_i R_{i,t}) = -\gamma_i R_{i,t}(-\gamma_i(\pi_{i,t}^*\sigma_t\rho_{S,t} + Z_{i,t})dW_{Y,t}^{\lambda_i^*} - \gamma_i\pi_{i,t}^*\sigma_t\hat{\rho}_{S,t}dW_{S,t}^{\lambda_i^*}).
$$

By Girsanov's theorem, $(W_S^{Q_i^{\lambda_i^*}}, W_Y^{Q_i^{\lambda_i^*}})$ defined by

$$
\begin{aligned}
dW_{Y,t}^{Q_i^{\lambda_i^*}} &= dW_{Y,t}^{\lambda_i^*} + \gamma_i(\pi_{i,t}^*\sigma_t\rho_{S,t} + Z_{i,t})dt, \\
dW_{S,t}^{Q_i^{\lambda_i^*}} &= dW_{S,t}^{\lambda_i^*} + \gamma_i\pi_{i,t}^*\sigma_t\hat{\rho}_{S,t}dt,
\end{aligned}
$$

is a $Q_i^{\lambda_i^*}$-Brownian motion.

Then, by $\pi^*_{i,t} = \frac{1}{\gamma_i \sigma_t}(\theta_t + \rho_{S,t}\lambda^\dagger_{Y,i,t}\sqrt{Y_t} + \hat{\rho}_{S,t}\lambda^\dagger_{S,i,t}\sqrt{Y_t} - \gamma_i \rho_{S,t} a^*_{i,t}\sigma_{Y,t}\sqrt{Y_t}) = \frac{1}{\gamma_i \sigma_t}(\theta^{\lambda^*_i}_t - \gamma_i \rho_{S,t} a^*_{i,t}\sigma_{Y,t}\sqrt{Y_t})$,

$$\rho_{S,t} dW^{\lambda^*_i}_{Y,t} + \hat{\rho}_{S,t} dW^{\lambda^*_i}_{S,t} = \rho_{S,t} dW^{Q^{\lambda^*_i}_i}_{Y,t} + \hat{\rho}_{S,t} dW^{Q^{\lambda^*_i}_i}_{S,t} - \theta^{\lambda^*_i}_t dt,$$

and thus by

$$\begin{aligned} dX^{\pi_i}_t &= \pi_{i,t}\sigma_t(\theta_t + \rho_{S,t}\lambda^\dagger_{Y,i,t}\sqrt{Y_t} + \hat{\rho}_{S,t}\lambda^\dagger_{S,i,t}\sqrt{Y_t})dt + \pi_{i,t}\sigma_t(\rho_{S,t} dW^{\lambda^*_i}_{Y,t} + \hat{\rho}_{S,t} dW^{\lambda^*_i}_{S,t}) \\ &= \pi_{i,t}\sigma_t(\theta^{\lambda^*_i}_t dt + \rho_{S,t} dW^{\lambda^*_i}_{Y,t} + \hat{\rho}_{S,t} dW^{\lambda^*_i}_{S,t}), \end{aligned} \tag{46}$$

we have

$$dX^{\pi_i}_t = \pi_{i,t}\sigma_t(\rho_{S,t} dW^{Q^{\lambda^*_i}_i}_{Y,t} + \hat{\rho}_{S,t} dW^{Q^{\lambda^*_i}_i}_{S,t}).$$

By the assumption, for $\pi_i \in \mathcal{A}_i$, X^{π_i} is a $Q^{\lambda^*_i}_i$-supermartingale. Also, $X^{\pi^*_i}$ is a $Q^{\lambda^*_i}_i$-martingale since $E^{Q^{\lambda^*_i}_i}[\int_0^T (\pi^*_{i,t})^2 \sigma^2_t dt] < \infty$, where $\pi^*_{i,t}\sigma_t = \frac{1}{\gamma_i}(\theta_t + \rho_{S,t}\lambda^\dagger_{Y,i,t}\sqrt{Y_t} + \hat{\rho}_{S,t}\lambda^\dagger_{S,i,t}\sqrt{Y_t} - \gamma_i \rho_{S,t}\sigma_{Y,t} a^*_{i,t}\sqrt{Y_t})$, $\theta_t = \sum_{j=1}^{\bar{I}} \frac{1}{\gamma_j}\Gamma(-\rho_{S,t}\lambda^\dagger_{Y,j,t}\sqrt{Y_t} - \hat{\rho}_{S,t}\lambda^\dagger_{S,j,t}\sqrt{Y_t} + \gamma_j \rho_{S,t} a^*_{j,t}\sigma_{Y,t}\sqrt{Y_t})$, which is due to the integrability of Y and the fact that $a^*_j,\ j = 1, \ldots, I$ are continuous functions bounded on $[0, T]$.

Moreover, we remark that

$$\begin{aligned} dS_{1,t} &= \mu_t S_{1,t} dt + \sigma_t S_{1,t}(\rho_{S,t} dW_{Y,t} + \hat{\rho}_{S,t} dW_{S,t}) \\ &= (\theta_t + (\rho_{S,t}\lambda^\dagger_{Y,i,t}\sqrt{Y_t} + \hat{\rho}_{S,t}\lambda^\dagger_{S,i,t}\sqrt{Y_t}))\sigma_t S_{1,t} dt + \sigma_t S_{1,t}(\rho_{S,t} dW^{\lambda^*_i}_{Y,t} + \hat{\rho}_{S,t} dW^{\lambda^*_i}_{S,t}) \\ &= \sigma_t S_{1,t}(\rho_{S,t} dW^{Q^{\lambda^*_i}_i}_{Y,t} + \hat{\rho}_{S,t} dW^{Q^{\lambda^*_i}_i}_{S,t}), \end{aligned} \tag{47}$$

which indicates that S_1 is also a $Q^{\lambda^*_i}_i$-local martingale.

Then, by using the convex duality argument, we show that π^*_i attains the supremum.

Specifically, we show

$$\begin{aligned} &E^{P^{\lambda^*_i}}[-\exp(-\gamma_i(X^{\pi_i}_T + Y_T))] \\ &\leq E^{P^{\lambda^*_i}}[-\exp(-\gamma_i(X^{\pi^*_i}_T + Y_T))], \forall \pi_i \in \mathcal{A}_i, \end{aligned}$$

by a convex duality argument.

We note that the following properties on the convex duality hold.

Let

$$\tilde{u}_i(y) = \sup_{x \in \mathcal{R}}(u_i(x) - xy),$$

for all $y > 0$, where $u_i(x) = -\exp(-\gamma_i x)$.

Then, for all $x \in \mathcal{R}$, $y > 0$,

$$u_i(x) \leq \tilde{u}_i(y) + yx, \tag{48}$$

$$\tilde{u}_i(u'_i(x)) + u'_i(x)x = u_i(x). \tag{49}$$

By (48),

$$u_i(X_T^{\pi_i}+Y_T) \leq \tilde{u}_i(E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_T)]\frac{dQ_i^{\lambda_i^*}}{dP^{\lambda_i^*}}) + E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_T)]\frac{dQ_i^{\lambda_i^*}}{dP^{\lambda_i^*}}(X_T^{\pi_i}+Y_T),$$

where we set

$$x = X_T^{\pi_i}+Y_T,$$
$$y = E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_T)]\frac{dQ_i^{\lambda_i^*}}{dP^{\lambda_i^*}}.$$

$$E^{P^{\lambda_i^*}}[-\exp(-\gamma_i(X_T^{\pi_i}+Y_T))] = E^{P^{\lambda_i^*}}[u_i(X_T^{\pi_i}+Y_T)]$$
$$\leq E^{P^{\lambda_i^*}}[\tilde{u}_i(E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_T)]\frac{dQ_i^{\lambda_i^*}}{dP^{\lambda_i^*}})] + E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_T)]E^{P^{\lambda_i^*}}[\frac{dQ_i^{\lambda_i^*}}{dP^{\lambda_i^*}}(X_T^{\pi_i}+Y_T)]$$
$$= E^{P^{\lambda_i^*}}[\tilde{u}_i(u_i'(X_T^{\pi_i^*}+Y_T))] + E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_T)]E^{Q_i^{\lambda_i^*}}[(X_T^{\pi_i}+Y_T)] \tag{50}$$
$$\leq E^{P^{\lambda_i^*}}[\tilde{u}_i(u_i'(X_T^{\pi_i^*}+Y_T))] + E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_T)]E^{Q_i^{\lambda_i^*}}[(X_T^{\pi_i^*}+Y_T)] \tag{51}$$
$$= E^{P^{\lambda_i^*}}[\tilde{u}_i(u_i'(X_T^{\pi_i^*}+Y_T))] + E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_T)(X_T^{\pi_i^*}+Y_T)] \tag{52}$$
$$= E^{P^{\lambda_i^*}}[-\exp(-\gamma_i(X_T^{\pi_i^*}+Y_T))]. \tag{53}$$

(51) follows since X^{π_i} is a $Q_i^{\lambda_i^*}$-supermartingale and $X^{\pi_i^*}$ is a $Q_i^{\lambda_i^*}$-martingale, (50) and (52) from the definition of $Q_i^{\lambda_i^*}$ in (45), and (53) from (49).

Step 2. Next, we show that $\lambda_{i,t}^* = (+\lambda_{Y,i,t}^{\dagger}\sqrt{Y_t}, +\lambda_{S,i,t}^{\dagger}\sqrt{Y_t})^\top$ attains $\inf_{\lambda_i\in\Lambda_i} E^{P^{\lambda_i}}[-\exp(-\gamma_i(X_T^{\pi_i^*}+Y_T))]$. That is, (39) holds.

Note that by Lemma 1, $R_{i,t} = -\exp(-\gamma_i(X_t^{\pi_i^*}+V_{i,t}))$ is a martingale under $P^{\lambda_i^*}$ satisfying an SDE

$$dR_{i,t} = \mathcal{Z}_{S,i,t}dW_{S,t}^{\lambda_i^*} + \mathcal{Z}_{Y,i,t}dW_{Y,t}^{\lambda_i^*}, \tag{54}$$

where

$$\mathcal{Z}_{S,i,t} = -\gamma_i R_{i,t}(\pi_{i,t}^*\sigma_t\rho_{S,t} + a_{i,t}^*\sigma_{Y,t}\sqrt{Y_t}),$$
$$\mathcal{Z}_{Y,i,t} = -\gamma_i R_{i,t}(\pi_{i,t}^*\sigma_t\hat{\rho}_{S,t}).$$

Then,

$$E^{P^{\lambda_i^*}}[-\exp(-\gamma_i(X_T^{\pi_i^*}+Y_T))] = E^{P^{\lambda_i^*}}[R_{i,T}] = R_{i,0}. \tag{55}$$

By the localization argument in the following, the inequality (39) holds.

First, for $\mathcal{Z}_i = (\mathcal{Z}_{Y,i}, \mathcal{Z}_{S,i})^\top$ we define a sequence of stopping times

$$\tau_j := j \wedge \inf\{t \geq 0 | \int_0^t |\mathcal{Z}_{i,s}|^2 ds \geq j\},\ j = 1, 2, \dots, \tag{56}$$

that satisfies

$$\tau_1 \le \tau_2 \le \dots, \text{ and } \lim_{j\to\infty} \tau_j = \infty, \tag{57}$$

in particular $\lim_{j\to\infty} R_{i,t\wedge\tau_j} = R_{i,t}$.

Since

$$dR_{i,t} = \mathcal{Z}_{Y,i,t} dW^{\lambda_i^*}_{Y,t} + \mathcal{Z}_{S,i,t} dW^{\lambda_i^*}_{S,t}, \tag{58}$$

where

$$\begin{aligned} \mathcal{Z}_{Y,i,t} &= -\gamma_i R_{i,t}(\pi^*_{i,t}\sigma_t \hat{\rho}_{S,t}), \\ \mathcal{Z}_{S,i,t} &= -\gamma_i R_{i,t}(\pi^*_{i,t}\sigma_t \rho_{S,t} + a^*_{i,t}\sigma_{Y,t}\sqrt{Y_t}), \end{aligned} \tag{59}$$

we have

$$\begin{aligned} R_{i,t} &= R_{i,0} + \int_0^t \mathcal{Z}_{Y,i,s} dW^{\lambda_i^*}_{Y,s} + \int_0^t \mathcal{Z}_{S,i,s} dW^{\lambda_i^*}_{S,s} \\ &= R_{i,0} + \int_0^t (-(\lambda^*_{i,s} - \lambda_{i,s})^\top (\mathcal{Z}_{Y,i,s}, \mathcal{Z}_{S,i,s})^\top) ds + \int_0^t \mathcal{Z}_{Y,i,s} dW^{\lambda_i}_{Y,s} + \int_0^t \mathcal{Z}_{S,i,s} dW^{\lambda_i}_{S,s}. \end{aligned} \tag{60}$$

Taking the expectation under P^{λ_i} for the stopped process

$$R_{i,T\wedge\tau_j} = R_{i,0} + \int_0^{T\wedge\tau_j} (-(\lambda^*_{i,s} - \lambda_{i,s})^\top (\mathcal{Z}_{Y,i,s}, \mathcal{Z}_{S,i,s})^\top) ds + \int_0^{T\wedge\tau_j} \mathcal{Z}_{Y,i,s} dW^{\lambda_i}_{Y,s} + \int_0^{T\wedge\tau_j} \mathcal{Z}_{S,i,s} dW^{\lambda_i}_{S,s}, \tag{61}$$

we obtain

$$\begin{aligned} &E^{P^{\lambda_i}}[R_{i,T\wedge\tau_j}] \\ &= R_{i,0} + E^{P^{\lambda_i}}[\int_0^{T\wedge\tau_j} (-(\lambda^*_{i,s} - \lambda_{i,s})^\top (\mathcal{Z}_{Y,i,s}, \mathcal{Z}_{S,i,s})^\top) ds] \ge R_{i,0}. \end{aligned} \tag{62}$$

Here we used the fact that $-(\lambda^*_{i,s} - \lambda_{i,s}) \ge 0$ and $\mathcal{Z}_{Y,i,s}, \mathcal{Z}_{S,i,s} \ge 0, 0 \le \forall s \le T$. We remark that $\mathcal{Z}_{Y,i,s}, \mathcal{Z}_{S,i,s} \ge 0, 0 \le \forall s \le T$ follows from conditions (28) and (29) by substituting the expressions of π^*_i and θ into (59).

By the reverse Fatou's lemma, we have

$$\begin{aligned} E^{P^{\lambda_i}}[R_{i,T}] = E^{P^{\lambda_i}}[\overline{\lim_{j\to\infty}} R_{i,T\wedge\tau_j}] &\ge \overline{\lim_{j\to\infty}} E^{P^{\lambda_i}}[R_{i,T\wedge\tau_j}] \\ &\ge R_{i,0} = E^{P^{\lambda_i^*}}[R_{i,T}]. \end{aligned} \tag{63}$$

Therefore, $\inf_{\lambda_i\in\Lambda_i} J_i(\lambda_i, \pi^*_i)$ is attained at $\lambda_i = \lambda^*_i$.

3.1.2 Proof for (π^*_i, λ^*_i) attaining the individual optimization problem for the aggressive and the neutral case

The fact that (π^*_i, λ^*_i) attains the individual optimization problem for the aggressive case is also proved in the following way. For the neutral case, we only consider the sup part for the strategy. Thus, it is proved in the same way as in Step 1 of the aggressive case below. In the following, for fixed λ_i, we first obtain $\pi_i^{l,*}$ that makes $R_{i,t}^l = -\exp(\gamma_i(X_t^{\pi_i^{l,*}} + V_{i,t}^l))$ a P^{λ_i}-martingale as in Step 1 of the conservative case in Section 3.1.1, and confirm that $\pi_i^{l,*}$ is optimal for λ_i in Step 1 below. Then, we consider supremum with respect to λ_i in Step 2 by utilizing a comparison theorem for ODEs.

First, by applying Girsanov's theorem to (12) with $\lambda_i = (l_{Y,i,t}\sqrt{Y_t}, l_{S,i,t}\sqrt{Y_t})^\top$, we have

$$Y_T = y_0 + \int_0^T ((\mu_{Y,1,t} + l_{Y,i,t}\sigma_{Y,t})Y_t + \mu_{Y,2,t})dt + \int_0^T \sigma_{Y,t}\sqrt{Y_t}dW_{Y,t}^{\lambda_i}.$$

In the same way, as in the conservative case in Section 3.1.1, we consider a BSDE

$$\begin{cases} dV_{i,t}^l = -f_i^l(Z_{i,t})dt + Z_{i,t}dW_{Y,t}^{\lambda}, \\ V_{i,T}^l = Y_T, \end{cases} \tag{64}$$

with

$$f_i^l(Z_{i,t}) = \frac{1}{2\gamma_i}(\theta_t^{\lambda_i} - \gamma_i\rho_{S,t}Z_{i,t})^2 - \frac{1}{2}\gamma_i Z_{i,t}^2, \tag{65}$$

$$\theta_t^{\lambda_i} = \theta_t + \rho_{S,t}l_{Y,i,t}\sqrt{Y_t} + \hat{\rho}_{S,t}l_{S,i,t}\sqrt{Y_t},$$

$$\theta_t = \sum_{j=1}^{\bar{I}} \Gamma\frac{1}{\gamma_j}(-\rho_{S,t}\lambda_{Y,j,t}^\dagger - \hat{\rho}_{S,t}\lambda_{S,j,t}^\dagger + \gamma_j\rho_{S,t}a_{j,t}^*\sigma_{Y,t})\sqrt{Y_t}, \tag{66}$$

which is solved as

$$V_{i,t}^l = a_{i,t}^l Y_t + b_{i,t}^l,\ a_{i,T}^l = 1, b_{i,T}^l = 0,$$

$$Z_{i,t} = a_{i,t}^l\sigma_{Y,t}\sqrt{Y_t}, \tag{67}$$

where $a_{i,t}^l, b_{i,t}^l$ are nonrandom processes differentiable with respect to t satisfying Riccati equations

$$-\dot{b}_{i,t}^l = a_{i,t}^l\mu_{Y,2,t},$$
$$-\dot{a}_{i,t}^l$$
$$= \Bigg(\frac{1}{2\gamma_i}(\sum_{j=1}^{\bar{I}}\frac{1}{\gamma_j}\Gamma(-\rho_{S,t}\lambda_{Y,j,t}^\dagger - \hat{\rho}_{S,t}\lambda_{S,j,t}^\dagger + \gamma_j\rho_{S,t}a_{j,t}^*\sigma_{Y,t}) + \rho_{S,t}l_{Y,i,t} + \hat{\rho}_{S,t}l_{S,i,t} - \gamma_i\rho_{S,t}a_{i,t}^l\sigma_{Y,t})^2$$
$$-\frac{1}{2}\gamma_i(a_{i,t}^l)^2\sigma_{Y,t}^2\Bigg) + a_{i,t}^l(\mu_{Y,t} + l_{Y,i,t}\sigma_{Y,t}). \tag{68}$$

Step 1. First, for $\lambda_{i,t} = (l_{Y,i,t}\sqrt{Y_t}, l_{S,i,t}\sqrt{Y_t})^\top$, we show that $\pi_i^{l,*}$ where $\pi_{i,t}^{l,*} = \frac{1}{\gamma_i\sigma_t}(\theta_t + \rho_{S,t}l_{Y,i,t}\sqrt{Y_t} + \hat{\rho}_{S,t}l_{S,i,t}\sqrt{Y_t} - \gamma_i\rho_{S,t}a_{i,t}^l\sigma_{Y,t}\sqrt{Y_t})$ attains the sup.

We consider

$$\sup_{\pi_i\in\mathcal{A}_i(\lambda_i)} E^{P^{\lambda_i}}[-\exp(-\gamma_i(X_T^{\pi_i} + Y_T))],$$

where

$$dY_t = (\mu_{Y,1,t} + \sigma_{Y,t} l_{Y,i,t}) Y_t + \mu_{Y,2,t} dt + \sigma_{Y,t} \sqrt{Y_t} dW^{\lambda_i}_{Y,t},$$
$$dX^{\pi_i}_t = \pi_{i,t}\sigma_t(\theta_t + \rho_{S,t} l_{Y,i,t}\sqrt{Y_t} + \hat{\rho}_{S,t} l_{S,i,t}\sqrt{Y_t})dt + \pi_{i,t}\sigma_t(\rho_{S,t} dW^{\lambda_i}_{Y,t} + \hat{\rho}_{S,t} dW^{\lambda_i}_{S,t}),$$

$$dW^{\lambda_i}_{Y,t} = dW_{Y,t} - l_{Y,i,t}\sqrt{Y_t}dt,$$
$$dW^{\lambda_i}_{S,t} = dW_{S,t} - l_{S,i,t}\sqrt{Y_t}dt.$$

Then, by showing that $-\exp(-\gamma_i(X^{\pi^{l,*}_i}_t + V^l_{i,t}))$ is a P^λ-martingale, using the convex duality argument in the same way as in Section 3.1.1, we show that $\pi^{l,*}_i$ attains the sup.

Step 2. Next, we show that $\lambda^*_{i,t} = (+\lambda^\star_{Y,i,t}\sqrt{Y_t}, +\lambda^\star_{S,i,t}\sqrt{Y_t})^\top$ attains $\sup_{\lambda_i\in\Lambda_i} E^{P^{\lambda_i}}[-\exp(-\gamma_i(X^{\pi^{l,*}_i}_T + Y_T))]$ for $i = I+1, \ldots, I+I'$.

Note that $\pi^{l,*}_i$ and f^l_i are chosen so that $R^l_{i,t} = -\exp(-\gamma_i(X^{\pi^{l,*}_i}_t + V^l_{i,t}))$ is a martingale under P^{λ_i} satisfying an SDE

$$dR^l_{i,t} = \mathcal{Z}_{S,i,t} dW^{\lambda_i}_{S,i,t} + \mathcal{Z}_{Y,i,t} dW^{\lambda_i}_{Y,i,t}, \tag{69}$$

where

$$\mathcal{Z}_{S,i,t} = -\gamma_i R^l_{i,t}(\pi^{l,*}_{i,t}\sigma_t\rho_{S,t} + a^l_{i,t}\sigma_{Y,t}\sqrt{Y_t}),$$
$$\mathcal{Z}_{Y,i,t} = -\gamma_i R^l_{i,t}(\pi^{l,*}_{i,t}\sigma_t\hat{\rho}_{S,t}).$$

Then,

$$\begin{aligned} E^{P^{\lambda_i}}[-\exp(-\gamma_i(X^{\pi^{l,*}_i}_T + Y_T))] &= E^{P^{\lambda_i}}[R^l_{i,T}] = R^l_{i,0} \\ &= -\exp(-\gamma_i(x_{i,0} + V^l_{i,0})) \\ &= -\exp(-\gamma_i(x_{i,0} + a^l_{i,0}y_0 + b^l_{i,0})). \end{aligned} \tag{70}$$

By applying a comparison theorem for ODEs (e.g., Theorem 1.3 in Teschl [33]) to two Riccati ODEs, the second equation of (68) and the one we obtain by plugging $l_{Y,i} = \lambda^\dagger_{Y,i}, l_{S,i} = \lambda^\dagger_{S,i}$ in the second equation of (68), we observe that

$$\begin{aligned} a^*_{i,0} &\geq a^l_{i,0}, \\ b^*_{i,0} &\geq b^l_{i,0}, \end{aligned} \tag{71}$$

which implies that (70) is maximized at λ^*.

In detail, for two systems of Riccati ODEs,

$$\begin{aligned} &-\dot{a}^l_{i,t} \\ &= \frac{1}{2\gamma_i}\Big(\sum_{j=1}^{\bar{I}}\frac{1}{\gamma_j}\Gamma(-\rho_{S,t}\lambda^\dagger_{Y,j,t} - \hat{\rho}_{S,t}\lambda^\dagger_{S,j,t} + \gamma_j\rho_{S,t}a^*_{j,t}\sigma_{Y,t}) + \rho_{S,t}l_{Y,i,t} + \hat{\rho}_{S,t}l_{S,i,t} - \gamma_i\rho_{S,t}a^l_{i,t}\sigma_{Y,t}\Big)^2 \\ &\quad -\frac{1}{2}\gamma_i(a^l_{i,t})^2\sigma^2_{Y,t} + a^l_{i,t}(\mu_{Y,t} + l_{Y,i,t}\sigma_{Y,t}) \\ &= g_l(t, a^l_{i,t}), \end{aligned} \tag{72}$$

and

$$-\dot{a}^*_{i,t}$$

$$= \frac{1}{2\gamma_i}(\sum_{j=1}^{\bar{I}} \frac{1}{\gamma_j}\Gamma(-\rho_{S,t}\lambda^\dagger_{Y,j,t} - \hat{\rho}_{S,t}\lambda^\dagger_{S,j,t} + \gamma_j\rho_{S,t}a^*_{j,t}\sigma_{Y,t}) + \rho_{S,t}\lambda^\dagger_{Y,j,t} + \hat{\rho}_{S,t}\lambda^\dagger_{S,i,t} - \gamma_i\rho_{S,t}a^*_{i,t}\sigma_{Y,t})^2$$

$$-\frac{1}{2}\gamma_i(a^*_{i,t})^2\sigma^2_{Y,t} + a^*_{i,t}(\mu_{Y,t} + \lambda^\dagger_{Y,i,t}\sigma_{Y,t})$$

$$= g_*(t, a^*_{i,t}), \tag{73}$$

where

$$g_*(t, v)$$

$$= \frac{1}{2\gamma_i}(\sum_{j=1}^{\bar{I}} \frac{1}{\gamma_j}\Gamma(-\rho_{S,t}\lambda^\dagger_{Y,j,t} - \hat{\rho}_{S,t}\lambda^\dagger_{S,j,t} + \gamma_j\rho_{S,t}a^*_{j,t}\sigma_{Y,t}) + \rho_{S,t}\lambda^\dagger_{Y,i,t} + \hat{\rho}_{S,t}\lambda^\dagger_{S,i,t} - \gamma_i\rho_{S,t}v\sigma_{Y,t})^2$$

$$-\frac{1}{2}\gamma_i v^2\sigma^2_{Y,t} + v(\mu_{Y,1,t} + \lambda^\dagger_{Y,i,t}\sigma_{Y,t}),$$

$$g_l(t, v)$$

$$= \frac{1}{2\gamma_i}(\sum_{j=1}^{\bar{I}} \frac{1}{\gamma_j}\Gamma(-\rho_{S,t}\lambda^\dagger_{Y,j,t} - \hat{\rho}_{S,t}\lambda^\dagger_{S,j,t} + \gamma_j\rho_{S,t}a^*_{j,t}\sigma_{Y,t}) + \rho_{S,t}l_{Y,i,t} + \hat{\rho}_{S,t}l_{S,i,t} - \gamma_i\rho_{S,t}v\sigma_{Y,t})^2$$

$$-\frac{1}{2}\gamma_i v^2\sigma^2_{Y,t} + v(\mu_{Y,1,t} + l_{Y,i,t}\sigma_{Y,t}),$$

(71) holds since

$$g_*(t, a^*_{i,t}) \geq g_l(t, a^*_{i,t}),\ 0 \leq t \leq T, \tag{74}$$

Here, (74) follows since $\frac{\partial}{\partial l_Y} g_l(t, a^*_{i,t})$, $\frac{\partial}{\partial l_S} g_l(t, a^*_{i,t}) \geq 0$, for all $-\lambda^\star_{Y,i,t} \leq l_{Y,i,t} \leq +\lambda^\star_{Y,i,t}$, $-\lambda^\star_{S,i,t} \leq l_{S,i,t} \leq +\lambda^\star_{S,i,t}$, which is satisfied under the conditions $\rho_{S,t}, \hat{\rho}_{S,t} \geq 0$, (28), and (29).

3.2 An example of the positive supply case

This section provides an example of the positive supply case in the square-root model in Section 3.

To introduce a positive supply $\pi^s > 0$ of the risky asset with keeping a system of Riccati equations similar to Eq.(27)-(30) in Theorem 1, we particularly need an equilibrium Sharpe ratio θ_t to be the form of $\theta_t = c_t\sqrt{Y_t}$ with a positive nonrandom process c. Concretely, we proceed as follows: First, given a $\pi^s > 0$, let us define the volatility of the risky asset price S as $\sigma = \frac{\alpha\sqrt{Y}}{\pi^s}$, where α is a given positive nonrandom process.

Also, by using the individual optimal demand (25) for the risky asset, we have the optimal demand for all agents as follows:

$$\sum_{i=1}^{\bar{I}} \pi^*_{i,t} = \sum_{i=1}^{\bar{I}} \frac{1}{\gamma_i\sigma_t}(\theta_t + \sqrt{Y_t}(\rho_{S,t}\lambda^\dagger_{Y,i,t} + \hat{\rho}_{S,t}\lambda^\dagger_{S,i,t} - \gamma_i\rho_{S,t}a^*_{i,t}\sigma_{Y,t})). \tag{75}$$

Then, plugging $\sigma_t = \frac{\alpha_t\sqrt{Y_t}}{\pi_t^s}$ and the market clearing condition for the risky asset, i.e., $\sum_{i=1}^{\bar{I}} \pi_{i,t}^* = \pi_t^s$ into (75), we obtain the equilibrium Sharpe ratio θ_t as

$$\theta_t = \alpha_t \Gamma \sqrt{Y_t} + \sum_{j=1}^{\bar{I}} \Gamma \frac{1}{\gamma_j}(-\rho_{S,t}\lambda_{Y,j,t}^{\dagger} - \hat{\rho}_{S,t}\lambda_{S,j,t}^{\dagger} + \gamma_j \rho_{S,t} a_{j,t}^* \sigma_{Y,t})\sqrt{Y_t},$$

where $a_{j,t}^*$ are the solutions of the system of Riccati equations (83)-(86) below.

In addition, the zero-net supply of the riskless asset implies that the optimal aggregate wealth π_t^d is equal to the all agents' optimal demand for the risky asset and hence, to the total supply of the risky asset:

$$\pi_t^d = \sum_{i=1}^{\bar{I}} X_t^{\pi_i^*} = \sum_{i=1}^{\bar{I}} (X_t^{\pi_i^*} - \pi_{i,t}^*) + \sum_{i=1}^{\bar{I}} \pi_{i,t}^* = \sum_{i=1}^{\bar{I}} \pi_{i,t}^* = \pi_t^s. \tag{76}$$

Moreover, using the dynamics of π^d, namely, with (2) (the dynamics of $X_t^{\pi_i^*}$),

$$\begin{aligned} d\pi_t^d &= d\sum_{i=1}^{\bar{I}} X_t^{\pi_i^*} \\ &= \sum_{i=1}^{\bar{I}} \pi_{i,t}^* \sigma_t \theta_t dt + \sum_{i=1}^{\bar{I}} \pi_{i,t}^* \sigma_t (\rho_{S,t} dW_{Y,t} + \hat{\rho}_{S,t} dW_{S,t}) \\ &= \pi_t^d \sigma_t \{\theta_t dt + (\rho_{S,t} dW_{Y,t} + \hat{\rho}_{S,t} dW_{S,t})\}, \\ \pi_0^d &= \sum_{i=1}^{\bar{I}} X_0^{\pi_i^*} = \sum_{i=1}^{\bar{I}} x_{i,0}, \end{aligned}$$

and $\pi^d \sigma = \pi^s \sigma = \alpha\sqrt{Y}$, the supply of the risky asset π^s should be generated as

$$\begin{aligned} &d\pi_t^s = \\ &Y_t \alpha_t \Gamma \left[\alpha_t - \sum_{i=1}^{\bar{I}} \frac{1}{\gamma_i}(\rho_{S,t}\lambda_{Y,i,t}^{\dagger} + \hat{\rho}_{S,t}\lambda_{S,i,t}^{\dagger} - \gamma_i \rho_{S,t}\sigma_{Y,t} a_{i,t}^*) \right] dt + \alpha_t \sqrt{Y_t}(\rho_{S,t} dW_{Y,t} + \hat{\rho}_{S,t} dW_{S,t}); \\ &\pi_0^s = \sum_{i=1}^{\bar{I}} x_{i,0}, \end{aligned}$$

where we suppose $\pi_t^s > 0$ for all $t \in [0, T]$.

Motivated by this observation, we define the supply process π^s as above to obtain the next theorem, in which the excess return and volatility of the risky asset with each agent's optimal trading strategy in market equilibrium are obtained for three types of agents. The proof is omitted since it is done in the same way as in Theorem 1. For numerical examples, see Section 4.2.

Theorem 2 *Suppose that for the systems of Riccati equations (i) and (ii) in the following, each system has a unique global solution in* $[0, T]$.

Suppose also that the supply of the risky asset π^s is given as follows:

$$d\pi_t^s =$$

$$Y_t\alpha_t\Gamma\left[\alpha_t - \sum_{i=1}^{\bar{I}}\frac{1}{\gamma_i}(\rho_{S,t}\lambda_{Y,i,t}^{\dagger} + \hat{\rho}_{S,t}\lambda_{S,i,t}^{\dagger} - \gamma_i\rho_{S,t}\sigma_{Y,t}a_{i,t}^*)\right]dt + \alpha_t\sqrt{Y_t}(\rho_{S,t}dW_{Y,t} + \hat{\rho}_{S,t}dW_{S,t});$$

$$\pi_0^s = \sum_{i=1}^{\bar{I}} x_{i,0} > 0, \tag{77}$$

where we suppose $\pi_t^s > 0$ for all $t \in [0,T]$ and $a_{i,t}^*$, $i = 1,\dots,\bar{I}$ are a unique solution of the system of Riccati equations (i) below.

Then, for the volatility process σ and the expected return process μ in (11) given by

$$\sigma_t = \frac{\alpha_t\sqrt{Y_t}}{\pi_t^s}, \tag{78}$$

$$\mu_t = \frac{\alpha_t}{\pi_t^s}Y_t\Gamma\left[\alpha_t - \sum_{i=1}^{\bar{I}}\frac{1}{\gamma_i}(\rho_{S,t}\lambda_{Y,i,t}^{\dagger} + \hat{\rho}_{S,t}\lambda_{S,i,t}^{\dagger} - \gamma_i\rho_{S,t}\sigma_{Y,t}a_{i,t}^*)\right], \tag{79}$$

the process (π_i^*, λ_i^*) defined by

$$\pi_{i,t}^* = \frac{1}{\gamma_i\sigma_t}(\theta_t + \rho_{S,t}\lambda_{Y,i,t}^{\dagger}\sqrt{Y_t} + \hat{\rho}_{S,t}\lambda_{S,i,t}^{\dagger}\sqrt{Y_t} - \gamma_i\rho_{S,t}a_{i,t}^*\sigma_{Y,t}\sqrt{Y_t}), \tag{80}$$

with

$$\theta_t = \frac{\mu_t}{\sigma_t}, \tag{81}$$

and

$$\lambda_{i,t}^* = (+\lambda_{Y,i,t}^{\dagger}\sqrt{Y_t}, +\lambda_{S,i,t}^{\dagger}\sqrt{Y_t})^{\top}, \tag{82}$$

attains the sup-inf/inf-sup, the sup-sup, and the sup problem (14),(18),(21), and the market is in equilibrium.

Here, (i) for $i = 1,\dots,\bar{I}$ (conservative, aggressive, and neutral agents)

$$-\dot{a}_{i,t}^* = \frac{1}{2\gamma_i}\times\Bigg(\Gamma\alpha_t + \sum_{j=1}^{\bar{I}}\frac{1}{\gamma_j}\Gamma(-\rho_{S,t}\lambda_{Y,j,t}^{\dagger} - \hat{\rho}_{S,t}\lambda_{S,j,t}^{\dagger} + \gamma_j\rho_{S,t}a_{j,t}^*\sigma_{Y,t})$$

$$+\rho_{S,t}\lambda_{Y,i,t}^{\dagger} + \hat{\rho}_{S,t}\lambda_{S,i,t}^{\dagger} - \gamma_i\rho_{S,t}a_{i,t}^*\sigma_{Y,t}\Bigg)^2 + a_{i,t}^*(\mu_{Y,1,t} + \lambda_{Y,i,t}^{\dagger}\sigma_{Y,t}) - \frac{1}{2}\gamma_i(a_{i,t}^*)^2\sigma_{Y,t}^2,$$

$$a_{i,T}^* = 1,\ i = 1,\dots,\bar{I};\ \Gamma = \frac{1}{\sum_{k=1}^{\bar{I}}\frac{1}{\gamma_k}}, \tag{83}$$

subject to for $i = 1,\dots,I + I'$(conservative and aggressive agents)

$$\frac{\rho_{S,t}}{\gamma_i}\left(\Gamma\alpha_t + \sum_{j=1}^{\bar{I}}\frac{1}{\gamma_j}\Gamma(-\rho_{S,t}\lambda_{Y,j,t}^{\dagger} - \hat{\rho}_{S,t}\lambda_{S,j,t}^{\dagger} + \gamma_j\rho_{S,t}\sigma_{Y,t}a_{j,t}^*) - \rho_{S,t}\lambda_{Y,i,t}^{\star} - \hat{\rho}_{S,t}\lambda_{S,i,t}^{\star} - \gamma_i\rho_{S,t}\sigma_{Y,t}a_{i,t}^*\right)$$

$$+\sigma_{Y,t}a_{i,t}^* \geq 0, \tag{84}$$

$$\frac{\hat{\rho}_{S,t}}{\gamma_i}\left(\Gamma\alpha_t + \sum_{j=1}^{\bar{I}} \frac{1}{\gamma_j}\Gamma(-\rho_{S,t}\lambda^{\dagger}_{Y,j,t} - \hat{\rho}_{S,t}\lambda^{\dagger}_{S,j,t} + \gamma_j\rho_{S,t}\sigma_{Y,t}a^*_{j,t}) - \rho_{S,t}\lambda^{\star}_{Y,i,t} - \hat{\rho}_{S,t}\lambda^{\star}_{S,i,t} - \gamma_i\rho_{S,t}\sigma_{Y,t}a^*_{i,t}\right)$$

$$\geq 0,$$

$$i = 1, \dots I + I', \tag{85}$$

and (ii) for arbitrary $-\lambda^{\star}_{Y,i,t} \leq l_{Y,i,t} < \lambda^{\star}_{Y,i,t}, -\lambda^{\star}_{S,i,t} \leq l_{S,i,t} < \lambda^{\star}_{S,i,t}$, *for* $i = I+1, \dots, I+I'$ *(aggressive agents),*

$$-\dot{a}^l_{i,t}$$

$$= \frac{1}{2\gamma_i}\left(\sum_{j=1}^{\bar{I}} \frac{1}{\gamma_j}\Gamma(\rho_{S,t}\lambda^{\star}_{Y,j,t} + \hat{\rho}_{S,t}\lambda^{\star}_{S,j,t} + \gamma_j\rho_{S,t}a^*_{j,t}\sigma_{Y,t}) + \rho_{S,t}l_{Y,i,t} + \hat{\rho}_{S,t}l_{S,i,t} - \gamma_i\rho_{S,t}a^l_{i,t}\sigma_{Y,t})^2\right.$$

$$\left. + a^l_{i,t}(\mu_{Y,1,t} + l_{Y,i,t}\sigma_{Y,t}\right) - \frac{1}{2}\gamma_i(a^l_{i,t})^2\sigma^2_{Y,t},$$

$$a^l_{i,T} = 1. \tag{86}$$

4 Numerical example

This section presents numerical examples of the equilibrium trading strategies and the expected return process in the square-root case in Section 3. As described in Section 3, there are two cases concretely solved: One is the zero-net supply case in Sec.3.1, and the other is the positive supply case in Sec.3.2. In the following, we show the results of the zero-net supply and the positive supply cases in Sec.4.1 and Sec.4.2, respectively. Particularly, we show how heterogeneous views of the agents affect the expected return and their trading strategies in equilibrium, numerically solving the system of Riccati ODEs (83)-(85) in Sec.3.2, and provide explanation on the effect of the heterogeneous views.

First, we describe the common setting and numerical procedure: We consider the square-root case with two agents called agent 1 and agent 2 ($I + I' + I'' = 2$), and the following optimization problems as described in Section 3.

Then, we summarize our numerical procedures as follows.

1. We set exogenous parameters : y_0, $\mu_{Y,1}$, $\mu_{Y,2}$, σ_Y, $\rho_S(\hat{\rho}_S)$, T, α, and agent i's γ_i, $\lambda^{\dagger}_{Y,i}$, $\lambda^{\dagger}_{S,i}$ $(i = 1, 2)$. In numerical examples, we suppose that $\mu_{Y,1}$, $\mu_{Y,2}$, σ_Y, $\rho_S(\hat{\rho}_S)$, and $\lambda^{\dagger}_{Y,i}$, $\lambda^{\dagger}_{S,i}$ $(i = 1, 2)$ are constants.

2. We solve the following system of ODE numerically :

$$-\dot{a}^*_{i,t}$$

$$= \frac{1}{2\gamma_i}\left(\Gamma\alpha + \sum_{j=1}^{2} \frac{1}{\gamma_j}\Gamma(-\rho_S\lambda^{\dagger}_{Y,j} - \hat{\rho}_S\lambda^{\dagger}_{S,j} + \gamma_j\rho_S a^*_{j,t}\sigma_Y) + \rho_S\lambda^{\dagger}_{Y,i} + \hat{\rho}_S\lambda^{\dagger}_{S,i} - \gamma_i\rho_S a^*_{i,t}\sigma_Y\right)^2$$

$$+ a^*_{i,t}(\mu_{Y,1} + \lambda^{\dagger}_{Y,i}\sigma_Y) - \frac{1}{2}\gamma_i a^{*2}_{i,t}\sigma^2_Y, \;\; a^*_{i,T} = 1, i = 1, 2. \tag{87}$$

If agent i has some views $\lambda^{\dagger}_{Y,i}, \lambda^{\dagger}_{S,i} \neq 0$, we need to confirm that the solution (a^*_1, a^*_2) satisfies the following conditions in $t \in [0,T]:$ when we set $\lambda^{\star}_{Y,i} = |\lambda^{\dagger}_{Y,i}|$, $\lambda^{\star}_{S,i} = |\lambda^{\dagger}_{S,i}|$,

$$\frac{\rho_S}{\gamma_i}\left(\Gamma\alpha + \sum_{j=1}^{2}\frac{1}{\gamma_j}\Gamma(-\rho_S\lambda^{\dagger}_{Y,j} - \hat{\rho}_S\lambda^{\dagger}_{S,j} + \gamma_j\rho_S\sigma_Y a^*_{j,t}) - \rho_S\lambda^{\star}_{Y,i} - \hat{\rho}_S\lambda^{\star}_{S,i} - \gamma_i\rho_S\sigma_Y a^*_{i,t}\right)$$
$$+\sigma_Y a^*_{i,t} \geq 0, \tag{88}$$

and

$$\frac{\hat{\rho}_S}{\gamma_i}\left(\Gamma\alpha + \sum_{j=1}^{2}\frac{1}{\gamma_j}\Gamma(-\rho_S\lambda^{\dagger}_{Y,j} - \hat{\rho}_S\lambda^{\dagger}_{S,j} + \gamma_j\rho_S\sigma_Y a^*_{j,t}) - \rho_S\lambda^{\star}_{Y,i} - \hat{\rho}_S\lambda^{\star}_{S,i} - \gamma_i\rho_S\sigma_Y a^*_{i,t}\right) \geq 0. \tag{89}$$

We note that (87)-(89) are obtained from Theorem 2 in Sec.3.2, where those with $\alpha = 0$ become the corresponding equations in the equations of Theorem 1 in Sec.3.1. In our examples, we discretize $[0,T]$ with one time step of 1/250.

3. We obtain an equilibrium Sharpe ratio process θ is given by

$$\theta_t = \sqrt{Y_t}\Gamma\left[\alpha - \sum_{i=1}^{2}\frac{1}{\gamma_i}(\rho_S\lambda^{\dagger}_{Y,i} + \hat{\rho}_S\lambda^{\dagger}_{S,i} - \gamma_i\rho_S\sigma_Y a^*_{i,t})\right], \tag{90}$$

where we simulate 100,000 paths of Y according to Eq.(12) with a discretization of one-time step 1/250.

Then, we calculate the expected return process of the risky asset in equilibrium:

$$\mu_t = \sigma_t\theta_t, \tag{91}$$

where σ_t is given as (93) or (94) below.

4. To see agents' individual strategies, we calculate the optimal portfolio :

$$\pi^*_{i,t} = \frac{1}{\gamma_i\sigma_t}(\theta_t + \rho_S\lambda^{\dagger}_{Y,i}\sqrt{Y_t} + \hat{\rho}_S\lambda^{\dagger}_{S,i}\sqrt{Y_t} - \gamma_i\rho_S a^*_{i,t}\sigma_Y\sqrt{Y_t}), \tag{92}$$

where we set σ_t as follows:

- Zero-net supply case : for a given positive constant $\bar{\sigma} > 0$,

$$\sigma_t = \bar{\sigma}\sqrt{Y_t}. \tag{93}$$

- Positive supply case : for a given positive constant $\alpha > 0$ with (78) in Theorem 2,

$$\sigma_t = \frac{\alpha\sqrt{Y_t}}{\pi^s_t}, \tag{94}$$

where the risky asset supply π_t^s is given as in (77) in Theorem 2: For a given initial value $\pi_0^s > 0$,

$$d\pi_t^s = Y_t \alpha \Gamma \left[\alpha - \sum_{i=1}^{2} \frac{1}{\gamma_i} (\rho_S \lambda_{Y,i}^{\dagger} + \hat{\rho}_S \lambda_{S,i}^{\dagger} - \gamma_i \rho_S \sigma_Y a_{i,t}^*) \right] dt + \alpha \sqrt{Y_t} (\rho_S dW_{Y,t} + \hat{\rho}_S dW_{S,t}). \tag{95}$$

Then, for all simulation grids and paths, we confirm $\pi_t^s > 0$, which also implies that the volatility process of the risky asset $\sigma_t = \alpha \sqrt{Y_t} / \pi_t^s$ is positive.

4.1 Zero-net supply case

Firstly, we present the numerical example of a zero-net supply case. We suppose that agent 1 has conservative views on the fundamental risks, i.e., $\lambda_{Y,1}^{\dagger} = -0.2$, $\lambda_{S,1}^{\dagger} = -0.2$, while agent 2 has neutral views, i.e., $\lambda_{Y,2}^{\dagger}, \lambda_{S,2}^{\dagger} \equiv 0$. In the following, we investigate the effect of conservative views of an agent on the expected return of the risky asset and trading strategies by comparing them with the case where both agents have neutral views.

In Section 4.1, we set the parameters as follows. $\mu_{Y,1} = -1, \mu_{Y,2} = 1, \sigma_Y = 0.2, \rho_S = 0.5, \hat{\rho}_S = \sqrt{1 - \rho_S^2} = 0.866, \bar{\sigma} = 1, \gamma_1 = 1, \gamma_2 = 10, y_0 = 0.5, T = 1$.

Moreover, we compare the result to the case of both agents with neutral views, namely $\lambda_{Y,i}^{\dagger} = \lambda_{S,i}^{\dagger} = 0$, $i = 1, 2$. Thus, we set parameters about agents' views and ARAs in Table 1.

Table 1: Settings of parameters about different views.

	$\lambda_{Y,1}^{\dagger}$	$\lambda_{S,1}^{\dagger}$	$\lambda_{Y,2}^{\dagger}$	$\lambda_{S,2}^{\dagger}$	γ_1	γ_2
agent 1: conservative, agent 2: neutral	-0.2	-0.2	0	0	1	10
both agents: neutral	0	0	0	0	1	10

Figure 1 describes the numerical solutions of the Riccati ODEs in (87) and the optimal portfolio processes of agents 1 and 2, which compares the two cases where agent 1 has conservative or neutral views. From the left panel, as is easily observed with the comparison principle for the solution of the Riccati ODEs, $a_{1,t}^*$ with the conservative views is smaller than $a_{1,t}^*$ with the neutral views. Here, the conditions in (88) and (89) are satisfied in both cases where agent 1 has the conservative views and the neutral views.

For the optimal portfolio processes, we note that as the net position of agents 1 and 2 is zero because of the clearing condition (23), one of π_i^* is positive, and the other is negative. Thus, we observe that the optimal portfolio of agent 1 is a long position ($\pi_1^* > 0$), while and that of agent 2 is a short position ($\pi_2^* < 0$) in the right panel of Figure 1.

Next, let us rewrite both agents' optimal portfolios as

$$\pi_{1,t}^* = \frac{1}{\gamma_1 \bar{\sigma} \sqrt{Y_t}} (\theta_t + \rho_S \lambda_{Y,1}^{\dagger} \sqrt{Y_t} + \hat{\rho}_S \lambda_{S,1}^{\dagger} \sqrt{Y_t}) - \rho_S \frac{\sigma_Y}{\bar{\sigma}} a_{1,t}^*, \tag{96}$$

$$\pi_{2,t}^* = \frac{1}{\gamma_2 \bar{\sigma} \sqrt{Y_t}} \theta_t - \rho_S \frac{\sigma_Y}{\bar{\sigma}} a_{2,t}^*.$$

Then, the first term of $\pi^*_{i,t}$ $(i = 1, 2)$ stands for the mean-variance portfolio adjusted by the risk aversion parameter γ_i for both agents and also by the conservative views for agent 1, i.e., $(\rho_S \lambda^\dagger_{Y,1}\sqrt{Y_t} + \hat{\rho}_S \lambda^\dagger_{S,1}\sqrt{Y_t})$; the second term does the so-called hedging portfolio to mitigate the terminal wealth shock.

Under the current parameter setting, the first term is dominant for agent 1, which induces the long position, while due to the larger risk aversion parameter γ_2, the second term $-\rho_S \frac{\sigma_Y}{\bar{\sigma}} a^*_{2,t}$ is dominant for agent 2, which leads to the short position given $\rho_S, \sigma_Y, \bar{\sigma}, a^*_{2,t} > 0$.

We remark that in equilibrium, the Sharpe ratio θ_t, equivalently, the expected return $\mu_t = \sigma_t \theta_t$ with $\sigma_t > 0$ must be positive, since if it is negative, the positions of both agents are short and the market clearing condition (23) is not satisfied.

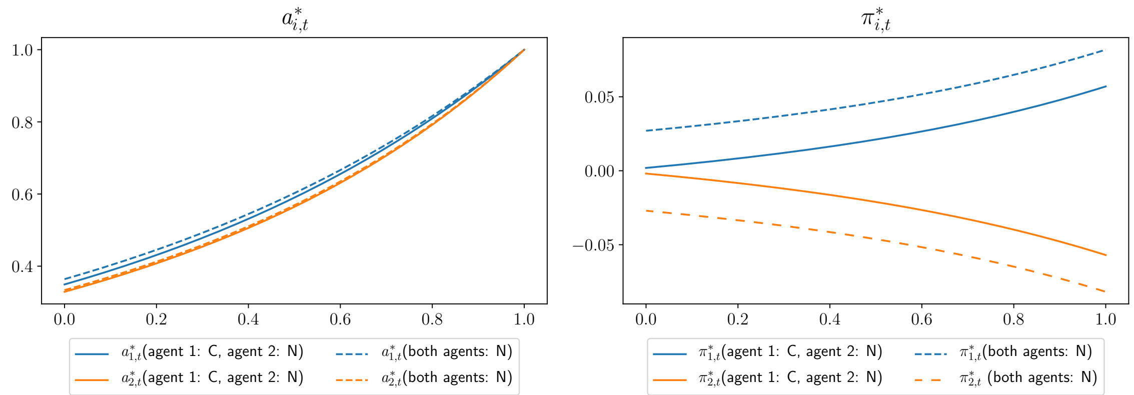

Figure 1: Zero-net supply case. Left panel: solutions of the ODE in (87). Right panel: optimal portfolio processes π^*_1 and π^*_2. Solid lines: agent 1 is conservative (C) and agent 2 is neutral (N). Dashed lines: both agents are neutral.

Secondly, we observe that the long amount of agent 1, π^*_1, is less when agent 1 has conservative views, which is explained as follows. The conservative sentiments make the long position of agent 1 π^*_1 less due to the presence of $\lambda^\dagger_{Y,1}$, $\lambda^\dagger_{S,1}$ in the mean-variance term in (96), which also makes the less short position π^*_2 for agent 2 because of the clearing condition.

Finally, Figure 2 exhibits the sample average $\bar{\mu}$ of the expected return process μ in both cases with and without conservative views for agent 1. The expected return is higher when the agent has conservative views due to the presence of $\lambda^\dagger_{Y,1} < 0$ and $\lambda^\dagger_{S,1} < 0$ in the expression of μ in the following:

$$\mu_t = \sigma_t \theta_t = -\bar{\sigma}\Gamma \sum_{j=1}^{2} \frac{1}{\gamma_j}(\rho_S \lambda^\dagger_{Y,j} + \hat{\rho}_S \lambda^\dagger_{S,j} + \gamma_j \rho_S a^*_{j,t}\sigma_Y) Y_t,$$

which can also be interpreted that agent 1 requires a higher expected return μ when agent 1 has a conservative view on the risky asset price to take a long position.

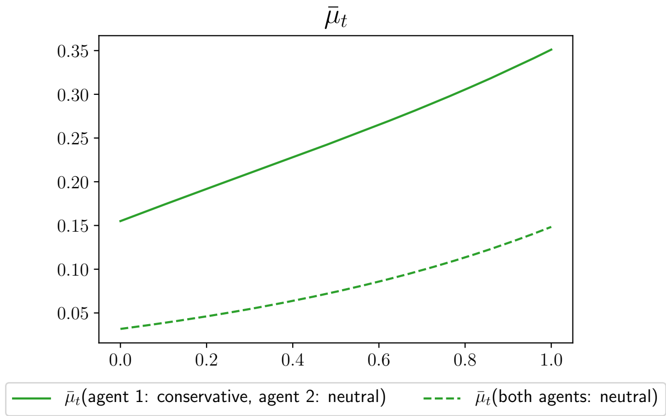

Figure 2: Sample average $\bar{\mu}$ of the expected returns in equilibrium when agent 1 has conservative views or neutral views.

4.2 Positive supply case

In this section, we present numerical examples in the positive supply case, where $\alpha > 0$.

Firstly, unless otherwise noted, we set baseline parameters as follows : $\mu_{Y,1} = -1,\ \mu_{Y,2} = 1.5,\ \sigma_Y = 0.2,\ \rho_S = 0.5,\ \hat{\rho}_S = \sqrt{1-\rho_S^2} = 0.866,\ \gamma_1 = 1,\ \gamma_2 = 1,\ y_0 = 1,\ \alpha = 1,\ \pi_0^s = 5,\ T = 1$. Then, we note $\Gamma = 1/(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}) = 0.5$.

In the following, we study the positive supply case in terms of agents' different views and the effect of the parameter α. Particularly, we investigate how the differences in views affect the expected return of the risky asset and the agents' trading positions in Section 4.2.1 and how the expected return and their positions shift when the total supply of the risky asset changes in Section 4.2.2.

4.2.1 Comparative study within different views' settings

We consider two cases in Table 2 with different agents' views. In Case 1, agent 1 has conservative views, while agent 2 has neutral views, i.e., $\lambda^\dagger_{Y,2}, \lambda^\dagger_{S,2} \equiv 0$ as in the zero-net supply case. In Case 2, agent 1 has conservative views, while agent 2 has aggressive views.

Table 2: Settings of parameters about different views.

	$\lambda^\dagger_{Y,1}$	$\lambda^\dagger_{S,1}$	$\lambda^\dagger_{Y,2}$	$\lambda^\dagger_{S,2}$
Case 1	-0.2	-0.2	0	0
Case 1 (both agents: neutral)	0	0	0	0
Case 2	-0.2	-0.2	0.2	0.2
Case 2 (agent 1: neutral, agent 2: aggressive)	0	0	0.2	0.2

First, Figure 3 shows the solutions of the ODE in (87) and agents' optimal strategies in Case 1. Here, we illustrate agent i's holding ratio against the total supply of risky assets, π_i^*/π_s, which is nonrandom :

$$\frac{\pi^*_{i,t}}{\pi^s_t} = \frac{\Gamma}{\gamma_i} + \frac{1}{\alpha}\left(\frac{1}{\gamma_i}(\rho_S\lambda^\dagger_{Y,i} + \hat{\rho}_S\lambda^\dagger_{S,i} - \gamma_i\rho_S\sigma_Y a^*_{i,t}) - \Gamma\frac{1}{\gamma_i}\sum_{j=1}^{2}\frac{1}{\gamma_j}(\rho_S\lambda^\dagger_{Y,j} + \hat{\rho}_S\lambda^\dagger_{S,j} - \gamma_j\rho_S\sigma_Y a^*_{j,t})\right). \tag{97}$$

In the right panel of Figure 3, conservative agent 1's ratio π_1^*/π_s in the solid line shows a decrease from the dashed line where both agents are neutral. Conversely, to satisfy market equilibrium, the ratio of agent 2, π_2^*/π_s, is increased. Compared with the zero-net supply case, we note that both agents take long positions in this positive supply case.

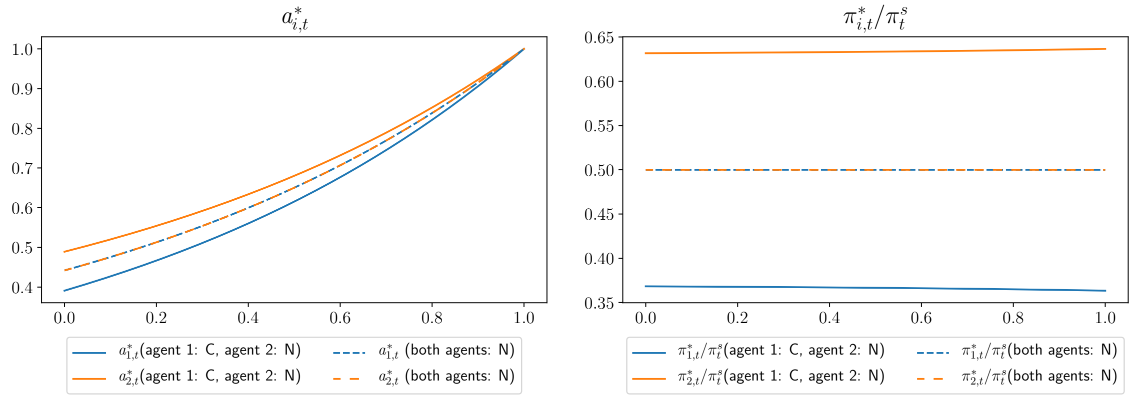

Figure 3: Case 1. Left panel: solutions of the ODE in (87). Right panel: holding ratios in (97). Solid lines: agent 1 is conservative (C) and agent 2 is neutral (N). Dashed lines: both agents are neutral and have the same ARAs, which shows overlapping of dashed lines.

Next, Figure 4 presents the results for Case 2. In Case 2, agent 1 is more conservative than in Case 1 due to agent 2's aggressive views. Consequently, agent 1's (2's) holding ratios in the right panel show a further decline (increase) compared to Case 1.

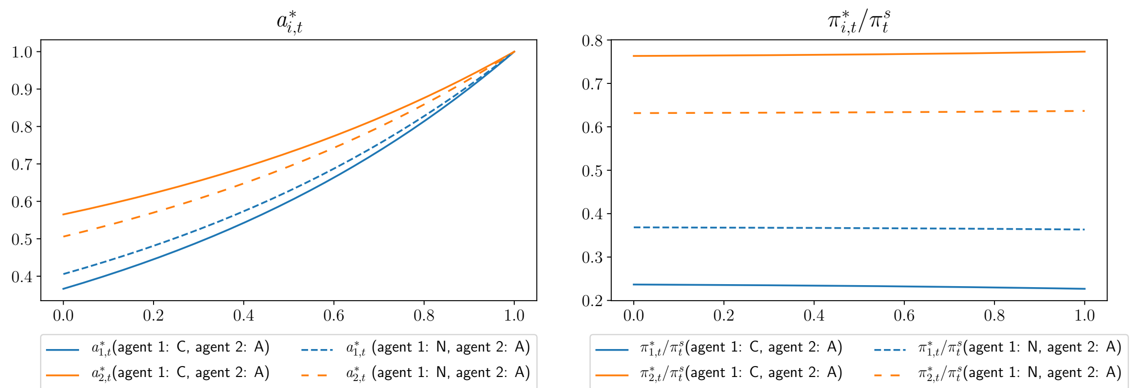

Figure 4: Case 2. Left panel: solutions of the ODE in (87). Right panel: holding ratios in (97). Solid lines: agent 1 is conservative (C), and agent 2 is aggressive (A). Dashed lines: agent 1 is neutral (N) and agent 2 is aggressive. The dashed lines are almost the same as the solid lines in Figure 3 for Case 1 since agent 1 is more conservative than agent 2 in both cases.

Finally, Figure 5 illustrates the sample average of simulated expected returns μ_t of Case 1 and Case 2. Case 1 shows a higher expected return than the case where both agents are neutral, which is explained as follows. In Case 1, agent 1 alone has conservative views, and its demand for the risky asset decreases. Thus, a higher expected return is needed to increase agent 2's demand so that the market is cleared. On the other hand, in Case 2, the expected return is almost the same as the dashed line. This is because agents have opposite views of the same size. Thus, the market equilibrium is satisfied without a change in expected returns.

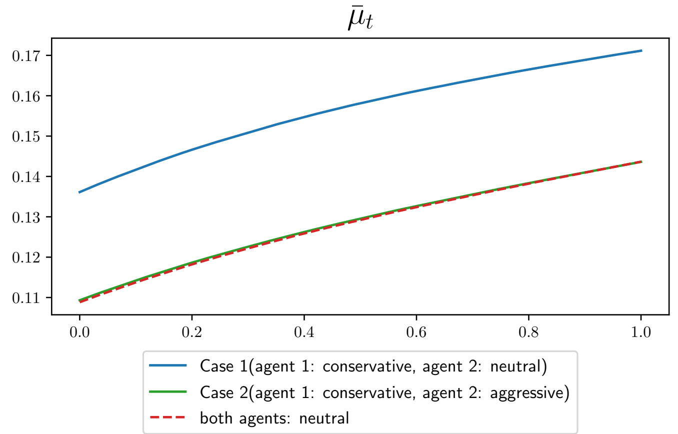

Figure 5: Sample average $\bar{\mu}$ of simulated expected returns. Solid lines: Case 1, Case 2. Dashed line: the case where both agents are neutral.

4.2.2 Comparative study about α

In this section, we set Case 3 in Table 3 to investigate the effect of the parameter α. The settings of agents' views in Case 3 are the same as in Case 1. Thus, agent 1 is conservative, and agent 2 is neutral. Figure 6 and Figure 7 study Case 3 where we increase $\alpha = 1.3$ from $\alpha = 1$ in Case 1.

Table 3: Settings of parameters about different views.

	$\lambda^{\dagger}_{Y,1}$	$\lambda^{\dagger}_{S,1}$	$\lambda^{\dagger}_{Y,2}$	$\lambda^{\dagger}_{S,2}$	α
Case 3	-0.2	-0.2	0	0	1.3
Case 3 (both agents: neutral)	0	0	0	0	1.3
Case 1	-0.2	-0.2	0	0	1.0

Firstly, the right panel of Figure 6 shows that agent 1's (2's) holding ratio in solid line becomes larger (smaller) than in Case 1 (see the solid lines in the right panel of Figure 3). This is because $1/\alpha$ in the second term of (97) suggests that an increase in α diminishes the impact of views.

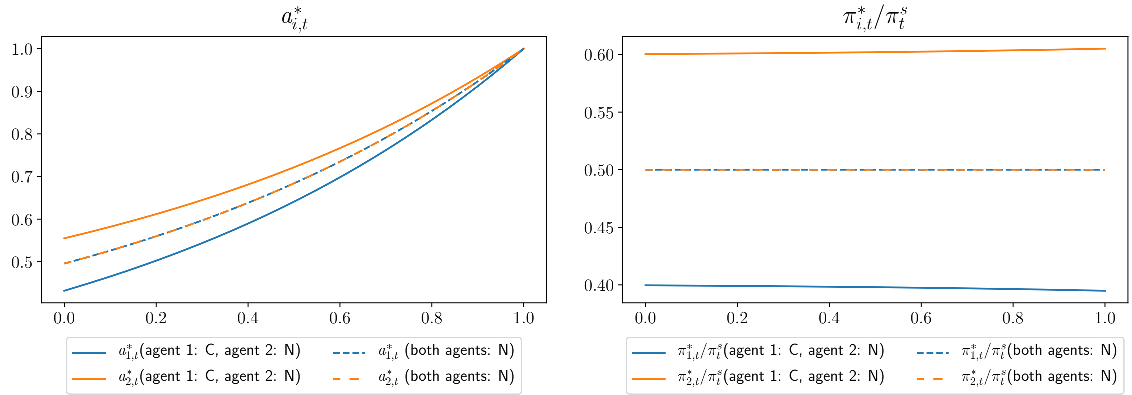

Figure 6: Case 3. Left panel: solutions of the ODE in (87). Right panel: holding ratios in (97). Solid lines: agent 1 is conservative (C), and agent 2 is neutral (N). Dashed lines: both agents are neutral and have the same ARAs. $\alpha = 1.3$ increased from $\alpha = 1$ in Case 1.

Secondly, Figure 7 indicates that the expected return in Case 3 is higher than in Case 1, which is explained as follows. With the increased total supply of risky assets π^s in (95) as α rises, the risky asset needs to become more attractive to clear the market, which results in a higher expected return. In fact, the rise in α is expected to enhance the Sharpe Ratio process θ in (90), which makes the risky asset more attractive. We remark that compared to $a^*_{i,t}$ in Figure 3, the increases in the solutions $a^*_{i,t}$ in the left panel of Figure 6 contribute to the rise in the Sharpe Ratio process θ in (90) to a certain extent.

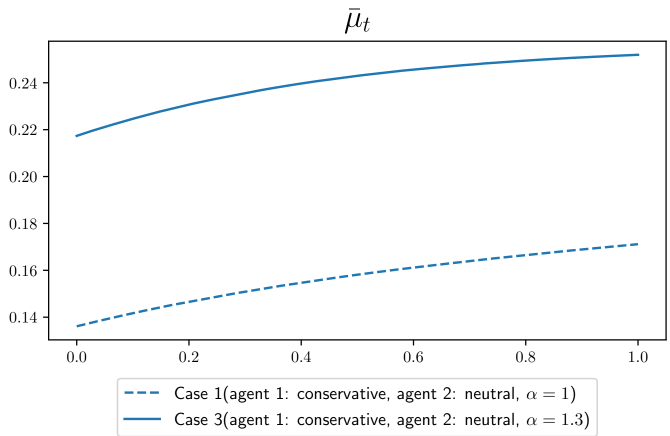

Figure 7: Sample average $\bar{\mu}$ of simulated expected returns. Solid line: Case 3 ($\alpha = 1.3$ increased from $\alpha = 1$ in Case 1), Dashed line: Case 1 (agent 1 is conservative, agent 2 is neutral, $\alpha = 1$).

5 The general procedure to confirm that the market is in equilibrium

In this section, we provide the excess return process in equilibrium in a general case where the state process is given by

$$dY_t = \mu_{Y,t}dt + \sigma_{Y,t}dW_{Y,t},\ Y_0 = y_0, \tag{98}$$

and the sup-inf/inf-sup, sup-sup, and sup individual optimization problems are solved. This includes the square-root state process case in Section 3, where the existence and uniqueness result and the comparison principle for the BSDE with a stochastic Lipschitz driver do not apply since the terminal condition is unbounded. Moreover, the general case includes a Gaussian state process case in Appendix B, where the existence and uniqueness result and the comparison principle for BSDEs with a standard Lipschitz driver apply.

5.1 Zero-net supply case

For exogenously given σ, endogenously determined θ or equivalently μ is confirmed to be in equilibrium under some conditions in the following theorem.

The individual optimization problem in an incomplete market setting in an exponential utility case generally reduces to solving a quadratic BSDE (qBSDE). Particularly, obtaining an equilibrium Sharpe ratio process requires solving a system of qBSDEs, which generally requires restrictive settings for the system to be solved, such as a bounded terminal condition.

Since our case with an unbounded terminal condition is not within the scope of the general theory of qBSDEs, and our setting includes the inf part on the random sentiments, the problem needs to be solved differently from the general theory for qBSDEs. As we have observed, in the square-root case, solving for an equilibrium reduces to solving a system of Riccati ODEs, which can be done numerically, and the existence and uniqueness of the solution are guaranteed up to a certain explosion time by the Picard Lindelöf theorem for ODEs. Thus, we summarize the outline of the procedure to solve for an equilibrium in a general framework. The following theorem concretely describes the procedure. For the proof, see Appendix A showing that (π_i^*, λ_i^*) attains the individual optimization problem in the conservative agent case.

Theorem 3 *Let* $\Gamma = \frac{1}{\sum_{k=1}^{\bar{I}} \frac{1}{\gamma_k}}$,

$$\bar{\lambda}^{\ddagger}_{Y,i,t} = \begin{cases} -\bar{\lambda}_{Y,i,t}\ (i = 1, \dots, I) \\ +\bar{\lambda}_{Y,i,t}\ (i = I+1, \dots, I+I') \\ 0\ (i = I+I'+1, \dots, \bar{I}) \end{cases}, \tag{99}$$

and

$$\bar{\lambda}^{\ddagger}_{S,i,t} = \begin{cases} -\bar{\lambda}_{S,i,t}\ (i = 1, \dots, I) \\ +\bar{\lambda}_{S,i,t}\ (i = I+1, \dots, I+I') \\ 0\ (i = I+I'+1, \dots, \bar{I}) \end{cases}. \tag{100}$$

We assume the conditions (i)-(iv) listed below.

Then, with the Sharpe ratio process θ in (102) below and (π_i^*, λ_i^*) in the following, the market is in equilibrium. That is, (π_i^*, λ_i^*) given by

$$\pi_{i,t}^* = \frac{1}{\gamma_i \sigma_t}(\theta_t + \rho_{S,t}\bar{\lambda}_{Y,i,t}^{\ddagger} + \hat{\rho}_{S,t}\bar{\lambda}_{S,i,t}^{\ddagger} - \gamma_i \rho_{S,t} Z_{i,t}), \tag{101}$$

and $\lambda_{i,t}^* = (\bar{\lambda}_{Y,i,t}^{\ddagger}, \bar{\lambda}_{S,i,t}^{\ddagger})^\top$ attains the sup-inf/inf-sup problem (4), (5) ($i = 1, \dots, I$), the sup-sup problem (6) ($i = I+1, \dots, I+I'$), and the sup problem (7) ($i = I+I'+1, \dots, \bar{I}$) for admissible strategies $\pi \in \mathcal{A}_i$, where the set of the admissible strategies is given by $\mathcal{A}_i = \{\pi | X^\pi \text{is a } Q_i\text{-supermartingale}\}$ and Q_i is defined by (iii) (a) below. For the aggressive agents where $i = I+1, \dots, I+I'$, the set of admissible strategies is instead given as $\mathcal{A}_i(\lambda) = \{\pi | X^\pi \text{is a } Q_i^\lambda\text{-supermartingale}\}$ where Q_i^λ is defined by (iii) (b).

(Conditions)

(i) (Solutions of BSDEs to define optimal portfolios and the Sharpe ratio process exist)

(a) Suppose that there exist (V_i, Z_i) $i = 1, \dots, \bar{I}$, that satisfy $E[\sup_{0\le s\le T} |V_{i,s}|^2] < \infty$, $E[\int_0^T Z_{i,s}^2 ds] < \infty$ and BSDEs

$$\begin{aligned} dV_{i,t} &= -(f_i(Z_{1,t}, \dots, Z_{\bar{I},t}) + \bar{\lambda}_{Y,i,t}^{\ddagger} Z_{i,t})dt + Z_{i,t} dW_{Y,t}, \\ V_{i,T} &= Y_T, \end{aligned}$$

where

$$\begin{aligned} &f_i(Z_{1,t}, \dots, Z_{\bar{I},t}) \\ &= \frac{1}{2\gamma_i}(\theta_t + \rho_{S,t}\bar{\lambda}_{Y,i,t}^{\ddagger} + \hat{\rho}_{S,t}\bar{\lambda}_{S,i,t}^{\ddagger} - \gamma_i \rho_{S,t} Z_{i,t})^2 - \frac{1}{2}\gamma_i Z_{i,t}^2, \end{aligned}$$

and

$$\theta_t = \sum_{i=1}^{\bar{I}} \frac{1}{\gamma_i} \Gamma(-\rho_{S,t}\bar{\lambda}_{Y,i,t}^{\ddagger} - \hat{\rho}_{S,t}\bar{\lambda}_{S,i,t}^{\ddagger} + \gamma_j \rho_{S,t} Z_{i,t}). \tag{102}$$

(b) Also, suppose that there exist $(V_i^\lambda, Z_i^\lambda)$ $i = I+1, \dots, I+I'$, that satisfy $E[\sup_{0\le s\le T} |V_{i,s}^\lambda|^2] < \infty$, $E[\int_0^T (Z_{i,s}^\lambda)^2 ds] < \infty$ and BSDEs

$$\begin{aligned} dV_{i,t}^\lambda &= -(\bar{f}_i(Z_{i,t}^\lambda) + \lambda_{Y,i,t} Z_{i,t}^\lambda)dt + Z_{i,t}^\lambda dW_{Y,t}, \\ V_{i,T} &= Y_T, \end{aligned}$$

where

$$\begin{aligned} &\bar{f}_i(Z_{i,t}^\lambda) \\ &= \frac{1}{2\gamma_i}(\theta_t + \rho_{S,t}\lambda_{Y,i,t} + \hat{\rho}_{S,t}\lambda_{S,i,t} - \gamma_i \rho_{S,t} Z_{i,t}^\lambda)^2 - \frac{1}{2}\gamma_i (Z_{i,t}^\lambda)^2. \end{aligned}$$

(ii) (The risky neutral probability measure Q_i is well-defined)

(a) We assume that

$$\left\{\exp\left(-\frac{1}{2}\int_0^t \gamma_i^2(\pi_{i,s}^*\sigma_s\rho_{S,s}+Z_{i,s})^2+(\gamma_i\pi_{i,s}^*\sigma_s\hat{\rho}_{S,s})^2ds+\int_0^t\gamma_i(\pi_{i,s}^*\sigma_s\rho_{S,s}+Z_{i,s})dW_{Y,s}^{\lambda_i^*}\right.\right.$$
$$\left.\left.+\int_0^t\gamma_i\pi_{i,s}^*\sigma_s\hat{\rho}_{S,s}dW_{S,s}^{\lambda_i^*}\right)\right\}_{0\le t\le T}, \tag{103}$$

is a $P^{\lambda_i^*}$-martingale, where $\lambda_{i,t}^*=(+\bar{\lambda}_{Y,i,t}^{\ddagger},+\bar{\lambda}_{S,i,t}^{\ddagger})^\top$, $\pi_i^*=\frac{1}{\gamma_i\sigma_t}(\theta_t+\rho_{S,t}\bar{\lambda}_{Y,i,t}^{\ddagger}+\hat{\rho}_{S,t}\bar{\lambda}_{S,i,t}^{\ddagger}-\gamma_i\rho_{S,t}Z_{i,t})$.

(b) In addition, in the aggressive case $i=I+1,\dots,I+I'$, we assume that

$$\left\{\exp\left(-\frac{1}{2}\int_0^t \gamma_i^2(\pi_{i,s}^{\lambda,*}\sigma_s\rho_{S,s}+Z_{i,s}^\lambda)^2+(\gamma_i\pi_{i,s}^{\lambda,*}\sigma_s\hat{\rho}_{S,s})^2ds+\int_0^t\gamma_i(\pi_{i,s}^{\lambda,*}\sigma_s\rho_{S,s}+Z_{i,s}^\lambda)dW_{Y,s}^{\lambda_i}\right.\right.$$
$$\left.\left.+\int_0^t\gamma_i\pi_{i,s}^{\lambda,*}\sigma_s\hat{\rho}_{S,s}dW_{S,s}^{\lambda_i}\right)\right\}_{0\le t\le T}, \tag{104}$$

is a P^{λ_i}-martingale, where $\lambda_{i,t}=(\lambda_{Y,i,t},\lambda_{S,i,t})^\top$, $\pi_i^{\lambda,*}=\frac{1}{\gamma_i\sigma_t}(\theta_t+\rho_{S,t}\lambda_{Y,i,t}+\hat{\rho}_{S,t}\lambda_{S,i,t}-\gamma_i\rho_{S,t}Z_{i,t}^\lambda)$.

(iii) ($X^{\pi_i^*}$ is a martingale under Q_i)

(a) Also, for a probability measure Q_i defined as

$$\frac{dQ_i}{dP^{\lambda_i^*}}=\frac{u_i'(X_T^{\pi_i^*}+Y_T)}{E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_T)]},$$

where

$$u_i'(x)=\gamma_i\exp(-\gamma_i x),$$

we assume that

$$E^{Q_i}\left[\int_0^T(\pi_{i,t}^*)^2\sigma_t^2dt\right]<\infty. \tag{105}$$

(b) In addition, in the aggressive case where $i=I+1,\dots,I+I'$, for a probability measure Q_i^λ defined as

$$\frac{dQ_i^\lambda}{dP^{\lambda_i}}=\frac{u_i'(X_T^{\pi_i^{\lambda,*}}+Y_T)}{E^{P^{\lambda_i}}[u_i'(X_T^{\pi_i^{\lambda,*}}+Y_T)]},$$

where we assume that

$$E^{Q_i^\lambda}\left[\int_0^T(\pi_{i,t}^{\lambda,*})^2\sigma_t^2dt\right]<\infty. \tag{106}$$

(iv) (Optimality of λ^* when the optimal portfolio is given)

(a) Moreover, in the conservative case $i=1,\dots,I$, we suppose that for all $\lambda\in\Lambda_i$, a BSDE

$$d\mathcal{V}_t^\lambda=-(\lambda_{S,t}\mathcal{Z}_{S,t}^\lambda+\lambda_{Y,t}\mathcal{Z}_{Y,t}^\lambda)dt+\mathcal{Z}_{S,t}^\lambda dW_{S,t}+\mathcal{Z}_{Y,t}^\lambda dW_{Y,t},$$
$$\mathcal{V}_T^\lambda=\exp(-\gamma_i(X_T^{\pi_i^*}+Y_T)), \tag{107}$$

has a unique solution $(\mathcal{V}^\lambda, \mathcal{Z}^\lambda)$ *satisfying* $E[\int_0^T((\mathcal{Z}^\lambda_{S,t})^2+(\mathcal{Z}^\lambda_{Y,t})^2)dt] < \infty$ *and* $E[\sup_{0\le t\le T}|\mathcal{V}^\lambda_t|^2] < \infty$*, and the inf is attained at* λ^*_i*. That is,* $\mathcal{V}^{\lambda^*}_t \le \mathcal{V}^\lambda_t,\ \forall\lambda\in\Lambda_i$*.*

(b) Also, in the aggressive case $i = I+1,\dots,I+I'$*, we suppose that for all* $\lambda\in\Lambda_i$*, a BSDE*

$$
\begin{aligned}
d\bar{\mathcal{V}}^\lambda_t &= -(\lambda_{S,t}\bar{\mathcal{Z}}^\lambda_{S,t} + \lambda_{Y,t}\bar{\mathcal{Z}}^\lambda_{Y,t})dt + \bar{\mathcal{Z}}^\lambda_{S,t}dW_{S,t} + \bar{\mathcal{Z}}^\lambda_{Y,t}dW_{Y,t},\\
\bar{\mathcal{V}}^\lambda_T &= \exp(-\gamma_i(X_T^{\pi_i^{\lambda,*}} + Y_T)),
\end{aligned}
\tag{108}
$$

has a unique solution $(\bar{\mathcal{V}}^\lambda, \bar{\mathcal{Z}}^\lambda)$ *satisfying* $E[\int_0^T((\bar{\mathcal{Z}}^\lambda_{S,t})^2+(\bar{\mathcal{Z}}^\lambda_{Y,t})^2)dt] < \infty$ *and* $E[\sup_{0\le t\le T}|\bar{\mathcal{V}}^\lambda_t|^2] < \infty$*, and the sup is attained at* λ^*_i*. That is,* $\bar{\mathcal{V}}^\lambda_t \le \bar{\mathcal{V}}^{\lambda^*}_t\ \forall\lambda\in\Lambda_i$*.*

5.2 Positive supply case

In a positive supply case, instead of assuming the volatility process exogenously, we assume the volatility of the supply in absolute terms, i.e., the standard deviation of the supply in a unit of time, and obtain the expected return μ and the volatility process σ, or equivalently, the stock price process S in equilibrium. Let $\mathcal{P}^s$ be the volatility in absolute terms of the supply for the risky asset. Then, given the supply volatility in absolute terms, the agents' utility, and their views on fundamental risks, we obtain the expected return process, the volatility process, or equivalently, the stock price process in equilibrium as in the following theorem. In this section, we define the clearing condition of the risky asset as

$$
\pi^s_t = \sum_{i=1}^{\bar{I}} \pi^*_{i,t},\ 0\le t\le T, \tag{109}
$$

instead of (23). In the following, for exogenously given $\bar{\lambda}_{Y,i}, \bar{\lambda}_{S,i}$, the parameters for Y, i.e., y_0, μ_Y, σ_Y, and the volatility of net supply of the risky asset in absolute term $\mathcal{P}^s$, we obtain the expected return μ and the volatility σ in equilibrium. The proof is omitted since it is done in the same way as in Theorem 3.

Theorem 4 *Let* $\Gamma = \frac{1}{\sum_{k=1}^{\bar{I}}\frac{1}{\gamma_k}}$*,*

$$
\bar{\lambda}^\ddagger_{Y,i,t} = \begin{cases} -\bar{\lambda}_{Y,i,t}\ (i=1,\dots,I)\\ +\bar{\lambda}_{Y,i,t}\ (i=I+1,\dots,I+I')\\ 0\ (i=I+I'+1,\dots,\bar{I})\end{cases}, \tag{110}
$$

and

$$
\bar{\lambda}^\ddagger_{S,i,t} = \begin{cases} -\bar{\lambda}_{S,i,t}\ (i=1,\dots,I)\\ +\bar{\lambda}_{S,i,t}\ (i=I+1,\dots,I+I')\\ 0\ (i=I+I'+1,\dots,\bar{I})\end{cases}. \tag{111}
$$

Given $\{\mathcal{F}_t\}$*-progressively measurable process* $\mathcal{P}^s$*, we assume the conditions (i)-(iv) listed below.*

If π^s *defined as*

$$\pi_t^s = \sum_{i=1}^{\bar{I}} x_{i,0} + \int_0^t \mathcal{P}_s^s(\theta_s ds + (\rho_{S,s} dW_{Y,s} + \hat{\rho}_{S,s} dW_{S,s})), \tag{112}$$

is positive for P*-almost surely, where* $x_{i,0} \geq 0,\ i = 1, \ldots, \bar{I}$ *are the initial wealth of agent* i*, and*

$$\theta_t = \Gamma(\mathcal{P}_t^s - \sum_{i=1}^{\bar{I}} (\frac{1}{\gamma_i}(+\rho_{S,t}\bar{\lambda}_{Y,i,t}^{\ddagger} + \hat{\rho}_{S,t}\bar{\lambda}_{S,i,t}^{\ddagger} - \gamma_i \rho_{S,t} Z_{i,t}))), \tag{113}$$

then, the Sharpe ratio process θ *in (113) and the volatility process* σ *defined as*

$$\sigma_t = \frac{\mathcal{P}_t^s}{\pi_t^s},\ 0 \leq t \leq T, \tag{114}$$

are in equilibrium. That is, (π_i^*, λ_i^*) *given by*

$$\pi_{i,t}^* = \frac{1}{\gamma_i \sigma_t}(\theta_t + \rho_{S,t}\bar{\lambda}_{Y,i,t}^{\ddagger} + \hat{\rho}_{S,t}\bar{\lambda}_{S,i,t}^{\ddagger} - \gamma_i \rho_{S,t} Z_{i,t}), \tag{115}$$

and

$$\lambda_{i,t}^* = (\bar{\lambda}_{Y,i,t}^{\ddagger}, \bar{\lambda}_{S,i,t}^{\ddagger})^\top, \tag{116}$$

attain the sup-inf/inf-sup problem (4), (5) $(i = 1, \ldots, I)$*, the sup-sup problem (6)* $(i = I+1, \ldots, I+I')$*, and the sup problem (7)* $(i = I+I'+1, \ldots, \bar{I})$ *for admissible strategies* $\pi \in \mathcal{A}_i$ *and the clearing conditions (9) and (109) are satisfied.*

Here, the set of the admissible strategies is given by $\mathcal{A}_i = \{\pi | X^\pi \text{is a } Q_i\text{-supermartingale}\}$ *and* Q_i *is defined by (iii) (a) below. For the aggressive case where* $i = I+1, \ldots I+I'$*, the admissible strategies is instead* $\mathcal{A}_i(\lambda) = \{\pi | X^\pi \text{is a } Q_i^\lambda\text{-supermartingale}\}$ *where* Q_i^λ *is defined by (iii) (b).*

(Conditions)

(i) (Solutions of BSDEs to define optimal portfolios and the Sharpe ratio process exist)

(a) Suppose that there exist (V_i, Z_i) $i = 1, \ldots, \bar{I}$*, that satisfy* $E[\sup_{0 \leq s \leq T} |V_{i,s}|^2] < \infty$, $E[\int_0^T Z_{i,s}^2 ds] < \infty$ *and BSDEs*

$$dV_{i,t} = -(f_i(Z_{1,t}, \ldots, Z_{\bar{I},t}) + \bar{\lambda}_{Y,i,t}^{\ddagger} Z_{i,t})dt + Z_{i,t} dW_{Y,t},$$
$$V_{i,T} = Y_T,$$

where

$$f_i(Z_{1,t}, \ldots, Z_{\bar{I},t})$$
$$= \frac{1}{2\gamma_i}(\theta_t + \rho_{S,t}\bar{\lambda}_{Y,i,t}^{\ddagger} + \hat{\rho}_{S,t}\bar{\lambda}_{S,i,t}^{\ddagger} - \gamma_i \rho_{S,t} Z_{i,t})^2 - \frac{1}{2}\gamma_i Z_{i,t}^2,$$

and

$$\theta_t = \Gamma(\mathcal{P}_t^s - \sum_{i=1}^{\bar{I}} \frac{1}{\gamma_i}(+\rho_{S,t}\bar{\lambda}_{Y,i,t}^{\ddagger} + \hat{\rho}_{S,t}\bar{\lambda}_{S,i,t}^{\ddagger} - \gamma_i \rho_{S,t} Z_{i,t})).$$

(b) Also, suppose that there exist $(V_i^\lambda, Z_i^\lambda)$ $i = I+1, \ldots, I+I'$, *that satisfy* $E[\sup_{0\le s\le T} |V_{i,s}^\lambda|^2] < \infty$, $E[\int_0^T (Z_{i,s}^\lambda)^2 ds] < \infty$ *and BSDEs*

$$dV_{i,t}^\lambda = -(\bar{f}_i(Z_{i,t}^\lambda) + \lambda_{Y,i,t} Z_{i,t}^\lambda)dt + Z_{i,t}^\lambda dW_{Y,t},$$
$$V_{i,T} = Y_T,$$

where

$$\bar{f}_i(Z_{i,t}^\lambda)$$
$$= \frac{1}{2\gamma_i}(\theta_t + \rho_{S,t}\lambda_{Y,i,t} + \hat{\rho}_{S,t}\lambda_{S,i,t} - \gamma_i \rho_{S,t} Z_{i,t}^\lambda)^2 - \frac{1}{2}\gamma_i (Z_{i,t}^\lambda)^2.$$

(ii) (The risky neutral probability measure Q_i *is well-defined)*

(a) We assume that

$$\Big\{\exp\Big(-\frac{1}{2}\int_0^t \gamma_i^2(\pi_{i,s}^*\sigma_s\rho_{S,s} + Z_{i,s})^2 + (\gamma_i\pi_{i,s}^*\sigma_s\hat{\rho}_{S,s})^2 ds + \int_0^t \gamma_i(\pi_{i,s}^*\sigma_s\rho_{S,s} + Z_{i,s})dW_{Y,s}^{\lambda_i^*} + \int_0^t \gamma_i \pi_{i,s}^*\sigma_s\hat{\rho}_{S,s} dW_{S,s}^{\lambda_i^*}\Big)\Big\}_{0\le t\le T}, \tag{117}$$

is a $P^{\lambda_i^*}$*-martingale, where* $\lambda_{i,t}^* = (+\bar{\lambda}_{Y,i,t}^\ddagger, +\bar{\lambda}_{S,i,t}^\ddagger)^\top$, $\pi_i^* = \frac{1}{\gamma_i\sigma_t}(\theta_t + \rho_{S,t}\bar{\lambda}_{Y,i,t}^\ddagger + \hat{\rho}_{S,t}\bar{\lambda}_{S,i,t}^\ddagger - \gamma_i\rho_{S,t}Z_{i,t})$.

(b) In addition, in the aggressive case $i = I+1, \ldots, I+I'$, *we assume that*

$$\Big\{\exp(-\frac{1}{2}\int_0^t \gamma_i^2(\pi_{i,s}^{\lambda,*}\sigma_s\rho_{S,s} + Z_{i,s}^\lambda)^2 + (\gamma_i\pi_{i,s}^{\lambda,*}\sigma_s\hat{\rho}_{S,s})^2 ds + \int_0^t \gamma_i(\pi_{i,s}^{\lambda,*}\sigma_s\rho_{S,s} + Z_{i,s}^\lambda)dW_{Y,s}^{\lambda_i^*} + \int_0^t \gamma_i\pi_{i,s}^{\lambda,*}\sigma_s\hat{\rho}_{S,s} dW_{S,s}^{\lambda_i^*})\Big\}_{0\le t\le T}, \tag{118}$$

is a P^{λ_i}*-martingale, where* $\lambda_{i,t} = (\lambda_{Y,i,t}, \lambda_{S,i,t})^\top$, $\pi_i^{\lambda,*} = \frac{1}{\gamma_i\sigma_t}(\theta_t + \rho_{S,t}\lambda_{Y,i,t} + \hat{\rho}_{S,t}\lambda_{S,i,t} - \gamma_i\rho_{S,t}Z_{i,t}^\lambda)$.

(iii) ($X^{\pi_i^*}$ *is a martingale under* Q_i*)*

(a) Also, for a probability measure Q_i *defined as*

$$\frac{dQ_i}{dP^{\lambda_i^*}} = \frac{u_i'(X_T^{\pi_i^*} + Y_T)}{E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*} + Y_T)]},$$

where

$$u_i'(x) = \gamma_i \exp(-\gamma_i x),$$

we assume that

$$E^{Q_i}\Big[\int_0^T (\pi_{i,t}^*)^2\sigma_t^2 dt\Big] < \infty. \tag{119}$$

(b) In addition, in the aggressive case where $i = I+1, \ldots, I+I'$, for a probability measure Q_i^λ defined as

$$\frac{dQ_i^\lambda}{dP^{\lambda_i}} = \frac{u_i'(X_T^{\pi_i^{\lambda,*}} + Y_T)}{E^{P^{\lambda_i}}[u_i'(X_T^{\pi_i^{\lambda,*}} + Y_T)]},$$

where we assume that

$$E^{Q_i^\lambda}\left[\int_0^T (\pi_{i,t}^{\lambda,*})^2 \sigma_t^2 dt\right] < \infty. \tag{120}$$

(iv) (Comparison results hold)

(a) Moreover, for the conservative agents i, $i = 1, \ldots, I$, we suppose that for all $\lambda \in \Lambda_i$, a BSDE

$$\begin{aligned} & d\mathcal{V}_t^\lambda \\ &= -(\lambda_{S,t}\mathcal{Z}_{S,t}^\lambda + \lambda_{Y,t}\mathcal{Z}_{Y,t}^\lambda)dt + \mathcal{Z}_{S,t}^\lambda dW_{S,t} + \mathcal{Z}_{Y,t}^\lambda dW_{Y,t}, \\ & \mathcal{V}_T^\lambda = \exp(-\gamma_i(X_T^{\pi_i^*} + Y_T)), \end{aligned} \tag{121}$$

has a unique solution $(\mathcal{V}, \mathcal{Z}^\lambda)$ satisfying $E[\int_0^T ((\mathcal{Z}_{S,t}^\lambda)^2 + (\mathcal{Z}_{Y,t}^\lambda)^2)dt] < \infty$ and $E[\sup_{0\le t\le T} |\mathcal{V}_t^\lambda|^2] < \infty$, and a comparison result holds. That is, $\mathcal{V}_t^\lambda \le \mathcal{V}_t^{\lambda^*}$ $\forall \lambda \in \Lambda_i$.

(b) For the aggressive agents i, $i = I+1, \ldots, I+I'$, for all $\lambda \in \Lambda_i$, a BSDE

$$\begin{aligned} & d\bar{\mathcal{V}}_t^\lambda \\ &= -(\lambda_{S,t}\bar{\mathcal{Z}}_{S,t}^\lambda + \lambda_{Y,t}\bar{\mathcal{Z}}_{Y,t}^\lambda)dt + \bar{\mathcal{Z}}_{S,t}^\lambda dW_{S,t} + \bar{\mathcal{Z}}_{Y,t}^\lambda dW_{Y,t}, \\ & \bar{\mathcal{V}}_T^\lambda = \exp(-\gamma_i(X_T^{\pi_i^{\lambda,*}} + Y_T)), \end{aligned} \tag{122}$$

has a unique solution $(\bar{\mathcal{V}}, \bar{\mathcal{Z}}^\lambda)$ satisfying $E[\int_0^T ((\bar{\mathcal{Z}}_{S,t}^\lambda)^2 + (\bar{\mathcal{Z}}_{Y,t}^\lambda)^2)dt] < \infty$ and $E[\sup_{0\le t\le T} |\bar{\mathcal{V}}_t^\lambda|^2] < \infty$, and a comparison result holds. That is, $\bar{\mathcal{V}}_t^\lambda \le \bar{\mathcal{V}}_t^{\lambda^*}$ $\forall \lambda \in \Lambda_i$.

Remark 4 *The zero-net supply corresponds to the case where $\pi^s = \mathcal{P}^s = 0$. In this case, instead of obtaining σ by (114) in the positive supply case, we exogenously specify σ and obtain the expected return μ in equilibrium.*

6 Conclusion

In this study, we have investigated a multi-agent equilibrium model with heterogeneous views on fundamental risks in an incomplete market setting. We have obtained the expressions of the expected return process in equilibrium in the cases of the square-root state case with the random bound for the views on Brownian motions and a general state process, where the sup-inf/inf-sup, sup-sup, or sup type individual optimization problems are solved. We have also presented numerical examples.

The implications of this study are as follows. Firstly, by utilizing the expected return process in equilibrium, traders can predict how the expected return on the risky asset changes when the sentiments of the market participants shift and construct a profitable trading strategy for investment. Also, policymakers such as central banks can control market sentiments as a result of their announcement of monetary policies so that it affects the risky asset price process they target by influencing the bandwidth of the sentiments of the market participants. Secondly, as a theoretical implication, the result shows that the market equilibrium can be obtained in the incomplete market setting with multiple agents with heterogeneous views on fundamental risks.

For limitations and future research, we have shown that the individual optimization problems are solved in the cases of the square-root state process and a general state process, assuming the one-time wealth shock depending on the state process, which is common among the agents and can be taken as a linear functional of the state process, and supposing the interest rate as zero. Extending the state process to a multi-dimensional one, investigating the case where the one-time wealth shock is a nonlinear functional of the state process, and solving for the equilibrium interest rate along with the excess return process are the next future research topics. Also, applying the model to security pricing under heterogeneous views on fundamental risks in an incomplete market is another future research topic.

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A Proof for the individual optimization problem of the conservative agents in Theorem 3 for the general procedure in the zero-net supply case in Section 5

In the following, we prove that the individual optimization problem of the conservative agents is attained with (π_i^*, λ_i^*). The aggressive case and the neutral agent case, as well as the fact that the market clearing conditions are satisfied, are proved in the same way as Theorem 1 in Section 3.

Let

$$J_i(\pi_i, \lambda) = E^{P^\lambda}[-\exp(-\gamma_i(X_T^{\pi_i} + Y_T))].$$

If (π_i^*, λ_i^*) is a saddle point that satisfies

$$J_i(\pi_i, \lambda_i^*) \leq J_i(\pi_i^*, \lambda_i^*) \leq J_i(\pi_i^*, \lambda),$$

for all $\pi_i \in \mathcal{A}_i$ and $\lambda \in \Lambda_i$, (π_i^*, λ_i^*) attains the sup-inf (4) and the inf-sup (5).

We show that for given λ_i^*, $\pi = \pi_i^*$ attains the sup by the following convex dual argument.

Proposition 1 *Under assumptions of Theorem 3, for given* $\lambda_i^* = (\bar{\lambda}^\ddagger_{Y,i,t}, \bar{\lambda}^\ddagger_{S,i,t})^\top$, $\pi = \pi_i^*$ *attains* $\sup_{\pi\in\mathcal{A}_i} J_i(\pi, \lambda_i^*)$.

Proof.
We consider

$$\sup_{\pi_i\in\mathcal{A}_i} E^{P^{\lambda_i^*}}[-\exp(-\gamma_i(X_T^{\pi_i} + Y_T))],$$

where

$$\begin{aligned} dY_t &= (\mu_{Y,t} + \sigma_{Y,t}\bar{\lambda}^\ddagger_{Y,i,t})dt + \sigma_{Y,t} dW^{\lambda_i^*}_{Y,t}, \\ dX_t^{\pi_i} &= \pi_{i,t}\sigma_t(\theta_t + \rho_{S,t}\bar{\lambda}^\ddagger_{Y,i,t} + \hat{\rho}_{S,t}\bar{\lambda}^\ddagger_{S,t})dt + \pi_{i,t}\sigma_t(\rho_{S,t} dW^{\lambda_i^*}_{Y,t} + \hat{\rho}_{S,t} dW^{\lambda_i^*}_{S,t}), \end{aligned} \tag{123}$$

$$\begin{aligned} dW^{\lambda_i^*}_{Y,t} &= dW_{Y,t} - \bar{\lambda}^\ddagger_{Y,i,t}dt, \\ dW^{\lambda_i^*}_{S,t} &= dW_{S,t} - \bar{\lambda}^\ddagger_{S,i,t}dt. \end{aligned}$$

We show that π_i^* attains the sup.
First, we let

$$R_{i,t} = -\exp(-\gamma_i(X_t^{\pi_i^*} + V_{i,t})),$$

where

$$\pi_{i,t}^* = \frac{1}{\gamma_i\sigma_t}(\theta_t^{\lambda_i^*} - \gamma_i\rho_{S,t}Z_{i,t}), \tag{124}$$

(V_i, Z_i) $i = 1, \dots, I$ are solutions of BSDEs

$$\begin{cases} dV_{i,t} = -f_i(Z_{1,t}, \dots, Z_{I,t})dt + Z_{i,t}dW_{Y,t}^{\lambda_i^*}, \\ V_{i,T} = Y_T, \end{cases}$$

with

$$f_i(Z_{1,t}, \dots, Z_{I,t}) = \frac{1}{2\gamma_i}(\theta_t^{\lambda_i^*} - \gamma_i \rho_{S,t} Z_{i,t})^2 - \frac{1}{2}\gamma_i Z_{i,t}^2,$$

$$\theta_t^{\lambda_i^*} = \theta_t + \rho_{S,t}\bar{\lambda}_{Y,i,t}^{\ddagger} + \hat{\rho}_{S,t}\bar{\lambda}_{S,i,t}^{\ddagger}.$$

Then,

$$\begin{aligned} dR_{i,t} &= -\gamma_i R_{i,t} d(X_t^{\pi_i^*} + V_{i,t}) + \frac{1}{2}\gamma_i^2 R_{i,t} d\langle X^{\pi_i^*} + V_i \rangle_t \\ &= -\gamma_i R_{i,t}\Bigg((\pi_{i,t}^* \sigma_t \theta_t^{\lambda_i^*} - \frac{1}{2}\gamma_i ((\pi_{i,t}^* \sigma_t \rho_{S,t} + Z_{i,t})^2 \\ &+ (\pi_{i,t}^* \sigma_t \hat{\rho}_{S,t})^2) - f_i(Z_{1,t}, \dots, Z_{I,t}))dt + (\pi_{i,t}^* \sigma_t \rho_{S,t} + Z_{i,t}) dW_{Y,t}^{\lambda_i^*} + \pi_{i,t}^* \sigma_t \hat{\rho}_{S,t} dW_{S,t}^{\lambda_i^*} \Bigg) \\ &= -\gamma_i R_{i,t}\Bigg((\pi_{i,t}^* \sigma_t \rho_{S,t} + Z_{i,t}) dW_{Y,t}^{\lambda_i^*} + \pi_{i,t}^* \sigma_t \hat{\rho}_{S,t} dW_{S,t}^{\lambda_i^*} \Bigg), \end{aligned} \tag{125}$$

since the drift part is

$$\begin{aligned} &(\pi_{i,t}^* \sigma_t \theta_t^{\lambda^*} - \frac{1}{2}\gamma_i ((\pi_{i,t}^* \sigma_t \rho_{S,t} + Z_{i,t})^2 + (\pi_{i,t}^* \sigma_t \hat{\rho}_{S,t})^2) - f_i(Z_{1,t}, \dots, Z_{I,t})) \\ = &-\frac{1}{2}\gamma_i \sigma_t^2 (\pi_{i,t}^* - \frac{1}{\gamma_i \sigma_t}(\theta_t^{\lambda_i^*} - \gamma_i \rho_{S,t} Z_{i,t}))^2 + \frac{1}{2\gamma_i}(\theta_t^{\lambda^*} - \gamma_i \rho_{S,t} Z_{i,t})^2 - \frac{1}{2}\gamma_i Z_{i,t}^2 - f_i(Z_{1,t}, \dots, Z_{I,t}) = 0. \end{aligned}$$

Next, we define a probability measure Q_i by

$$\frac{dQ_i}{dP^{\lambda_i^*}} = \frac{u_i'(X_T^{\pi_i^*} + Y_T)}{E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*} + Y_T)]}, \tag{126}$$

where

$$u_i'(x) = \gamma_i \exp(-\gamma_i x).$$

We remark that Q_i is well defined since $u_i'(x) > 0$ and $E^{P^{\lambda_i^*}}[\frac{dQ_i}{dP^{\lambda_i^*}}] = 1$. Since

$$\begin{aligned} u_i'(X_t^{\pi_i^*} + V_t) &= \gamma_i \exp(-\gamma_i (X_t^{\pi_i^*} + V_t)) \\ &= -\gamma_i R_{i,t}, \end{aligned}$$

and by (125)

$$d(-\gamma_i R_{i,t}) = -\gamma_i R_{i,t}(-\gamma_i (\pi_{i,t}^* \sigma_t \rho_{S,t} + Z_{i,t}) dW_{Y,t}^{\lambda_i^*} - \gamma_i \pi_{i,t}^* \sigma_t \hat{\rho}_{S,t} dW_{S,t}^{\lambda_i^*}),$$

we can apply Girsanov's theorem because R_i is a $P^{\lambda_i^*}$-martingale by (103), and $(W_Y^{Q_i}, W_S^{Q_i})$ defined by

$$dW_{Y,t}^{Q_i} = dW_{Y,t}^{\lambda_i^*} + \gamma_i(\pi_{i,t}^*\sigma_t\rho_{S,t} + Z_{i,t})dt,$$
$$dW_{S,t}^{Q_i} = dW_{S,t}^{\lambda_i^*} + \gamma_i\pi_{i,t}^*\sigma_t\hat{\rho}_{S,t}dt,$$

is a Q_i-Brownian motion.

Then, by (124)

$$\rho_{S,t}dW_{Y,t}^{\lambda_i^*} + \hat{\rho}_{S,t}dW_{S,t}^{\lambda_i^*}$$
$$= \rho_{S,t}dW_{Y,t}^{Q_i} + \hat{\rho}_{S,t}dW_{S,t}^{Q_i} - \theta_t^{\lambda_i^*}dt,$$

and thus by (123)

$$dX_t^{\pi_i} = \pi_{i,t}\sigma_t(\rho_{S,t}dW_{Y,t}^{Q_i} + \hat{\rho}_{S,t}dW_{S,t}^{Q_i}).$$

By (105), it follows that for $\pi_i^* \in \mathcal{A}_i$, $X^{\pi_i^*}$ is a Q_i-martingale.

Finally, we show

$$E^{P^{\lambda_i^*}}[-\exp(-\gamma_i(X_T^{\pi_i} + Y_T))]$$
$$\leq E^{P^{\lambda_i^*}}[-\exp(-\gamma_i(X_T^{\pi_i^*} + Y_T))],$$

by a convex duality argument.

We note that the following properties on the convex duality hold.

Let

$$\tilde{u}_i(y) = \sup_{x\in\mathcal{R}}(u_i(x) - xy),$$

for all $y > 0$, where $u_i(x) = -\exp(-\gamma_i x)$.

Then, for all $x \in \mathcal{R}$, $y > 0$,

$$u_i(x) \leq \tilde{u}_i(y) + yx, \tag{127}$$
$$\tilde{u}_i(u_i'(x)) + u_i'(x)x = u_i(x). \tag{128}$$

By (127),

$$u_i(X_T^{\pi_i} + Y_T) \leq \tilde{u}_i(E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i} + Y_T)]\frac{dQ_i}{dP^{\lambda_i^*}}) + E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i} + Y_T)]\frac{dQ_i}{dP^{\lambda_i^*}}(X_T^{\pi_i} + Y_T),$$

where we set

$$x = X_T^{\pi_i} + Y_T,$$
$$y = E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*} + Y_T)]\frac{dQ_i}{dP^{\lambda_i^*}}.$$

Hence

$$E^{P^{\lambda_i^*}}[-\exp(-\gamma_i(X_T^{\pi_i}+Y_T))] = E^{P^{\lambda_i^*}}[u_i(X_T^{\pi_i}+Y_T)]$$

$$\leq E^{P^{\lambda_i^*}}[\tilde{u}_i(E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_{i,T})]\frac{dQ_i}{dP^{\lambda_i^*}})] + E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_{i,T})]E^{P^{\lambda_i^*}}[\frac{dQ_i}{dP^{\lambda_i^*}}(X_T^{\pi_i}+Y_T)]$$

$$= E^{P^{\lambda_i^*}}[\tilde{u}_i(u_i'(X_T^{\pi_i^*}+Y_T))] + E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_T)]E^{Q_i}[(X_T^{\pi_i}+Y_T)] \tag{129}$$

$$\leq E^{P^{\lambda_i^*}}[\tilde{u}_i(u_i'(X_T^{\pi_i^*}+Y_T))] + E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_T)]E^{Q_i}[(X_T^{\pi_i^*}+Y_T)] \tag{130}$$

$$= E^{P^{\lambda_i^*}}[\tilde{u}_i(u_i'(X_T^{\pi_i^*}+Y_T))] + E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_T)(X_T^{\pi_i^*}+Y_T)] \tag{131}$$

$$= E^{P^{\lambda_i^*}}[-\exp(-\gamma_i(X_T^{\pi_i^*}+Y_T))]. \tag{132}$$

(130) follows since X^{π_i} is a Q_i-supermartingale and $X^{\pi_i^*}$ is a Q_i-martingale. (129) and (131) are due to the definition of Q_i in (126), and (132) is obtained from (128).

For given π_i^*, $\lambda = \lambda_i^*$ attains the inf. Thus, the proof is complete. ■

B The Gaussian case where the individual optimization problems for the conservative agents are solved

In this section, we solve the individual optimization problem for given expected return process μ of the risky asset process S_1 in (1) when Y in (3) is a Gaussian process, where $\bar{\lambda}_{Y,i}$, $\bar{\lambda}_{S,i}$, μ_Y, σ_Y and ρ_S are deterministic processes and $\bar{I} = I$, that is, the agents are conservative. Although we limit the case where the agents are conservative and the zero-net supply case for simplicity, the result can be extended to include the aggressive and the neutral agent case, and the positive supply case. In this case, the system of BSDEs reduces to separate BSDEs since Z in the BSDE can be specified as a volatility process of Y.

The following theorem holds for the expected return process and the trading strategy of the individual optimization problem (4) and (5) in equilibrium for the Gaussian case. We let $\Gamma = \frac{1}{\sum_{k=1}^{I}\frac{1}{\gamma_k}}$, $\bar{\lambda}^{\ddagger}_{Y,j,t} = -\bar{\lambda}_{Y,j,t}$, and $\bar{\lambda}^{\ddagger}_{S,j,t} = -\bar{\lambda}_{S,j,t}$ in the following.

Theorem 5 *Suppose that the following conditions hold.*

$$\frac{\rho_{S,t}}{\gamma_i}\left(\sum_{j=1}^{I}\frac{1}{\gamma_j}\Gamma(-\rho_{S,t}\bar{\lambda}^{\ddagger}_{Y,j,t} - \hat{\rho}_{S,t}\bar{\lambda}^{\ddagger}_{S,j,t} + \gamma_j\rho_{S,t}\sigma_{Y,t}) + \rho_{S,t}\bar{\lambda}^{\ddagger}_{Y,i,t} + \hat{\rho}_{S,t}\bar{\lambda}^{\ddagger}_{S,i,t} - \gamma_i\rho_{S,t}\sigma_{Y,t}\right) + \sigma_{Y,t}$$
$$= \pi^*_{i,t}\sigma_t\rho_{S,t} + \sigma_{Y,t} \geq 0, \tag{133}$$

$$\frac{\hat{\rho}_{S,t}}{\gamma_i}\left(\sum_{j=1}^{I}\frac{1}{\gamma_j}\Gamma(-\rho_{S,t}\bar{\lambda}^{\ddagger}_{Y,j,t} - \hat{\rho}_{S,t}\bar{\lambda}^{\ddagger}_{S,j,t} + \gamma_j\rho_{S,t}\sigma_{Y,t})\right)$$
$$= \pi^*_{i,t}\sigma_t\hat{\rho}_{S,t} \geq 0. \tag{134}$$

Then, the expected return process μ in equilibrium is given by $\mu_t = \sigma_t\theta_t$ where

$$\theta_t = \sum_{j=1}^{I}\frac{1}{\gamma_j}\Gamma(-\rho_{S,t}\bar{\lambda}^{\ddagger}_{Y,j,t} - \hat{\rho}_{S,t}\bar{\lambda}^{\ddagger}_{S,j,t} + \gamma_j\rho_{S,t}\sigma_{Y,t}), \tag{135}$$

and (π_i^*, λ_i^*) *in equilibrium is given by* $\pi_{i,t}^* = \frac{1}{\gamma_i \sigma_t}(\theta_t + \rho_{S,t}\bar{\lambda}_{Y,i,t}^{\ddagger} + \hat{\rho}_{S,t}\bar{\lambda}_{S,i,t}^{\ddagger} - \gamma_i \rho_{S,t}\sigma_{Y,t})$. *That is,* $\lambda_{i,t}^* = (+\bar{\lambda}_{Y,i,t}^{\ddagger}, +\bar{\lambda}_{S,i,t}^{\ddagger})^{\top}$ *attains the sup-inf/inf-sup problem (4), (5) for admissible strategies* $\pi \in \mathcal{A}_i$*, where the set of the admissible strategies is given by* $\mathcal{A}_i = \{\pi | X^{\pi} \text{is a } Q_i\text{-supermartingale}\}$*, where a probability measure* Q_i *is defined as*

$$\frac{dQ_i}{dP^{\lambda_i^*}} = \frac{u_i'(X_T^{\pi_i^*} + Y_T)}{E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*} + Y_T)]}.$$

Moreover, the market clearing conditions (22) and (23) hold.

Remark 5 *We remark that the following case, where there are two agents and one agent has neutral views, is an example that satisfies the conditions (133) and (134). Let* $I = 2$*,* $\gamma_1, \gamma_2 > 0$*,* $\rho_{S,t}, \hat{\rho}_{S,t} > 0$*,* $\sigma_{Y,t} > 0$*. We assume* $\bar{\lambda}_{Y,2}, \bar{\lambda}_{S,2} \equiv 0$*. Then, the conditions (133) and (134) become*

$$\frac{\rho_{S,t}}{\gamma_1}\left(\sum_{j=1}^{2} \frac{\frac{1}{\gamma_j}}{\sum_{k=1}^{2}\frac{1}{\gamma_k}}(-\rho_{S,t}\bar{\lambda}_{Y,j,t}^{\ddagger} - \hat{\rho}_{S,t}\bar{\lambda}_{S,j,t}^{\ddagger} + \gamma_j \rho_{S,t}\sigma_{Y,t}) - (-\rho_{S,t}\bar{\lambda}_{Y,1,t}^{\ddagger} - \hat{\rho}_{S,t}\bar{\lambda}_{S,1,t}^{\ddagger} + \gamma_1 \rho_{S,t}\sigma_{Y,t})\right) + \sigma_{Y,t}$$
$$= \pi_{1,t}^* \sigma_t \rho_{S,t} + \sigma_{Y,t} \geq 0,$$

and

$$\frac{\hat{\rho}_{S,t}}{\gamma_1}\left(\sum_{j=1}^{2} \frac{\frac{1}{\gamma_j}}{\sum_{k=1}^{2}\frac{1}{\gamma_k}}(-\rho_{S,t}\bar{\lambda}_{Y,j,t}^{\ddagger} - \hat{\rho}_{S,t}\bar{\lambda}_{S,j,t}^{\ddagger} + \gamma_j \rho_{S,t}\sigma_{Y,t}) - (-\rho_{S,t}\bar{\lambda}_{Y,1,t}^{\ddagger} - \hat{\rho}_{S,t}\bar{\lambda}_{S,1,t}^{\ddagger} + \gamma_1 \rho_{S,t}\sigma_{Y,t})\right)$$
$$= \pi_{1,t}^* \sigma_t \hat{\rho}_{S,t} \geq 0.$$

Proof of Theorem 5.

Let

$$J_i(\pi_i, \lambda) = E^{P^{\lambda}}[-\exp(-\gamma_i(X_T^{\pi_i} + Y_T))].$$

If (π_i^*, λ_i^*) is a saddle point that satisfies

$$J_i(\pi_i, \lambda_i^*) \leq J_i(\pi_i^*, \lambda_i^*) \leq J_i(\pi_i^*, \lambda),$$

for all $\pi_i \in \mathcal{A}_i$ and $\lambda \in \Lambda_i$, (π_i^*, λ_i^*) attains the sup-inf in (4) and the inf-sup in (5).

First, we show that for given λ_i^*, $\pi_i = \pi_i^*$ attains the sup as follows using a supermartingale property.

Lemma 2 *Under assumptions of Theorem 5, for given* $\lambda_i^* = (+\bar{\lambda}_{Y,i,t}^{\ddagger}, +\bar{\lambda}_{S,i,t}^{\ddagger})^{\top}$*,* $\pi_i = \pi_i^*$ *attains* $\sup_{\pi_i \in \mathcal{A}_i} J_i(\pi_i, \lambda_i^*)$*.*

Proof.

We consider

$$\sup_{\pi_i \in \mathcal{A}_i} E^{P^{\lambda_i^*}}[-\exp(-\gamma_i(X_T^{\pi_i} + Y_T))],$$

where

$$\begin{aligned} dY_t &= (\mu_{Y,t} + \sigma_{Y,t}\bar{\lambda}^{\ddagger}_{Y,i,t})dt + \sigma_{Y,t}dW^{\lambda^*_i}_{Y,t}, \\ dX^{\pi_i}_t &= \pi_{i,t}\sigma_t(\theta_t + \rho_{S,t}\bar{\lambda}^{\ddagger}_{Y,i,t} + \hat{\rho}_{S,t}\bar{\lambda}^{\ddagger}_{S,i,t})dt \\ &\quad + \pi_{i,t}\sigma_t(\rho_{S,t}dW^{\lambda^*_i}_{Y,t} + \hat{\rho}_{S,t}dW^{\lambda^*_i}_{S,t}), \end{aligned} \tag{136}$$

$$\begin{aligned} dW^{\lambda^*_i}_{Y,t} &= dW_{Y,t} - (+\bar{\lambda}^{\ddagger}_{Y,i,t})dt, \\ dW^{\lambda^*_i}_{S,t} &= dW_{S,t} - (+\bar{\lambda}^{\ddagger}_{S,i,t})dt. \end{aligned}$$

We show that π^*_i attains the sup.

First, we let

$$R_{i,t} = -\exp(-\gamma_i(X^{\pi}_t + V_{i,t})).$$

Here, $V_{i,t},\ i = 1, \dots, I$ are given by

$$V_{i,t} = Y_T + \int_t^T f_i(\sigma_{Y,t})dt - \int_t^T \sigma_{Y,t}dW^{\lambda^*_i}_{Y,t}, \tag{137}$$

with

$$f_i(\sigma_{Y,t}) = \frac{1}{2\gamma_i}(\theta^{\lambda^*_i}_t - \gamma_i\rho_{S,t}\sigma_{Y,t})^2 - \frac{1}{2}\gamma_i\sigma^2_{Y,t},$$

$$\theta^{\lambda^*_i}_t = \theta_t + \rho_{S,t}\bar{\lambda}^{\ddagger}_{Y,i,t} + \hat{\rho}_{S,t}\bar{\lambda}^{\ddagger}_{S,i,t}.$$

$$\begin{aligned} dR_{i,t} &= -\gamma_i R_{i,t} d(X^{\pi_i}_t + V_{i,t}) + \frac{1}{2}\gamma_i^2 R_{i,t} d\langle X^{\pi_i} + V_i\rangle_t \\ &= -\gamma_i R_{i,t}\Bigg((\pi_{i,t}\sigma_t\theta^{\lambda^*_i}_t - \frac{1}{2}\gamma_i((\pi_{i,t}\sigma_t\rho_{S,t} + \sigma_{Y,t})^2 + (\pi_{i,t}\sigma_t\hat{\rho}_{S,t})^2) - f_i(\sigma_{Y,t}))dt \\ &\qquad + (\pi_{i,t}\sigma_t\rho_{S,t} + \sigma_{Y,t})dW^{\lambda^*_i}_{Y,t} + \pi_{i,t}\sigma_t\hat{\rho}_{S,t}dW^{\lambda^*_i}_{S,t}\Bigg) \\ &= -\gamma_i R_{i,t}\Bigg(-\frac{1}{2}\gamma_i\sigma^2_t(\pi_{i,t} - \frac{1}{\gamma_i\sigma_t}(\theta^{\lambda^*_i}_t - \gamma_i\rho_{S,t}\sigma_{Y,t}))^2 dt + (\pi_{i,t}\sigma_t\rho_{S,t} + \sigma_{Y,t})dW^{\lambda^*_i}_{Y,t} + \pi_{i,t}\sigma_t\hat{\rho}_{S,t}dW^{\lambda^*_i}_{S,t}\Bigg), \end{aligned} \tag{138}$$

since the drift part is

$$\begin{aligned} &\Bigg(\pi_{i,t}\sigma_t\theta^{\lambda^*}_t - \frac{1}{2}\gamma_i((\pi_{i,t}\sigma_t\rho_{S,t} + \sigma_{Y,t})^2 + (\pi_{i,t}\sigma_t\hat{\rho}_{S,t})^2) - f_i(\sigma_{Y,t})\Bigg) \\ &= -\frac{1}{2}\gamma_i\sigma^2_t(\pi_{i,t} - \frac{1}{\gamma_i\sigma_t}(\theta^{\lambda^*_i}_t - \gamma_i\rho_{S,t}\sigma_{Y,t}))^2 + \frac{1}{2\gamma_i}(\theta^{\lambda^*}_t - \gamma_i\rho_{S,t}\sigma_{Y,t})^2 - \frac{1}{2}\gamma_i\sigma^2_{Y,t} - f_i(\sigma_{Y,t}) \\ &= -\frac{1}{2}\gamma_i\sigma^2_t(\pi_{i,t} - \frac{1}{\gamma_i\sigma_t}(\theta^{\lambda^*_i}_t - \gamma_i\rho_{S,t}\sigma_{Y,t}))^2, \end{aligned}$$

which is maximized at

$$\pi_i^* = \frac{1}{\gamma_i \sigma_t}(\theta_t^{\lambda_i^*} - \gamma_i \rho_{S,t}\sigma_{Y,t}). \tag{139}$$

Therefore, R_i is a supermartingale and particularly a martingale when $\pi_i = \pi_i^*$. Hence,

$$\begin{aligned} &E^{P^{\lambda_i^*}}[-\exp(-\gamma_i(X_T^{\pi_i} + Y_T))] \\ &\leq E^{P^{\lambda_i^*}}[-\exp(-\gamma_i(X_T^{\pi_i^*} + Y_T))]. \end{aligned}$$

■

Next, for given π_i^*, we show that $\lambda = \lambda_i^*$ attains the inf by a BSDE approach.

Lemma 3 *Under assumptions of Theorem 5, for given* $\pi_{i,t}^* = \frac{1}{\gamma_i\sigma_t}(\theta_t + \rho_{S,t}\bar{\lambda}_{Y,i,t}^{\ddagger} + \hat{\rho}_{S,t}\bar{\lambda}_{S,i,t}^{\ddagger} - \gamma_i\rho_{S,t}\sigma_{Y,t})$, $\lambda = \lambda_i^* = (+\bar{\lambda}_{Y,i,t}^{\ddagger}, +\bar{\lambda}_{S,i,t}^{\ddagger})^\top$ *attains* $\inf_{\lambda\in\Lambda_i} J_i(\pi_i^*, \lambda)$.

Proof.
Firstly, for $\lambda \in \Lambda_i$, we consider a BSDE

$$\begin{aligned} d\mathcal{V}_t^\lambda &= \mathcal{Z}_{S,t}^\lambda(dW_{S,t} - \lambda_{S,t}dt) + \mathcal{Z}_{Y,t}^\lambda(dW_{Y,t} - \lambda_{Y,t}dt) \\ &= -(\lambda_{S,t}\mathcal{Z}_{S,t}^\lambda + \lambda_{Y,t}\mathcal{Z}_{Y,t}^\lambda)dt + \mathcal{Z}_{S,t}^\lambda dW_{S,t} + \mathcal{Z}_{Y,t}^\lambda dW_{Y,t}, \\ \mathcal{V}_T^\lambda &= R_{i,T}. \end{aligned}$$

Also, we note that $\mathcal{V}_0^\lambda = E^{P^\lambda}[R_{i,T}]$ and $\mathcal{V}_0^\lambda$ is minimized at $(\lambda_{Y,t}^*, \lambda_{S,t}^*)$, $\lambda_{Y,t}^* = +\bar{\lambda}_{Y,i,t}^{\ddagger} sgn(\mathcal{Z}_{Y,t}^{\lambda^*})$, $\lambda_{S,t}^* = +\bar{\lambda}_{S,i,t}^{\ddagger} sgn(\mathcal{Z}_{S,t}^{\lambda^*})$, which satisfies $\lambda_{Y,t}^*\mathcal{Z}_{Y,t}^{\lambda^*} = +\bar{\lambda}_{Y,i,t}^{\ddagger}|\mathcal{Z}_{Y,t}^{\lambda^*}|$, $\lambda_{S,t}^*\mathcal{Z}_{S,t}^{\lambda^*} = +\bar{\lambda}_{S,i,t}^{\ddagger}|\mathcal{Z}_{S,t}^{\lambda^*}|$, by the comparison principle for BSDEs (e.g., Theorem 6.2.2 in Pham [27]).

In the following, we first presuppose that $\lambda_{Y,t}^*\mathcal{Z}_{Y,t}^{\lambda^*} = +\bar{\lambda}_{Y,i,t}^{\ddagger}$, $\lambda_{S,t}^*\mathcal{Z}_{S,t}^{\lambda^*} = +\bar{\lambda}_{S,i,t}^{\ddagger}$, then confirm $\mathcal{Z}_{Y,t}^{\lambda^*}, \mathcal{Z}_{S,t}^{\lambda^*} \geq 0$.

Let $R_{i,t} = -\exp(-\gamma_i(X_t^{\pi_i^*} + V_{i,t}))$, where $V_{i,t},\ i = 1,\dots,I$ are given by $V_{i,t} = Y_T + \int_t^T f_i(\sigma_{Y,s})ds - \int_t^T \sigma_{Y,s}dW_{Y,s}^{\lambda_i^*}$.

Since $R_{i,t} = -\exp(-\gamma_i(X_t^{\pi_i^*} + V_{i,t}))$ is a martingale under $P^{\lambda_i^*}$ satisfying an SDE

$$dR_{i,t} = \mathcal{Z}_{S,i,t}dW_{S,i,t}^{\lambda_i^*} + \mathcal{Z}_{Y,i,t}dW_{Y,i,t}^{\lambda_i^*},$$

where

$$\begin{aligned} \mathcal{Z}_{S,i,t} &= -\gamma_i R_{i,t}(\pi_{i,t}^*\sigma_t\rho_{S,t} + \sigma_{Y,t}), \\ \mathcal{Z}_{Y,i,t} &= -\gamma_i R_{i,t}(\pi_{i,t}^*\sigma_t\hat{\rho}_{S,t}), \end{aligned}$$

we have only to confirm $\mathcal{Z}_{S,i,t}, \mathcal{Z}_{Y,i,t} \geq 0$, namely,

$$\begin{aligned} \frac{\rho_{S,t}}{\gamma_i}(\theta_t^{\lambda_i^*} - \gamma_i\rho_{S,t}\sigma_{Y,t}) + \sigma_{Y,t} \geq 0, \\ \frac{\hat{\rho}_{S,t}}{\gamma_i}(\theta_t^{\lambda_i^*} - \gamma_i\rho_{S,t}\sigma_{Y,t}) \geq 0, \end{aligned}$$

which is satisfied by conditions (133) and (134). ■

Thus, (π_i^*, λ_i^*) is a saddle point and (π_i^*, λ_i^*) attains the sup-inf in (4) and the inf-sup in (5). Finally, we confirm that when the expected return process μ of the risky asset price process S_1 is given by $\mu_t = \sigma_t \theta_t$ with θ in (135), the market is in equilibrium, that is, the market clearing conditions

$$\sum_{i=1}^{I} \pi_{i,t}^* = 0, \tag{140}$$

and

$$\sum_{i=1}^{I} (X_t^{\pi_i^*} - \pi_{i,t}^*) = 0, \tag{141}$$

hold.

Lemma 4 *Under assumptions of Theorem 5, for the given expected return process μ, where $\mu_t = \sigma_t \theta_t$ with θ in (135), the market clearing conditions (140) and (141) hold.*

Proof. Since

$$\begin{aligned}
\pi_{i,t}^* &= \frac{1}{\gamma_i \sigma_t}(\theta_t^{\lambda_i^*} - \gamma_i \rho_{S,t} \sigma_{Y,t}) \\
&= \frac{1}{\gamma_i \sigma_t}(\theta_t + \rho_{S,t} \bar{\lambda}_{Y,i,t}^{\ddagger} + \hat{\rho}_{S,t} \bar{\lambda}_{S,i,t}^{\ddagger} - \gamma_i \rho_{S,t} \sigma_{Y,t}),
\end{aligned}$$

we have

$$\begin{aligned}
\sigma_t \sum_{i=1}^{I} \pi_{i,t}^* &= \sum_{i=1}^{I} \frac{1}{\gamma_i}(\theta_t + \rho_{S,t} \bar{\lambda}_{Y,i,t}^{\ddagger} + \hat{\rho}_{S,t} \bar{\lambda}_{S,i,t}^{\ddagger} - \gamma_i \rho_{S,t} \sigma_{Y,t}) \\
&= (\sum_{i=1}^{I} \frac{1}{\gamma_i})\theta_t - \sum_{i=1}^{I} \frac{1}{\gamma_i}(-\rho_{S,t} \bar{\lambda}_{Y,i,t}^{\ddagger} - \hat{\rho}_{S,t} \bar{\lambda}_{S,i,t}^{\ddagger} + \gamma_i \rho_{S,t} \sigma_{Y,t}) \\
&= 0.
\end{aligned}$$

Thus, $\sum_{i=1}^{I} \pi_{i,t}^* = 0$.

Also, (141) follows from (2) and (140).

Thus, the proof of Theorem 5 is completed. ■