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Super-Long Discount Rates for Insurers in Incomplete Markets with Bond Supply Control

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Abstract

This paper presents a multi-agent equilibrium model within an incomplete market framework, introducing a super-long discount rate for insurance companies while incorporating the dynamics of government financing and central bank operations in purchasing government bonds.

The financial market under examination includes dividend-paying securities that represent the market value of outstanding government bonds, categorized by their time to maturity. The model considers the optimal consumption and portfolio problems of agents, who possess varying risk sentiments and heterogeneous time preferences for consumption.

Applying a convex dual problem approach, we derive expressions for the equilibrium interest rate and market price of risk associated with government bonds, segmented by their remaining time to maturity. Additionally, we associate exogenously determined dividend processes, which reflect notional adjustments due to the central bank's bond purchases, government bond issuances, and coupon payments, in order to derive the yield curve for government bonds in equilibrium.

Our study's contribution lies in incorporating changes in the supply of government bonds in the secondary market into the incomplete market equilibrium model. This approach reveals the heterogeneous perspectives among agents regarding fundamental risks, represented by Brownian motion, and constructs a model that explicitly calculates the impact of net supply changes on super-long discount rates.

Lastly, we examine how variations in the supply of government and central bank bonds impact the pricing of insurance products, including death benefits and pension annuities, through changes in the super-long discount rates. This analysis is vital for insurance pricing and the evaluation of government bonds on the asset side of insurance companies.

Key words: Incomplete Market; Super-Long Discount Rates; Government Bond Supply; Insurance Asset-Liability Management
JEL classification codes: C61; D91; E52; G12; H63.

1 Introduction

Life insurance companies require super-long maturity bonds in their asset portfolios to match the cash flows generated by their liabilities, such as repayments for life and pension insurance. However, government bonds often have limited maturities, and ideal super-long maturity bonds, such as those with maturities of 50 to 100 years, are not available for effective asset-liability management within these companies. For instance, the longest maturity for Japanese government bonds is 40 years (Ministry of Finance, Japan [1]), while the longest maturity for U.S. Treasury bonds is 30 years. Even bonds with a maturity of 30 years experience liquidity issues, as life insurance companies are typically the only significant investors interested in such long-term investments.

As a result, insurance companies must develop methods to estimate the super-long interest rate for discounting purposes when pricing their insurance products. This may involve extrapolating their yield curves despite the lack of tradable super-long maturity assets, which would typically assist in determining the discount rate for pricing insurance products. This situation not only leads to a mismatch between the asset and liability sides of life insurance companies but also complicates the pricing of life insurance products. Since the longer the maturity of insurance products, the greater the impact of discount rate differences on their present value, assumptions about the super-long interest rate for discounting are critical for these companies.

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Moreover, it is increasingly vital for insurance companies to account for the effects of central bank policies on the supply of government bonds when estimating the super-long discount rate used in insurance pricing. Recently, central banks have implemented measures to control long-term interest rates through the management of government bond supply in the secondary market, which primarily involves institutional investors. For instance, the Bank of Japan has conducted yield curve control, purchasing long-term government bonds aggressively to maintain low long-term yields from 2017 to 2024 (Bank of Japan [2], [3]). Even after the policy's termination, they have also indicated the possibility of increasing the purchase amount if government bond yields rise due to inflation (Kihara [4]). Similarly, the European Central Bank has initiated unconventional monetary easing policies and asset purchase programs, buying assets, including government bonds, to keep yields low (Andrade et al. [5]). While traditional monetary policy focuses on controlling short-term interest rates, during periods of monetary easing, central banks often purchase long-term bonds and adjust their supply as part of their easing strategies. This control over the supply of long-term government bonds significantly influences the long-term interest rates that insurance companies rely on when establishing rates for future insurance payouts (see Bank of Japan [6]; Pelizzon et al. [7]).

Recognizing the significance of the super-long discount rate for insurance pricing and the central banks' control over government bond supplies, we propose a multi-agent equilibrium model. This model aims to determine super-long interest rates for discounting insurance product pricing, incorporating the bond supply control policies of the government and central banks. The agents within the model represent institutional investors' shareholders, who possess diverse perspectives on risk and varying time preferences for consumption.

Using this model, we can derive a long-term equilibrium interest rate that is useful for estimating the discount rate for insurance product payoffs. By incorporating dividend processes that reflect government bond supply influenced by monetary policy, we calculate the equilibrium interest rate, the market price of risk, and the resulting state price density processes. This allows us to derive the prices of zero-coupon bonds for varying maturities and construct the corresponding yield curve. The super-long interest rate, which has implications for the pricing of life and pension insurance products - potentially with maturities exceeding 100 years when considering increasing life expectancy in the future (Ministry of Health, Labor and Welfare, Japan [8]) - is determined endogenously through the multi-agent equilibrium model. This model incorporates the time preferences and perspectives of institutional investors' representative shareholders, as well as the dividend processes reflecting government policy actions.

We remark that our model utilizes a time and state-dependent expected utility, in which the utility function of agent i is defined as $u(t, \omega, c) = \exp(-\beta_i t)\eta^i_t \log c$. This function incorporates heterogeneous time preferences β_i and views λ^i on the fundamental risks represented by Brownian motion. Here, $\beta_i > 0$ is the time preference parameter, and η^i is an exponential martingale of the form $\eta^i_t = \exp\left(-\frac{1}{2}\int_0^t |\lambda^i_s|^2\, ds + \int_0^t \lambda^i_s\, dW_s\right)$, which will be explained in Section 2.2 in detail. Due to η^i_t in the expectation, this can be considered the subjective probability of the agent. This time and state-dependent utility function expresses agents' varying time preferences and reflects their heterogeneous views on fundamental risks. Additionally, this utility is analytically tractable, enabling us to determine an equilibrium state price density process for agents in an incomplete market setting.

Numerical examples illustrate how policy changes by the authorities and sentiment changes of the institutional investors' representative shareholders affect insurance pricing through the change in the super-long discount rate. A decrease in bond supply in the secondary market, typically resulting from the central bank's bond purchasing operations in monetary easing, for instance, tends to lower interest rates and increase insurance prices. Our model can estimate the effects of changes in bond market supply on insurance companies' insurance pricing through the change in the super-long discount rates.

For related literature on equilibrium models that derive the term structure of interest rates and address optimal consumption problems in an incomplete market setting, Vasicek [9], [10] developed a term structure of interest rates under complete market equilibrium, involving heterogeneous agents and a production process. Kizaki et al. [11] proposed a multi-agent equilibrium model in a complete market setting, where agents have differing sentiments about the fundamental risks represented by Brownian motion. Karatzas et al. [12] tackled the optimal consumption and portfolio problem for an agent in an incomplete market model using a convex dual approach, which incorporates fictitious securities. In research concerning an incomplete market equilibrium model, Kizaki et al. [13] explored multi-agent equilibrium scenarios in an incomplete market context. This research focused on obtaining risk-neutral measures for different agents and applying these measures to price reinsurance claims and analyze life-cycle investments under exponential utility, considering varying risk attitudes and views on fundamental risks.

While Kizaki et al. [13] model the stock market within a financial market framework, our study aims to explicitly model the bond market. We categorize government bonds by their time to maturity and analyze how changes in their supply, influenced by unhedgeable economic factors, affect insurance pricing through variations in super-long discount rates.

Moreover, in empirical studies examining the effect of central banks' outright purchases on the term structure of interest rates using a reduced-form model, Nakano et al. [14] analyzed the impact of the Bank of Japan's (BOJ) purchases of Japanese government bonds (JGBs) during the Qualitative and Quantitative Easing (QQE) period on the yield curve. They estimated this effect by directly linking the purchase amount to the yield curve using a state-space model. Koeda and Sekine [15] evaluated the dynamic Nelson-Siegel model for Japanese government bond yields, suggesting that QQE negatively affects the time-dependent decay factor in Nelson's model. Jarrow and Li [16] investigated the effect of quantitative easing on the term structure in the US by estimating an arbitrage-free term structure model that integrates the price impacts of central bank purchases of government bonds. Ray et al. [17] examined preferred habitat theory as a policy channel for quantitative easing, analyzing demand shocks within an equilibrium model. Additionally, Joyce et al. [18] studied the Bank of England's gilt purchases using an event study on announcements integrated with a portfolio balance model.

While these empirical studies assess the impact of monetary policies on yield curves, our research is novel in constructing a model that examines these effects through a general equilibrium framework, allowing for endogenous observation of impacts.

Regarding research on insurance problems, particularly concerning the optimal portfolio issues faced by insurance companies in asset-liability management (ALM), Thomson [19] examined market-consistent pricing of financial institutions' liabilities within an incomplete market that is in equilibrium. Wang and Zhang [20] addressed ALM challenges in incomplete markets and derived robust optimal investment strategies. Additionally, Wang et al. [21] investigated equilibrium investment strategies for defined contribution (DC) pension plans amid stochastic markets and variable contribution rates within a game-theoretic framework. For yield curve analysis related to insurance companies' asset-liability management, Vereda et al. [22] demonstrated that incorporating macroeconomic variables and market data enhances the estimation of an existing VAR model's yield curve. Zhao et al. [23] constructed a yield curve featuring endogenously determined very long-term interest rates, consistent with market conditions within an optimization framework.

Our study distinguishes itself from previous research by deriving equilibrium super-long discount rates within an incomplete market model that incorporates agents' heterogeneous time preferences and views on fundamental risks. This study aims to elucidate the intrinsic effects of a multi-agent equilibrium model on the yield curve by modeling the trading behaviors of institutional investors' representative shareholders as individual optimal consumption and investment problems. By this approach, we can determine the super-long discount rate in market equilibrium when the total supply of government bonds aligns with the demand from representative shareholders of institutional investors. While we have presented numerical examples illustrating the model's impacts on insurance pricing through changes in the super-long discount rate based on specified parameter sets, conducting an empirical analysis based on this model remains one of our future research objectives.

The organization of this paper is as follows: Section 2 introduces the incomplete market model, detailing the securities that represent the secondary market outstanding values of government bonds, categorized by time to maturity, along with the dividend processes that reflect the net supply changes of government bonds influenced by the central bank and government. This section also addresses an individual optimal consumption and portfolio problem within the incomplete market setting. Section 3 presents the calculations of interest rates and the market risk premium. Section 4 provides numerical examples illustrating how changes in the supply of government bonds and agents' sentiments impact insurance pricing through super-long discount rates. Finally, Section 5 concludes the paper.

2 Discrete cash flow model

This section introduces a discrete cash flow model for multi-agent equilibrium in an incomplete market setting. This model incorporates the government and central bank's control of bond supply to the secondary market, which comprises representative shareholders from banks and institutional investors such as life insurance companies and pension funds.

First, we suppose a financial market consisting of N groups of the government bonds categorized by time to maturity, one representative stock index in the country, and a money market account. Thus, $N+1$ securities and money market account are traded.

For a trading period $[0,T]$, $T > 0$, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, P)$ be a filtered probability space satisfying the usual conditions. Let W be a d-dimensional Brownian motion where $d \ge N+1$ and $\{\mathcal{F}_t\}_{0\le t\le T}$ be an augmented filtration generated by the Brownian motion W. Let $\delta^j, r, \mu_S^j, \sigma_S^j$ be $\mathcal{R}$-valued ($\mathcal{R}^{1\times d}$-valued for σ_S^j) $\{\mathcal{F}_t\}$-progressively measurable processes.

We also suppose that security j, $j = 1, \dots, N+1$, that is, government bonds in the zones categorized by time to maturity $\mathcal{T}_j$, $0 < \mathcal{T}_1 < \cdots < \mathcal{T}_N \le T$ and the stock index for security $N+1$, pay out δ^j as dividends at K discrete times $\{t_1, t_2, \dots, t_K\}$, $0 < t_1 < t_2, \dots, < t_K \le T$.

Namely, let $S^j,\ j=1,\ldots,N+1$ be the market value process of the j-th security satisfying an SDE

$$\begin{aligned} dS^j_t &= \mu^j_{S,t} S^j_t dt + S^j_t \sigma^j_{S,t} dW_t - dD^j_t \\ &= r_t S^j_t dt + S^j_t \sigma^j_{S,t}(\theta_t dt + dW_t) - dD^j_t, \end{aligned} \tag{1}$$

where θ is an $\mathcal{R}^d$-valued $\{\mathcal{F}_t\}$-progressively measurable process defined as

$$\theta = \sigma_S^\top(\sigma_S\sigma_S^\top)^{-1}(\mu_{S,t} - r) \in Range(\sigma_S^\top), \tag{2}$$

and D^j is a cumulative dividend process defined as $D^j_t = \sum_{k:t_k\le t}^K \delta^j_{t_k}$, and $dD^j_t := D^j_t - D^j_{t-}$. Here, $\sigma_S = (\sigma_S^{1\top}\ \ldots\ \sigma_S^{N+1,\top})^\top$, $\mu_S = (\mu_1 \ldots \mu_{N+1})^\top$, and $Range(\sigma_S^\top)$ is a linear space spanned by $\sigma_S^1\ \ldots\ \sigma_S^{N+1}$.

Also, we denote the price process of a money market account with instantaneous interest rate r by B, i.e.,

$$B_t = e^{\int_0^t r_s ds}. \tag{3}$$

2.1 Interpretation of the market outstanding value processes $S^j,\ j=1,\ldots,N+1$

In detail, we interpret the secondary market outstanding values $S^j,\ j=1,\ldots,N+1$ of the securities in the model as follows. We consider S^j as a secondary market outstanding value of the government bonds with time to maturity in zone $\mathcal{T}_j$ for $j=1,\ldots,N$ and the stock index in the secondary market for $j=N+1$. $\mu_S^j\ j=1,\ldots,N$ is the expected return of the secondary market outstanding value of the government bonds in the zone $\mathcal{T}_j$ categorized by time to maturity. Thus, we may understand μ_S^j as the instantaneous return of the government bonds categorized in the zone with time to maturity $\mathcal{T}_j$, where $0<\mathcal{T}_1<\mathcal{T}_2<\cdots<\mathcal{T}_N$. We consider $\delta^j_{t_k}$ as the net receipt amount for the representative shareholders of institutional investors, derived from the redemption minus issuance of government bonds and coupons from the government bonds within the zone categorized as security j. This amount is controlled exogenously by the government and central bank. Additionally, it includes the roll-down effect from long maturity to short maturity, where, as time passes, the bond's time to maturity shortens, causing it to be regarded as a bond in a zone of shorter time to maturity.

In summary, the outstanding market values of government bonds, categorized by time to maturity in the secondary market comprising agents such as representative shareholders of institutional investors, are denoted as $S^j,\ j=1,\ldots,N$. Additionally, the central bank's monetary policy and the government's fiscal policy are reflected in the dividend process $\delta^j,\ j=1,\ldots,N$.

We remark that the agents representing the shareholders of institutional investors trade the government bonds in the zones as the baskets of the bonds with constant maturities, which pay dividends expressing the coupon payments, redemption by the government, and the outright purchase by the central bank.

Remark 1 *More in detail, the finance and treasury department's bond issuing and the central bank's bond outright purchasing operations take place as follows.*

Ministry of Finance, the finance and treasury department, submits the issuing plan for the government bond every year. They investigate the average duration of the outstanding bonds they will need to redeem in the future and the coupons they will need to pay. Then, they decide how much and how long they borrow from the market by newly issuing bonds in addition to reissuing bonds for the redeemed amount. In detail, they determine the notional they issue along with the maturity of the bonds to compensate for the redemption.

On the other hand, the central bank also decides for which sector of time to maturity they will purchase the bonds by considering the outstanding notional of the government bonds in the sector and the yield to maturity. Then, they determine the amount they purchase for the bonds in the zones categorized by time to maturity. They buy government bonds called current bonds, issued in the latest among the bonds with the same time to maturity. For instance, in a simple case, if the central bank aims to buy bonds with a 9-year time to maturity, it would buy a 10-year bond issued one year ago, which is most recent, rather than a 20-year bond issued 11 years ago. Here, the 10-year bond issued one year ago is the current bond.

Moreover, representative shareholders of institutional investors, such as pension funds and life insurance companies, manage their portfolios by monitoring the notional amounts of bonds categorized by remaining time to maturity and their corresponding yields.

For instance, a representative shareholder of a large institutional investor may hold 1 trillion yen of JGBs with a time to maturity within five years, yielding 0.5%. They may also hold 2 trillion yen of JGBs with a time to maturity of 5 to 10 years, yielding 1.0%; 3 trillion yen of JGBs with a time to maturity of 10 to 20 years, yielding 1.5%; and 2 trillion yen of JGBs with a time to maturity exceeding 20 years, yielding 2.0%. These shareholders continuously monitor the yield curve of JGBs, which represents the yields corresponding to these different times to maturity.

Remark 2 *The primary feature of the model is the incorporation of trading strategies from the perspectives of institutional investors' representative shareholders. These shareholders manage their government bond portfolios by* *categorizing bond positions based on time to maturity and monitoring their yield to maturity. The model is also aligned with the perspectives of the government and the central bank, which issue or purchase government bonds according to their plans. They monitor the outstanding notional amounts of bonds by time to maturity and their yields.*

Although one straightforward way of modeling their strategy is implementing the issuing, redemption, and purchasing of government bonds, modeling in the way accompanies too many factors to be considered. First, each time, bonds with different maturities are issued or redeemed. Second, though various bonds have been issued in the past at one point, the bonds with the same time to maturity must have the same yield to maturity due to the arbitrage-free condition. For example, the ten-year bond, the 20-year bond issued ten years ago, and the 30-year bond issued 20 years ago have the same bond price if it is quoted in terms of zero coupon bond prices and the same yield to maturity. Thus, modeling all series of bonds issued during the period is excessively complex, and we need to focus on a reasonable way to capture the features of the model we deal with.

Thus, we develop a model that is useful for portfolio management by institutional investors' representative shareholders. This model focuses on bonds categorized by time to maturity and their yields, incorporating the impacts of outright purchases by the central bank and reissuances by the government.

This modeling is unique in that it captures institutional investors' representative shareholders' portfolio management, who aim to maximize their expected utility. It also incorporates the central bank's outright bond purchases and the government's issuance schedule management, focusing on the time to maturity, outstanding volume, and yield curve of government bonds. Thus, we can monitor the impact of the government's and the central bank's supply control of the government bonds on the term structure of interest rates.

2.2 Individual optimization problem of the agents

Next, we introduce the individual optimization problems of the I agents. We suppose that there are I $(I \geq 2)$ agents who have the log utility on K discrete-time consumption with heterogeneous beliefs on the Brownian motion W. Let π^i and $\pi^{i,0}$, $\{\mathcal{F}_t\}$-progressively measurable $\mathcal{R}^{N+1}$-valued and $\mathcal{R}$-valued processes satisfying $\int_0^T |\pi_t^i|^2 ds, \int_0^T |\pi_t^{i,0}|^2 ds, < \infty$, $P-a.s.$, be agent i's position of the $N+1$ securities and the money market, respectively, and X^i be the wealth process of agent i, the total value of agent i's portfolio. We also suppose that agent i invests π^i on value basis in the $N+1$ securities and the rest $\pi^{i,0}$ in the money market account and continuously rebalance its position. At K discrete times $t_1, \ldots, t_K$, agent i consumes c_k^i where $c_k^i,\ k = 1, \ldots, K$ are $\mathcal{F}_{t_k}$-measurable non-negative random variables satisfying $E[\sum_{k=1}^K (c_k^i)^2] < \infty$,

$$X_t^i = \pi_t^{i,0} + \pi_t^{i\top}\mathbf{1}, \tag{4}$$

where $\mathbf{1}$ is an $N+1$ dimensional column vector whose elements are 1. Firstly, the wealth process X^i satisfies the following SDE,

$$\begin{aligned} dX_t^i &= \pi_t^{i,0}\frac{dB_t}{B_t} + \sum_{j=1}^{N+1} \pi_t^{i,j}\frac{dS_t^j + dD_t^j}{S_t^j} - dC_t^i \\ &= r_t X_t^i dt + \pi_t^{i\top}\sigma_{S,t}(\theta_t dt + dW_t) - dC_t^i,\ X_0^i = x_0^i > 0, \end{aligned} \tag{5}$$

where

$$dC_t^i = C_t^i - C_{t-}^i;\ C_t^i = \sum_{k: t_k \leq t} c_k^i. \tag{6}$$

Moreover, we impose

$$X_t^i \geq 0, 0 \leq \forall t \leq T,\ P-a.s., \tag{7}$$

which indicates that agent i's wealth is always non-negative, that is, agent i does not go bankrupt.

Remark 3 *(5) indicates that the agents' holding position for security j changes in proportion to securities j's aggregate market value S^j at the discrete times $t_k,\ k = 1, \ldots, K$. That is, if one percent of the outstanding notional of security j is purchased back by the central bank or newly issued by the government, then one percent of the agent's holding security j is also bought back or added. In detail, let π_i^j be the value of agent i's position on the j-th zone of government bonds categorized by time to maturity. At the discrete time $t = t_k$, for example, agent i's position on security j on value basis also changes by $-\pi_{t_k}^{i,j}\frac{\delta_{t_k}^j}{S_{t_k}^j}$.*

Secondly, we consider the following set of consumption processes c^i as

$$\mathcal{A}^i = \left\{ c^i = \{c_k^i\}_{k=1,\ldots,K} \mid E\left[\sum_{k=1}^K \frac{Z_{t_k}^{\theta^i}}{B_{t_k}} c_k^i\right] \leq x_0^i, \text{for all probability densities} Z^{\theta^i} \text{for risk-neutral probability measures} \right\}, \tag{8}$$

where the probability density Z^{θ^i} is of the form

$$Z_t^{\theta^i} = e^{\left\{-\frac{1}{2}\int_0^t |\theta_s+\nu_s^i|^2 ds - \int_0^t (\theta_s+\nu_s^i)\cdot dW_s\right\}}, \ \sigma_{S,t}\nu_t^i = 0, \ \forall t \in [0,T], \tag{9}$$

where ν^i is an $\mathcal{R}^d$-valued $\{\mathcal{F}_t\}$-progressively measurable process satisfying a weak version of Novikov's condition (e.g., Corollary 3.5.14 in Karatzas and Shreve [24]).

Remark 4 *The above admissibility of the consumption process derives from the condition in (7) where agent i does not go bankrupt, i.e., its wealth is always non-negative. Since*

$$X_t^i = x_0^i + \sum_{j=1}^{N+1} \int_0^t \pi_s^{i,j} \frac{dS_s^j + dD_s^j}{S_s^j} - \sum_{k:t_k<t} c_k^i \geq 0, \tag{10}$$

taking expectation with the state-price density $H_t^i = \frac{Z_t^{\theta^i}}{B_t}$ *for* $t = T$ *yields*

$$x_0^i \geq E\left[\sum_{k=1}^K c_k^i H_{t_k}^i\right]. \tag{11}$$

Thirdly, we assume that each agent has the log utility $u(x) = \log x$ on the consumption with subjective belief λ^i on the Brownian motion W and maximizes the sum of its expected utility on the discrete consumption with discounting for time-preference $\alpha_t^i = e^{-\beta_i t}$, where $\beta_i > 0$.

In detail, the probability density η_T^i for the subjective belief λ^i of agent i, is defined as

$$\eta_T^i = \exp\left(\int_0^T \lambda_s^i \cdot dW_t - \frac{1}{2}\int_0^T |\lambda_s^i|^2 ds\right), \tag{12}$$

where λ^i is an $\mathcal{R}^d$-valued $\{\mathcal{F}_t\}$-progressively measurable process satisfying a weak version of Novikov's condition (e.g., Corollary 3.5.14 in Karatzas and Shreve [24]). Here, λ^i represents agent i's subjective views on the Brownian motion W. Namely, for the probability measure P^i defined as

$$\frac{dP^i}{dP} = \eta_T^i, \tag{13}$$

by Girsanov's theorem, W^{P^i} defined as $dW_t = dW_t^{P^i} + \lambda^i dt$ is a P^i-Brownian motion and λ^i indicates agent i's bias on the Brownian motion W under the physical probability measure P.

Then, we consider the following individual optimization problem of agent i.

(**Individual Optimization Problem**)

Maximize

$$\sum_{k=1}^K E[\eta_{t_k}^i \alpha_{t_k}^i \log c_k^i], \tag{14}$$

with respect to $c^i \in \mathcal{A}^i$.

Specifically, c^i is subject to the budget constraint

$$E\left[\sum_{k=1}^K H_{t_k}^i c_k^i\right] \leq x_0^i, \tag{15}$$

where $\alpha_t^i = e^{-\beta^i t}$, and $H_t^i = \frac{Z_t^{\theta^i}}{B_t}$, for all density processes Z^{θ^i} for the risk-neutral probability measures of the form

$$Z_t^{\theta^i} = e^{\left\{-\frac{1}{2}\int_0^t |\theta_s+\nu_s^i|^2 ds - \int_0^t (\theta_s+\nu_s^i)\cdot dW_s\right\}}, \ \sigma_{S,t}\nu_t^i = 0, \ \forall t \in [0,T]. \tag{16}$$

This individual optimization problem indicates that agent i aims to maximize its expected utility on its consumption at discrete times with the time preference β^i by choosing the consumption amount while continuously trading on $N+1$ securities and the money market account.

2.3 Solving a dual problem for the individual optimization

This section solves a dual-problem for this individual optimization problem (14) and (15). In the following, noting that the individual optimization problem (14) and (15) is described as

$$\sup_{c^i} \inf_{y^i>0,\nu^i,\sigma_S\nu^i=0} \sum_{k=1}^K E[\eta_{t_k}^i \alpha_{t_k}^i \log c_k^i] + y^i\left(x_0^i - E[\sum_{k=1}^K H_{t_k}^i c_k^i]\right), \tag{17}$$

we solve the dual problem of this and later confirm that the obtained solution is optimal.

Remark 5 *The primal optimization problem describes the individual optimization problem (14) and (15) for the* *following reasons. If the given c^i does not satisfy the budget constraint (15) for some H^i, the inf part is $-\infty$ by taking y^i any large number. Thus, the sup part indicates that the sup is taken with respect to c^i satisfying the budget constraint (15). Hence, the primal problem expresses the individual optimization problem (14) and (15) where taking supremum on the expected utility with respect to c^i satisfying the budget constraint (15).*

As we will observe in Section 2.5 for arbitrary $\{c_k\}_{k=1,\dots,K}$, where $c_k,\ k=1\dots,K$ are non-negative $\mathcal{F}_{t_k}$-measurable random variables, we determine $(\pi^i,\pi^{i,0})$ so that $c^i\in\mathcal{A}_i$, we have only to consider the following dual problem.

(Dual problem)

$$\inf_{y^i>0,\nu^i,\sigma_S\nu^i=0}\sup_{c^i}\sum_{k=1}^K E[\eta^i_{t_k}\alpha^i_{t_k}\log c^i_k]+y^i\left(x^i_0-E[\sum_{k=1}^K H^i_{t_k}c^i_k]\right), \tag{18}$$

Proposition 1 *$c^{i,*}$, ν^i, y^i set as*

$$c^{i,*}_k=\frac{x^i_0}{\sum_{k=1}^K\alpha^i_{t_k}}\frac{\alpha^i_{t_k}Z^i_{t_k}B_{t_k}}{Z^\theta_{t_k}}>0, \tag{19}$$

$$\nu^i_t=-\hat{\lambda}^{i,\perp}_t, \tag{20}$$

$$y^i=\frac{\sum_{k=1}^K\alpha^i_{t_k}}{x^i_0}, \tag{21}$$

(22)

attain the inf-sup dual problem (18), where

$$Z^i_t=\exp\biggl(-\frac{1}{2}\int_0^t|\hat{\lambda}^i_s|^2ds+\int_0^t\hat{\lambda}^i_s\cdot dW_s\biggr), \tag{23}$$

$$Z^\theta_t=\exp\biggl(-\frac{1}{2}\int_0^t|\theta_s|^2ds-\int_0^t\theta_s\cdot dW_s\biggr). \tag{24}$$

Here, $\hat{\lambda}^{i,\perp}$ is an orthogonal part of λ^i to the linear space spanned by σ_S, i.e., $\lambda^i=\hat{\lambda}^i+\hat{\lambda}^{i,\perp}$, $\hat{\lambda}^i\in Range(\sigma_S^\top)$, $\sigma_S\hat{\lambda}^{i,\perp}=0$.

(Proof). First, for fixed y^i,ν^i, we consider for each k and sample $\omega\in\Omega$,

$$\sup_{c^i_k\ge 0}[\alpha^i_{t_k}\eta^i_{t_k}\log c^i_k-y^iH^i_{t_k}c^i_k]. \tag{25}$$

This supremum is attained at

$$c^{i,*}_k=\frac{\alpha^i_{t_k}\eta^i_{t_k}}{y^iH^i_{t_k}}. \tag{26}$$

Setting

$$\tilde{U}(y^iH^i_{t_k},t_k):=\alpha^i_{t_k}\eta^i_{t_k}\log c^{i,*}_k-y^iH^i_{t_k}c^{i,*}_k, \tag{27}$$

we consider

$$\inf_{\nu^i,\ \sigma_S\nu^i=0}E\left[\sum_{k=1}^K\tilde{U}(y^iH^i_{t_k},t_k)\right]. \tag{28}$$

First, we calculate

$$\begin{aligned}E\left[\sum_{k=1}^K\tilde{U}(y^iH^i_{t_k},t_k)\right]&=E\left[\sum_{k=1}^K\alpha^i_{t_k}\eta^i_{t_k}\log\left(\frac{\alpha_k\eta^i_{t_k}}{y^iH^i_{t_k}}\right)\right]-E\left[\sum_{k=1}^K y^iH^i_{t_k}\frac{\alpha^i_{t_k}\eta^i_{t_k}}{y^iH^i_{t_k}}\right]\\&=E\left[\sum_{k=1}^K\alpha^i_{t_k}\eta^i_{t_k}\{\log\alpha^i_{t_k}+\log\eta^i_{t_k}-\log y^i-\log H^i_{t_k}\}\right]-\sum_{k=1}^K\alpha^i_{t_k}.\end{aligned} \tag{29}$$

Thus, we have only to consider

$$\inf_{\nu^i,\ \sigma_S\nu^i=0}E\left[-\sum_{k=1}^K\eta^i_{t_k}\log H^i_{t_k}\right]= \tag{30}$$

$$\inf_{\nu^i,\ \sigma_S\nu^i=0}E\left[\sum_{k=1}^K\eta^i_{t_k}\left\{\int_0^{t_k}r_sds+\frac{1}{2}\int_0^{t_k}|\theta_s+\nu^i_s|^2ds+\int_0^{t_k}(\theta_s+\nu^i_s)\cdot dW_s\right\}\right] \tag{31}$$

Noting that $\theta \cdot \nu^i = \hat{\lambda}^i \cdot \nu^i = 0$, since

$$\inf_{\nu^i,\ \sigma_S \nu^i = 0} E^i \left[\int_0^T \left(\frac{1}{2} |\nu_s^i|^2 + \nu_s^i \cdot \lambda_s^i \right) ds \right] \tag{32}$$

is attained at $\nu^i = -\hat{\lambda}^{i,\perp}$, we observe that the infimum is attained at $\nu^i = -\hat{\lambda}^{i,\perp}$. Then, we have

$$\begin{aligned} H_t^i &= \exp\left[-\int_0^t r_s ds - \frac{1}{2} \int_0^t |\theta_s - \hat{\lambda}_s^{i,\perp}|^2 ds - \int_0^t (\theta_s - \hat{\lambda}_s^{i,\perp}) \cdot dW_s \right] \\ &= \frac{Z_t^\theta}{B_t \exp(-\frac{1}{2} \int_0^t |\hat{\lambda}_s^{i,\perp}|^2 ds - \int_0^t \hat{\lambda}_s^{i,\perp} \cdot dW_s)} \end{aligned} \tag{33}$$

where E^i denotes the expectation operator under the measure induced by λ^i, i.e., $\frac{dP^i}{dP} = \eta_T^i$, which implies that $dW_t = dW_t^i + \lambda_t^i dt$, where W^i is a Brownian motion under P^i, by Girsanov's theorem.

Finally, noting that consumption, dividend, and redemption occur at t_k, $k = 1, \cdots, K$ with $t_0 = 0$ and $t_K = T$, y^i is calculated as

$$\inf_{y^i} \sum_{k=1}^K E[\eta_{t_k}^i \alpha_{t_k}^i \log c_k^{i,*}] + y^i E\left[x_0^i - \sum_{k=1}^K H_{t_k}^i c_k^{i,*} \right], \tag{34}$$

where

$$c_k^{i,*} = \frac{\eta_{t_k}^i \alpha_{t_k}^i}{y^i H_{t_k}^i} = \frac{\alpha_{t_k}^i Z_{t_k}^i B_{t_k}}{y^i Z_{t_k}^\theta}, \tag{35}$$

where we used

$$\frac{\eta_{t_k}^i}{H_{t_k}^i} = \frac{Z_{t_k}^i B_{t_k}}{Z_{t_k}^\theta}. \tag{36}$$

Substituting (35) into (34), we observe that this infimum is attained at

$$y^i = \frac{\sum_{k=1}^K \alpha_{t_k}^i}{x_0^i}, \tag{37}$$

and thus

$$c_k^{i,*} = \frac{x_0^i}{\sum_{k=1}^K \alpha_{t_k}^i} \frac{\alpha_{t_k}^i Z_{t_k}^i B_{t_k}}{Z_{t_k}^\theta}. \tag{38}$$

□

2.4 Confirmation of the optimality by convex duality argument

Then, we confirm that the obtained solution, $c^{i,*}$, ν^i, and y^i, of the dual problem (18) is also a solution of the primal problem (17) as follows.

Theorem 1 *The solution of the dual problem (18), $c^{i,*}$, ν^i, y^i obtained as*

$$c_k^{i,*} = \frac{x_0^i}{\sum_{k=1}^K \alpha_{t_k}^i} \frac{\alpha_{t_k}^i Z_{t_k}^i B_{t_k}}{Z_{t_k}^\theta} > 0, \tag{39}$$

$$\nu_t^i = -\hat{\lambda}_t^{i,\perp},\ 0 \le t \le T \tag{40}$$

$$y^i = \frac{\sum_{k=1}^K \alpha_{t_k}^i}{x_0^i}, \tag{41}$$

$$\tag{42}$$

also attains the sup-inf of the primal problem (17).

(Proof). We show this by a convex duality argument. Noting that for $y > 0$, $u : \mathcal{R}^+ \to \mathcal{R}$ twice continuously differentiable with $u'(x) > 0, u''(x) < 0$, and $\tilde{u}(y) := \sup_x (u(x) - xy)$,

$$\begin{aligned} \tilde{u}(y) &\ge u(x) - xy \\ \tilde{u}(u'(x)) &= u(x) - xu'(x), \end{aligned} \tag{43}$$

we consider

$$\begin{aligned} u_k(x) &= \alpha_{t_k}^i \eta_{t_k}^i \log x, \\ u_k'(x) &= \frac{\alpha_{t_k}^i \eta_{t_k}^i}{x}, \end{aligned} \tag{44}$$

$$u_k(x) \le \sup_x (u_k(x) - xy) + xy$$
$$= \tilde{u}_k(y) + xy \tag{45}$$

For $c_k^{i,*} = \frac{\eta^i_{t_k}\alpha^i_{t_k}}{y^i H^i_{t_k}}$ and arbitrary c_k^i satisfying the budget constraint, with $y = y^i H^i_{t_k}$,

$$\eta^i_{t_k}\alpha^i_{t_k}\log(c_k^i) \le \tilde{u}_k(y^i H^i_{t_k}) + c_k^i y^i H^i_{t_k},$$
$$\tilde{u}_k(y^i H^i_{t_k}) = \eta^i_{t_k}\alpha^i_{t_k}\log(c_k^{i,*}) - c_k^{i,*} y^i H^i_{t_k}, \tag{46}$$

where the second equation follows since $u_k'(c_k^{i,*}) = y^i H^i_{t_k}$.

By the budget constraint, we have

$$\sum_{k=1}^K E[c_k y^i H^i_{t_k}] \le x_0^i, \tag{47}$$

Also, since

$$\sum_{k=1}^K E[c_k^{i,*} y^i H_{t_k}] = E\left[\eta^i_{t_k}\alpha^i_{t_k}\frac{H_{t_k}}{H^i_{t_k}}\right] = y^i \sum_{k=1}^K \alpha^i_{t_k} = x_0^i, \tag{48}$$

holds for all state price density processes H, $c_k^{i,*}$ satisfy the budget constraint for arbitrary state price density process H.

Therefore, we have

$$\sum_{k=1}^K E[\eta^i_{t_k}\alpha^i_{t_k}\log(c_k^i)] \le \sum_{k=1}^K E[\tilde{u}_k(y^i H^i_{t_k})] + \sum_{k=1}^K E[c_k^i y^i H^i_{t_k}]$$
$$\le \sum_{k=1}^K E[\tilde{u}_k(u_k'(c_k^{i,*}))] + E[c_k^{i,*} y^i H^i_{t_k}] = \sum_{k=1}^K E[\eta^i_{t_k}\alpha^i_{t_k}\log(c_k^{i,*})]. \tag{49}$$

□

2.5 Optimal wealth and portfolio processes of the agents

Finally, for the optimal consumption $c^{i,*}$ obtained as the solution of the primal problem, the corresponding portfolio process $(\pi^{i,*}, \pi^{i,0,*})$ satisfying the non-negative wealth process condition (7) is obtained as follows.

Theorem 2 *Under the assumption that* $\mathrm{rank}(\sigma_{S,t}) = N+1$ *for* $0 \le t \le T$*, the optimal wealth process* $X^{i,*}$ *and the portfolio process* $(\pi^{i,*}, \pi^{i,0,*})$ *corresponding to the optimal consumption* $c^{i,*}$ *in (39), i.e.,* $(\pi^{i,*}, \pi^{i,0,*})$ *that generates the wealth* $X^{i,*}$ *satisfying the non-negative wealth process condition in (7), for the individual optimization problem (15) are given as follows.*

$$X_t^{i,*} = \frac{B_t Z_t^i}{Z_t^\theta}\frac{x_0^i}{\sum_{k=1}^K \alpha^i_{t_k}}\sum_{k:t_k \ge t}^K \alpha^i_{t_k} > 0, \tag{50}$$

$$\pi_t^{i,*} = X_t^{i,*}(\sigma_{S,t}\sigma_{S,t}^\top)^{-1}\sigma_{S,t}(\hat{\lambda}_t^i + \theta_t), \tag{51}$$

and

$$\pi_t^{i,0,*} = X_t^{i,*} - \pi_t^{i,*\top}\mathbf{1}. \tag{52}$$

(Proof).

First, we note that if we find $X^{i,*}$ associated with $(\pi^{i,*}, \pi^{i,0,*})$ such that $X_t^{i,*}H_t^i + \sum_{k:t_k<t} c_k^{i,*}H^i_{t_k}$ is a martingale and $X_T^{i,*} = 0$, $c^{i,*}$ is in the admissible set $\mathcal{A}^i$, since

$$x_0 = E[\sum_{k:t_k<T} c_k^{i,*}H^i_{t_k}]. \tag{53}$$

We can find such wealth process of agent i $X^{i,*}$ by

$$X_t^{i,*} = \frac{1}{H_t^i}E[\sum_{k:t<t_k<T} c_k^{i,*}H^i_{t_k}|\mathcal{F}_t] \tag{54}$$

since

$$X_t^{i,*}H_t^i + \sum_{k:t_k<t} c_k^{i,*}H^i_{t_k} = E[\sum_{k:t_k<T} c_k^{i,*}H^i_{t_k}|\mathcal{F}_t]. \tag{55}$$

As agent i's optimal consumption is $c_k^{i,*} = \frac{\eta_k^i \alpha_{t_k}^i}{y^i H_{t_k}^i}$, we calculate

$$\frac{1}{H_t^i} E[H_{t_k}^i c_k^{i,*} | \mathcal{F}_t] = \frac{1}{H_t^i} E\left[\frac{\eta_k^i \alpha_{t_k}^i}{y^i} | \mathcal{F}_t\right] = \frac{\eta_t^i}{H_t^i} \frac{\alpha_{t_k}^i}{y^i}, \tag{56}$$

where

$$\eta_t^i = \exp\left[-\frac{1}{2}\int_0^t |\lambda_s^i|^2 ds + \int_0^t \lambda_s^i \cdot dW_s\right]. \tag{57}$$

Then, noting that

$$\frac{\eta_t^i}{H_t^i} = \frac{Z_t^i B_t}{Z_t^\theta}, \tag{58}$$

since

$$H_t^i = \frac{1}{B_t} \exp\left[-\frac{1}{2}\int_0^t |\theta_s - \hat{\lambda}_s^{i,\perp}|^2 ds - \int_0^t (\theta_s - \hat{\lambda}_s^{i,\perp}) \cdot dW_s\right] \tag{59}$$

$$= \frac{1}{B_t} \exp\left[-\frac{1}{2}\int_0^t (|\theta_s|^2 + |\hat{\lambda}_s^{i,\perp}|^2) ds + \int_0^t (-\theta_s + \hat{\lambda}_s^{i,\perp}) \cdot dW_s\right], \tag{60}$$

we have

$$\frac{1}{H_t^i} E[H_{t_k}^i c_k^{i,*} | \mathcal{F}_t] = \frac{\alpha_{t_k}^i B_t Z_t^i}{y^i Z_t^\theta}. \tag{61}$$

Thus, we obtain $X^{i,*}$ as

$$X_t^{i,*} = \sum_{k:t_k \geq t}^K \frac{1}{H_t^i} E[H_{t_k}^i c_k^{i,*} | \mathcal{F}_t] = \frac{B_t Z_t^i}{y^i Z_t^\theta} \sum_{k:t_k \geq t}^K \alpha_{t_k}^i, \tag{62}$$

$$= \frac{B_t Z_t^i}{Z_t^\theta} \frac{x_0^i}{\sum_{k=1}^K \alpha_{t_k}^i} \sum_{k:t_k \geq t}^K \alpha_{t_k}^i, \tag{63}$$

where we used

$$\frac{1}{y^i} = \frac{x_0^i}{\sum_{k=1}^K \alpha_{t_k}^i}. \tag{64}$$

Next, we calculate $\pi^{i,*}$ associated with the wealth process $X^{i,*}$ as follows.
Recalling

$$c_k^{i,*} = \frac{B_{t_k} Z_{t_k}^i}{Z_{t_k}^\theta} \frac{x_0^i}{\sum_{k=1}^K \alpha_{t_k}^i} \alpha_{t_k}^i, \tag{65}$$

$c_k^{i,*}$ is paid out from $X^{i,*}$ as consumption at each t_k.
Accordingly, with $C_t^{i,*} := \sum_{k:t_k \leq t}^K c_k^{i,*}$, and $dC_t^{i,*} := C_t^{i,*} - C_{t-}^{i,*}$, applying Ito's formula to (63), we have

$$dX_t^{i,*} = r_t X_t^{i,*} dt + X_t^{i,*}\{\theta_t \cdot (\hat{\lambda}_t^i + \theta_t) dt + (\hat{\lambda}_t^i + \theta_t) \cdot dW_t\} - dC_t^{i,*}, \tag{66}$$

and for $t \in (t_{k-1}, t_k)$,

$$dX_t^{i,*} = r_t X_t^{i,*} dt + X_t^{i,*}\{\theta_t \cdot (\hat{\lambda}_t^i + \theta_t) dt + (\hat{\lambda}_t^i + \theta_t) \cdot dW_t\}. \tag{67}$$

The optimal portfolio of agent i in equilibrium is calculated as follows.
Agent i's optimal portfolio should satisfy

$$(\pi_t^{i,*})^\top \sigma_{S,t} = X_t^{i,*} (\hat{\lambda}_t^i + \theta_t)^\top, \tag{68}$$

where $\sigma_{S,t} := \begin{pmatrix} \sigma_{S,t}^1 \\ \vdots \\ \sigma_{S,t}^{N+1} \end{pmatrix}$. Then, under the assumption that rank($\sigma_{S,t}$)$= N+1$ ($N+1 \leq d$), that is $\sigma_{S,t}^1, \cdots, \sigma_{S,t}^{N+1}$ are linearly independent, i's optimal portfolio $\pi_t^{i,*}$ is obtained as

$$\pi_t^{i,*} = X_t^{i,*} (\sigma_{S,t} \sigma_{S,t}^\top)^{-1} \sigma_{S,t} (\hat{\lambda}_t^i + \theta_t). \tag{69}$$

Finally, agent i's position on the money market account $\pi^{i,0,*}$ is $X_t^{i,*} - \pi_t^{i,*\top} \mathbf{1}$. □

Remark 6 *First, $X_t^{i,*}$ in (50) corresponding to $(c^{i,*}, \pi^{i,*}, \pi^{i,0,*})$ in (39),(51) and (52) satisfies $X_t^{i,*} > 0,\ 0 \leq \forall t \leq T,\ P-a.s.$ Since $c^{i,*}$ is the optimal consumption in the admissible set $\mathcal{A}_i$, in which all the consumption processes* *in (5), (6) with non-negative condition (7) are included as shown in Remark 4, it follows that* $(c^*, \pi^{i,*}, \pi^{i,0,*})$ *is the optimal consumption and portfolio among triplets* $(c, \pi^i, \pi^{i,0})$ *for a problem* $\sup_{(c^i,\pi^i,\pi^{i,0})} \sum_{k=1}^K E[\eta^i_{t_k} \alpha^i_{t_k} \log c^i_k]$ *subject to (5),(6), and the non-negative wealth condition (7).*

3 Market clearing condition and equilibrium

This section provides the expression of the interest rate r and the market price of risk θ in equilibrium using the optimal consumption processes of the agents $c^{i,*}, \; i = 1, \dots, I$ obtained in Theorem 1 and imposing the market clearing condition (70). Moreover, we show an expression of the outstanding market value of the government bonds in zone j categorized by time to maturity, S^j in equilibrium.

Specifically, we consider the following market clearing condition where the aggregate consumption over the agents is equal to the dividends from the securities. Here, we assume the aggregate redemption amount δ_{t_k} is positive, while each $\delta^j_{t_k}$ may not necessarily be positive.

(Market Clearing Condition)

At each discrete time $t_k, \; k = 1, \dots, K$, the following equation holds.

$$\sum_{i=1}^{I} c^{i,*}_k = \sum_{j=1}^{N+1} \delta^j_{t_k} \equiv \delta_{t_k}. \tag{70}$$

When the optimal consumption processes of the agents satisfy this clearing condition, we call that the market is in equilibrium.

Firstly, we obtain the following expressions for the interest rate r and the density process Z^θ for the common part of the risk-neutral probability measure, which includes the market price of risk process θ in equilibrium.

Theorem 3 *In equilibrium,* Z^θ *in (24) is expressed as*

$$Z^\theta_{t_k} = \frac{B_{t_k}}{\delta_{t_k}} \sum_{i=1}^{I} \frac{x^i_0 \alpha^i_{t_k} Z^i_{t_k}}{\sum_{k=1}^K \alpha^i_{t_k}} \tag{71}$$

for $k = 1, \dots, K$ *and for* $t \in (t_k, t_{k+1})$*,*

$$Z^\theta_t = E\left[\frac{B_{t_k}}{\delta_{t_k}} \sum_{i=1}^{I} \frac{x^i_0 \alpha^i_{t_k} Z^i_{t_k}}{\sum_{k=1}^K \alpha^i_{t_k}} | \mathcal{F}_t \right]. \tag{72}$$

Moreover, supposing that the interest rate process r *is piece-wise* $\mathcal{F}_{t_{k-1}}$*-measurable random variable between the discrete times* $(t_{k-1}, t_k]$*,* r *is expressed as*

$$r_t = \frac{1}{(t_k - t_{k-1})} \log \left[\frac{\frac{1}{\delta_{k-1}} \sum_{i=1}^{I} \frac{x^i_0 \alpha^i_{k-1} Z^i_{t_{k-1}}}{\sum_{k=1}^K \alpha^i_k}}{E\left[\frac{1}{\delta_{t_k}} \sum_{i=1}^{I} \frac{x^i_0 \alpha^i_{t_k} Z^i_{t_k}}{\sum_{k=1}^K \alpha^i_{t_k}} | \mathcal{F}_{t_{k-1}} \right]} \right] \quad \text{for } t \in (t_{k-1}, t_k], \tag{73}$$

(Proof).

By the market clearing condition (70),

$$\sum_{k=1}^{K} c^{i,*}_k = \frac{B_{t_k}}{Z^\theta_{t_k}} \sum_{i=1}^{I} \frac{x^i_0 \alpha^i_{t_k} Z^i_{t_k}}{\sum_{k=1}^K \alpha^i_{t_k}} = \delta_{t_k}. \tag{74}$$

$$\tag{75}$$

Then, for $k = 1, \dots, K$ we have

$$Z^\theta_{t_k} = \frac{B_{t_k}}{\delta_{t_k}} \sum_{i=1}^{I} \frac{x^i_0 \alpha^i_{t_k} Z^i_{t_k}}{\sum_{k=1}^K \alpha^i_{t_k}}. \tag{76}$$

Since Z^θ_t is a martingale, for $t \in (t_{k-1}, t_k]$,

$$Z^\theta_t = E\left[\frac{B_{t_k}}{\delta_{t_k}} \sum_{i=1}^{I} \frac{x^i_0 \alpha^i_{t_k} Z^i_{t_k}}{\sum_{k=1}^K \alpha^i_{t_k}} | \mathcal{F}_t \right]. \tag{77}$$

For the interest rate r in equilibrium, at t_{k-1}, noting that

$$\frac{B_{t_{k-1}}}{\delta_{k-1}} \sum_{i=1}^{I} \frac{x^i_0 \alpha^i_{k-1} Z^i_{t_{k-1}}}{\sum_{k=1}^K \alpha^i_k} = E\left[\frac{B_{t_k}}{\delta_{t_k}} \sum_{i=1}^{I} \frac{x^i_0 \alpha^i_{t_k} Z^i_{t_k}}{\sum_{k=1}^K \alpha^i_{t_k}} | \mathcal{F}_{t_{k-1}} \right] \tag{78}$$

and B_{t_k} is $\mathcal{F}_{t_{k-1}}$-measurable, since $B_{t_k} = B_{t_{k-1}} e^{\int_{t_{k-1}}^{t_k} r_s ds}$ with $B_0 = 1$, we have

$$\frac{B_{t_k}}{B_{t_{k-1}}} = e^{\int_{t_{k-1}}^{t_k} r_s ds} = \frac{\frac{1}{\delta_{k-1}} \sum_{i=1}^{I} \frac{x_0^i \alpha_{k-1}^i Z_{k-1}^i}{\sum_{k=1}^{K} \alpha_k^i}}{E\left[\frac{1}{\delta_{t_k}} \sum_{i=1}^{I} \frac{x_0^i \alpha_{t_k}^i Z_{t_k}^i}{\sum_{k=1}^{K} \alpha_{t_k}^i} | \mathcal{F}_{t_{k-1}}\right]}, \ B_0 = 1, \tag{79}$$

and equivalently,

$$B_{t_k} = B_{t_{k-1}} \frac{\frac{1}{\delta_{k-1}} \sum_{i=1}^{I} \frac{x_0^i \alpha_{k-1}^i Z_{t_{k-1}}^i}{\sum_{k=1}^{K} \alpha_k^i}}{E\left[\frac{1}{\delta_{t_k}} \sum_{i=1}^{I} \frac{x_0^i \alpha_{t_k}^i Z_{t_k}^i}{\sum_{k=1}^{K} \alpha_{t_k}^i} | \mathcal{F}_{t_{k-1}}\right]}, \ B_0 = 1. \tag{80}$$

Particularly, r_t is obtained as

$$r_t = \frac{1}{(t_k - t_{k-1})} \log \left[\frac{\frac{1}{\delta_{k-1}} \sum_{i=1}^{I} \frac{x_0^i \alpha_{k-1}^i Z_{k-1}^i}{\sum_{k=1}^{K} \alpha_k^i}}{E\left[\frac{1}{\delta_{t_k}} \sum_{i=1}^{I} \frac{x_0^i \alpha_{t_k}^i Z_{t_k}^i}{\sum_{k=1}^{K} \alpha_{t_k}^i} | \mathcal{F}_{t_{k-1}}\right]} \right] \text{ for } t \in (t_{k-1}, t_k], \tag{81}$$

$$B_t = B_{t_{k-1}} e^{r_t (t - t_{k-1})} \text{ for } t \in (t_{k-1}, t_k], \ B_0 = 1. \tag{82}$$

□

Next, we provide the expression of S^j, the aggregate market value of security j, in equilibrium.

Proposition 2 *The aggregate market value S^j associated with the dividend process δ^j in equilibrium is expressed as*

$$\begin{aligned} S_t^j &= \frac{B_t}{Z_t^\theta} \sum_{k:t_k \geq t}^{K} E\left[\frac{\delta_{t_k}^j}{\delta_{t_k}} \sum_{i=1}^{I} \frac{x_0^i \alpha_{t_k}^i Z_{t_k}^i}{\sum_{k=1}^{K} \alpha_{t_k}^i} | \mathcal{F}_t\right] \\ &= \frac{1}{E\left[\frac{1}{\delta_{t_k}} \sum_{i=1}^{I} \frac{x_0^i \alpha_{t_k}^i}{\sum_{k=1}^{K} \alpha_{t_k}^i} Z_{t_k}^i | \mathcal{F}_t\right]} \sum_{k:t_k \geq t}^{K} E\left[\frac{\delta_{t_k}^j}{\delta_{t_k}} \sum_{i=1}^{I} \frac{x_0^i \alpha_{t_k}^i Z_{t_k}^i}{\sum_{k=1}^{K} \alpha_{t_k}^i} | \mathcal{F}_t\right]. \end{aligned} \tag{83}$$

(Proof).

Since $S_t^j \frac{Z_t^\theta}{B_t} + \sum_{k:t_k < t} \delta_{t_k}^j \frac{Z_{t_k}^\theta}{B_{t_k}}$ is a martingale and $S_T^j = 0$, we have

$$S_t^j = \frac{B_t}{Z_t^\theta} \sum_{k:t_k \geq t}^{K} E\left[\frac{Z_{t_k}^\theta}{B_{t_k}} \delta_{t_k}^j | \mathcal{F}_t\right] = \frac{B_t}{Z_t^\theta} \sum_{k:t_k \geq t}^{K} E\left[\frac{\delta_{t_k}^j}{\delta_{t_k}} \sum_{i=1}^{I} \frac{x_0^i \alpha_{t_k}^i Z_{t_k}^i}{\sum_{k=1}^{K} \alpha_{t_k}^i} | \mathcal{F}_t\right]. \tag{84}$$

Particularly, for $t \in (t_{k-1}, t_k]$, $k = 1, \cdots, K$, since

$$Z_t^\theta = B_{t_k} E\left[\frac{1}{\delta_{t_k}} \sum_{i=1}^{I} \frac{x_0^i \alpha_{t_k}^i}{\sum_{k=1}^{K} \alpha_{t_k}^i} Z_{t_k}^i | \mathcal{F}_t\right], \tag{85}$$

$$\frac{B_t}{Z_t^\theta} = \frac{B_{t_k}}{Z_t^\theta} = \frac{1}{E\left[\frac{1}{\delta_{t_k}} \sum_{i=1}^{I} \frac{x_0^i \alpha_{t_k}^i}{\sum_{k=1}^{K} \alpha_{t_k}^i} Z_{t_k}^i | \mathcal{F}_t\right]}, \tag{86}$$

we have

$$S_t^j = \frac{1}{E\left[\frac{1}{\delta_{t_k}} \sum_{i=1}^{I} \frac{x_0^i \alpha_{t_k}^i}{\sum_{k=1}^{K} \alpha_{t_k}^i} Z_{t_k}^i | \mathcal{F}_t\right]} \sum_{k:t_k \geq t}^{K} E\left[\frac{\delta_{t_k}^j}{\delta_{t_k}} \sum_{i=1}^{I} \frac{x_0^i \alpha_{t_k}^i Z_{t_k}^i}{\sum_{k=1}^{K} \alpha_{t_k}^i} | \mathcal{F}_t\right]. \tag{87}$$

□

Finally, we show that in equilibrium, the total of the I agents' wealth is equal to the total of the aggregate market value of $N+1$ securities. That is, another feature of the market clearing holds. We denote S by the total of the aggregate market value of $N+1$ securities and X by the aggregate wealth process of I agents, that is,

$$S_t := \sum_{j=1}^{N+1} S_t^j, \tag{88}$$

and

$$X_t := \sum_{i=1}^{I} X_t^i. \tag{89}$$

Proposition 3 *The total of the aggregate market value of $N+1$ securities in equilibrium S has the following expression*

$$\begin{aligned} S_t = \sum_{j=1}^{N+1} S_t^j = \frac{B_t}{Z_t^\theta} \sum_{k:t_k \geq t}^{K} \sum_{i=1}^{I} \frac{x_0^i Z_t^i \alpha_{t_k}^i}{\sum_{k=1}^K \alpha_{t_k}^i} \\ = \frac{1}{E\left[\frac{1}{\delta_{k_t}} \sum_{i=1}^I \frac{x_0^i \alpha_{k_t}^i}{\sum_{k=1}^K \alpha_{t_k}^i} Z_{k_t}^i | \mathcal{F}_t\right]} \sum_{k:t_k \geq t}^{K} \sum_{i=1}^{I} \frac{x_0^i Z_t^i \alpha_{t_k}^i}{\sum_{k=1}^K \alpha_{t_k}^i}, \end{aligned} \tag{90}$$

where

$$k_t := \min\{k \in \{1, \cdots, K\} : k \geq t\}, \tag{91}$$

In particular, at $t = k_t$,

$$S_t = S_{k_t} = \frac{\delta_{k_t}}{\left[\sum_{i=1}^I \frac{x_0^i \alpha_{k_t}^i}{\sum_{k=1}^K \alpha_k^i} Z_{k_t}^i\right]} \sum_{k:t_k \geq k_t}^{K} \sum_{i=1}^{I} \frac{x_0^i Z_{k_t}^i \alpha_{t_k}^i}{\sum_{k=1}^K \alpha_{t_k}^i}. \tag{92}$$

Moreover, S, the total of the aggregate market values of the $N+1$ securities, equals to X, the aggregate wealth process of I agents.

(Proof).

First, by noting $\sum_{j=1}^{N+1} \delta^j = \delta$, the expression of the total of the aggregate market value of the securities (90) immediately follows from (83).

Next, by the expression of the optimal wealth (50) for the individual optimization problem, we have

$$\begin{aligned} X_t^* := \sum_{i=1}^{I} X_t^{i,*} = \sum_{i=1}^{I} \frac{B_t Z_t^i}{Z_t^\theta} \frac{x_0^i}{\sum_{k=1}^K \alpha_{t_k}^i} \sum_{\{k:t_k \geq t\}}^{K} \alpha_{t_k}^i \qquad (93) \\ = \frac{B_t}{Z_t^\theta} \sum_{k:t_k \geq t}^{K} \sum_{i=1}^{I} \frac{x_0^i Z_t^i \alpha_{t_k}^i}{\sum_{k=1}^K \alpha_{t_k}^i}. \qquad (94) \end{aligned}$$

Since

$$S_t = \frac{B_t}{Z_t^\theta} \sum_{k:t_k \geq t}^{K} \sum_{i=1}^{I} \frac{x_0^i Z_t^i \alpha_{t_k}^i}{\sum_{k=1}^K \alpha_{t_k}^i}, \tag{95}$$

we have $X_t = S_t$. □

Remark 7 *The market clearing condition also derives the clearing equations for each security and the money market account in the following way.*

For each agent i,

$$X_t^{i,*} = \pi_t^{i,0,*} + \sum_{j=1}^{N+1} \pi_t^{i,j,*}, \ i = 1, \cdots, I \tag{96}$$

Then, a consumption-financed strategy yields that

$$dX_t^{i,*} = r_t X_t^{i,*} dt - dC_t^{i,*} + \sum_{j=1}^{N+1} \pi_t^{i,j,*} \sigma_{S,t}^j dW_t^*, \ i = 1, \cdots, I, \tag{97}$$

where $dW_t^* = dW_t + \theta_t dt$, and with $C^* := \sum_{i=1}^I C^{i,*}$,

$$dX_t^* = r_t X_t^* dt - dC_t^* + \sum_{j=1}^{N+1} \sum_{i=1}^{I} \pi_t^{i,j,*} \sigma_{S,t}^j dW_t^*. \tag{98}$$

On the other hand, with $S = \sum_{j=1}^{N+1} S^j$ *and* $D := \sum_{j=1}^{N+1} D^j$,

$$dS_t = r_t S_t dt - dD_t + \sum_{j=1}^{N+1} S_t^j \sigma_{S,t}^j dW_t^*. \tag{99}$$

Since $C^* = D$ *from the consumption market clearing and* $X_t = S_t$, *it must hold that*

$$\sum_{j=1}^{N+1} (\sum_{i=1}^{I} \pi_t^{i,j,*} - S_t^j) \sigma_{S,t}^j = 0. \tag{100}$$

Then, if σ_S^j, $j = 1, \cdots, N+1$ *are independent vectors, we obtain the market clearing for every risky asset,*

$$\sum_{i=1}^{I} \pi_t^{i,j,*} - S_t^j = 0, \ j = 1, \cdots, N+1, \tag{101}$$

and also have the market clearing for the money market account $\sum_i^I \pi_t^{i,0,*} = 0$, *since*

$$S_t = X_t^* = \sum_{i=1}^{I} X_t^{i,*} = \sum_{j=0}^{N+1} \sum_{i=1}^{I} \pi_t^{i,j,*} = S_t + \sum_{i=1}^{I} \pi_t^{i,0,*}. \tag{102}$$

3.1 Dividend process as a realizations of a continuous stochastic process

Next, this section provides a case where the interest rate r, the market price of risk θ, and the aggregate market value of security j, S^j, are obtained explicitly. Particularly, we consider a case where the dividend processes $\delta_{t_k}^j$, $k = 1, \ldots, K$, $j = 1, \ldots, N+1$ are given as a realization of continuous processes δ^j satisfying an SDE.

In Theorem 3, we obtained a piece-wise $\mathcal{F}_{t_k}$-measurable version of the interest rate process r and the aggregate market value of security j, S^j, in equilibrium. In this section, we further obtain expressions of a continuous version of the interest rate process r, the market price of risk θ in equilibrium. Moreover, although the expressions include the volatility processes σ_S^j, which is also determined in equilibrium, we also show a condition where the volatility process σ_S^j is explicitly determined by the assumptions on the dividend processes δ^j.

First, we suppose that $\delta_{t_k}^j$, $j = 1, \ldots, K$ are driven by the common stochastic process δ^j satisfying the following SDEs. Here, we suppose $\sigma^{\delta,j}$, $j = 1, \ldots, N+1$ are $\mathcal{R}^{d\times 1}$-valued $\{\mathcal{F}_t\}$-progressively measurable process. We note that we set $\sigma^{\delta,j}$, $j = 1, \ldots, N+1$ to be $d \times 1$ column vector, while we assumed σ_S^j, $j = 1, \ldots, N+1$ to be $1 \times d$ row vectors.

Let $\delta_t = \sum_{j=1}^{N+1} \delta_t^j$ and

$$d\delta_t^j = \delta_t^j [\mu_t^{\delta,j} dt + \sigma_t^{\delta,j} \cdot dW_t], \ \sigma_t^{\delta,j} > 0, \tag{103}$$

$$d\delta_t = \delta_t \left[\left(\sum_{j=1}^{N+1} \frac{\delta_t^j}{\delta_t} \mu_t^{\delta,j} \right) dt + \left(\sum_{j=1}^{N+1} \frac{\delta_t^j}{\delta_t} \sigma_t^{\delta,j} \right) \cdot dW_t \right] \tag{104}$$

$$= \delta_t [\mu_t^\delta dt + \sigma_t^\delta \cdot dW_t], \tag{105}$$

where

$$\mu_t^\delta := \frac{1}{\delta_t} \left(\sum_{j=1}^{N+1} \delta_t^j \mu_t^{\delta,j} \right), \sigma_t^\delta := \frac{1}{\delta_t} \left(\sum_{j=1}^{N+1} \delta_t^j \sigma_t^{\delta,j} \right). \tag{106}$$

and we assume that $\delta_0 = \sum_{j=1}^{N+1} \delta_0^j$ satisfies

$$\delta_0 = \sum_{i=1}^{I} \frac{x_0^i}{\sum_{k=1}^{K} \alpha_{t_k}^i}. \tag{107}$$

Proposition 4 *Suppose that* $\lambda^i \in Range(\sigma_S^\top)$, $i = 1, \ldots, I$, *where the agents' biases on the Brownian motion can be hedged with the risks of the* $N+1$ *securities. With a money market account price process* B, *where* $B_t = \exp(\int_0^t r_s ds)$, *and the interest rate process* r *given as a continuous process*

$$r_t = \mu_t^\delta + \beta_t - |\sigma_t^\delta|^2 + \sigma_t^\delta \cdot \hat{\lambda}_t, \tag{108}$$

the probability density process for the risk neutral probability measure Z^θ *in equilibrium in (72) in Theorem 3 satisfies*

$$Z_t^\theta = \frac{B_t}{\delta_t} \sum_{i=1}^{I} \frac{\alpha_t^i x_0^i Z_t^i}{\sum_{k=1}^{K} \alpha_{t_k}^i}, \ 0 \le t \le T, \tag{109}$$

and particularly is an exponential martingale with an expression $Z^\theta_t = \exp(-\frac{1}{2}\int_0^t |\theta_s|^2 ds - \int_0^t \theta_s \cdot dW_s)$*, where the market price of risk* θ *is given by*

$$\theta_t = \sigma^\delta_t - \hat{\lambda}_t. \tag{110}$$

Here, we set the aggregate view $\hat{\lambda}$ *and the aggregate time preference* β *of the agents as*

$$\hat{\lambda}_t = \sum_{i=1}^I \hat{\lambda}^i_t w^i_t, \tag{111}$$

and

$$\beta_t = \sum_{i=1}^I \beta_i w^i_t, \tag{112}$$

where

$$w^i_t = \frac{\frac{\alpha^i_t x^i_0 Z^i_t}{\sum_{k=1}^K \alpha^i_{t_k}}}{\sum_{l=1}^I \frac{\alpha^l_t x^l_0 Z^l_t}{\sum_{k=1}^K \alpha^l_{t_k}}}. \tag{113}$$

Moreover, assuming that $E[\int_0^T (\sigma^{\delta,j}_s - \theta_s)^2 H_s^2 \delta^{j2}_s ds] < \infty$*, the volatility process of the j-th security* σ^j_S *in equilibrium is expressed as follows.*

$$\begin{aligned}\sigma^{j\top}_{S,t} = &\frac{\sum_{k:t_k>t} \delta^j_t H_t}{\sum_{k:t_k>t} \delta^j_t H_t + \sum_{k:t_k>t}\int_t^{t_k} E[(\mu^{\delta,j}_s - r_s - \sigma^{\delta,j}_s \cdot \theta_s)\delta^j_s H_s|\mathcal{F}_t]ds}\sigma^{\delta,j}_t \\ &+ \frac{\sum_{k:t_k>t}\int_t^{t_k} E[(\mu^{\delta,j}_s - r_s - \sigma^{\delta,j}_s \cdot \theta_s)\delta^j_s H_s|\mathcal{F}_t]ds}{\sum_{k:t_k>t} \delta^j_t H_t + \sum_{k:t_k>t}\int_t^{t_k} E[(\mu^{\delta,j}_s - r_s - \sigma^{\delta,j}_s \cdot \theta_s)\delta^j_s H_s|\mathcal{F}_t]ds}\theta_t \\ &+ \frac{\sum_{k:t_k>t}\int_t^{t_k} E[D_t\left((\mu^{\delta,j}_s - r_s - \sigma^{\delta,j}_s \cdot \theta_s)\delta^j_s H_s\right)|\mathcal{F}_t]ds}{\sum_{k:t_k>t} \delta^j_t H_t + \sum_{k:t_k>t}\int_t^{t_k} E[(\mu^{\delta,j}_s - r_s - \sigma^{\delta,j}_s \cdot \theta_s)\delta^j_s H_s|\mathcal{F}_t]ds},\end{aligned} \tag{114}$$

where D_t *is a Malliavin derivative with respect to the Brownian motion* W_t*, and* $H_t = \frac{Z^\theta_t}{B_t} = \frac{1}{\delta_t}\sum_{i=1}^I \frac{\alpha^i_t x^i_0 Z^i_t}{\sum_{k=1}^K \alpha^i_{t_k}}$ *is a state-price density process obtained by (109), corresponding to the market price of risk* θ *in equilibrium in (110).*

Remark 8 *The expression of* r *in (108), which is*

$$r_t = \mu^\delta_t + \beta_t - |\sigma^\delta_t|^2 + \sigma^\delta_t \cdot \hat{\lambda}_t, \tag{115}$$

indicates that equilibrium interest rate r *increases when the expected return of the total dividend process* μ^δ *increases. The second term indicates that if the time preferences of the agents increases, the interest rate also increases, and the third term implies that if the total dividend's volatility increases, the interest rate decreases due to the agents' risk aversion for uncertainty of the total dividend process.*

The fourth term also describes how the agents' aggregate view on the Brownian motion affects the interest rate risk. Here, we may interpret the Brownian motion as the risks related to the economic(growth) factor, inflation factor, fiscal condition(extent of deficit/surplus) factor, and factors characteristic to each market value process of security j*.*

The aggregate view of heterogeneous agents $\hat{\lambda}$ *in (111) is weighed averaged of each agent's view on the Brownian motion* $\hat{\lambda}^i$*, with the weight* $\frac{\frac{\alpha^i_t x^i_0 Z^i_t}{\sum_{k=1}^K \alpha^i_{t_k}}}{\sum_{l=1}^I \frac{\alpha^l_t x^l_0 Z^l_t}{\sum_{k=1}^K \alpha^l_{t_k}}}$*, consisting of* $\frac{\alpha^i_t}{\sum_{k=1}^K \alpha^i_{t_k}}$*, where* $\alpha^i_t = e^{-\beta_i t}$ *is the agent's discounting on its consumption with time preference parameter* β_i*, the initial wealth* x^i_0*, and the density process corresponding to the agent's view* Z^i_t*.*

Therefore, noting that the inner product $\sigma^\delta_t \hat{\lambda}_t = \sum_{j=1}^d \sigma^\delta_{j,t}\hat{\lambda}_{j,t}$ *can be interpreted as the total dividend volatility weighted aggregate view by considering* $\sigma^\delta_{j,t}$ *as the weight on the aggregate view on the j-th risk* $\hat{\lambda}_{j,t}$*, we observe that if the total divided volatility weighted aggregate view is positive/negative, then it affects positively/negatively the short rate* r *in equilibrium.*

Next, the expression of the market price of risk θ *in (110),*

$$\theta_t = \sigma^\delta_t - \hat{\lambda}_t. \tag{116}$$

This expression indicates that σ^δ*, the volatility of the total dividend process, is the required excess return for each fun-*

damental risk i.e., d-dimensional Brownian motion. Also, if the aggregate view on the Brownian motion is aggressive, the market only requires a smaller excess return, and vice versa.

Also, μ^j_S, the expected return of the security j, is expressed as

$$\mu^j_{S,t} = r_t + \sigma^j_{S,t}\theta_t. \tag{117}$$

The second term indicates if the inner product of the security j's market value process σ^j_S and the market price of risk θ is positive, the term has a positive effect on the expected return process, which will be further discussed in Section 3.2 as a term premium in the case of the central bank's purchasing reduction.

Remark 9 *Although Proposition 4 indicates that σ^S is also determined in equilibrium, even if $\lambda^i \in Range(\sigma_S^\top)$ is not satisfied, when $\hat{\lambda}^i$, the orthogonal projection of λ^i, is set as in (118), r, θ, and σ^j_S are explicitly determined and the market is in equilibrium. In detail, as long as $\hat{\lambda}^i$, $i = 1, \ldots, I$, given by*

$$\hat{\lambda}^i_t = a^i_t \sigma^\delta_t = \frac{a^i_t}{\sum_{j=1}^{N+1} \delta^j_t} \left(\sum_{j=1}^{N+1} \delta^j_t \sigma^{\delta,j}_t \right), \tag{118}$$

for some $\{\mathcal{F}_t\}$-adapted $\mathcal{R}$-valued process a^i_t, satisfy $\hat{\lambda}^i \in Range(\sigma_S^\top)$, and thus $\hat{\lambda}^i$, $i = 1, \ldots, I$ given by (118) can be the orthogonal projection of some belief of agent i λ^i onto the linear space spanned by σ^j_S, $j = 1, \ldots, N+1$, which is proved as follows.

If we set $\hat{\lambda}^i$ proportional to exogenously given σ^δ,

$$S_t = \sum_{j=1}^{N+1} S^j_t = \frac{B_t}{Z^\theta_t} \sum_{k:t_k \geq t}^{K} \sum_{i=1}^{I} \frac{\alpha^i_{t_k} x^i_0 Z^i_t}{\sum_{k=1}^K \alpha^i_{t_k}} \tag{119}$$

with

$$Z^\theta_t = \frac{B_t}{\delta_t} \sum_{i=1}^{I} \frac{\alpha^i_t x^i_0 Z^i_t}{\sum_{k=1}^K \alpha^i_{t_k}}, \tag{120}$$

we have

$$S_t = \delta_t \frac{\sum_{k:t_k \geq t}^{K} \sum_{i=1}^{I} \frac{\alpha^i_{t_k} x^i_0 Z^i_t}{\sum_{k=1}^K \alpha^i_{t_k}}}{\sum_{i=1}^{I} \frac{\alpha^i_t x^i_0 Z^i_t}{\sum_{k=1}^K \alpha^i_{t_k}}}. \tag{121}$$

This indicates that σ^S is a linear combination of σ^δ and $\hat{\lambda}^i$, $i = 1, \ldots, I$, and thus is proportional to σ^δ. Since σ^S is a linear combination of σ^S_j, $j = 1, \ldots, N+1$, for $\hat{\lambda}^i$ proportional to σ^δ, we have $\hat{\lambda}^i \in Range(\sigma_S^\top)$.

(Proof). Firstly, by applying Ito's formula to

$$\frac{Z^\theta_t}{B_t} = \frac{1}{\delta_t} \sum_{i=1}^{I} \frac{\alpha^i_t x^i_0 Z^i_t}{\sum_{k=1}^K \alpha^i_{t_k}}, \tag{122}$$

we observe that

$$\frac{d\left(\frac{Z^\theta_t}{B_t}\right)}{\frac{Z^\theta_t}{B_t}} = -\left(\mu^\delta_t + \beta_t - |\sigma^\delta_t|^2 + \sigma^\delta_t \cdot \hat{\lambda}_t\right) dt - \left(\sigma^\delta_t - \hat{\lambda}_t\right) \cdot dW_t. \tag{123}$$

Hence, if r and θ are given as in (108) and (110),

$$\frac{d\left(\frac{Z^\theta_t}{B_t}\right)}{\frac{Z^\theta_t}{B_t}} = -r_t dt - \theta_t \cdot dW_t, \tag{124}$$

and particularly, $Z^\theta_t = \exp(-\frac{1}{2}\int_0^t |\theta_s|^2 ds - \int_0^t \theta_s dW_s)$, which indicates that Z^θ is an exponential martingale.

Noting that

$$\frac{S^j_t Z^\theta_t}{B_t} = \sum_{k:t_k \geq t}^{K} E\left[\frac{Z^\theta_{t_k}}{B_{t_k}} \delta^j_{t_k} | \mathcal{F}_t\right], \tag{125}$$

and taking Malliavin derivatives of the both hand sides and comparing them, we obtain the expression of σ^j_S.

In detail,

$$\sum_{k:t_k\geq t}^{K}\frac{Z^\theta_{t_k}}{B_{t_k}}\delta^j_{t_k}$$

$$= \sum_{k:t_k\geq t}^{K}\frac{Z^\theta_t}{B_t}\delta^j_t + \sum_{k:t_k\geq t}^{K}\int_t^{t_k}(\mu^{\delta,j}_s - r_s - \sigma^{\delta,j}_s\cdot\theta_s)\frac{Z^\theta_s}{B_s}\delta^j_s ds + \sum_{k:t_k\geq t}^{K}\int_t^{t_k}\frac{Z^\theta_s}{B_s}\delta^j_s(\sigma^{\delta,j}_s - \theta_s)\cdot dW_s. \quad (126)$$

Then, taking conditional expectation $E[\cdot|\mathcal{F}_t]$, we have

$$\frac{S^j_t Z^\theta_t}{B_t} = \sum_{k:t_k\geq t}^{K}\frac{Z^\theta_t}{B_t}\delta^j_t + \sum_{k:t_k\geq t}^{K}E\left[\int_t^{t_k}(\mu^{\delta,j}_s - r_s - \sigma^{\delta,j}_s\cdot\theta_s)\frac{Z^\theta_s}{B_s}\delta^j_s|\mathcal{F}_t\right]ds. \quad (127)$$

Taking Malliavin derivative D_t on both hand sides, we have

$$D_t\left(\frac{S^j_t Z^\theta_t}{B_t}\right) = \left(\sum_{k:t_k\geq t}^{K}\frac{Z^\theta_t}{B_t}\delta^j_t\right)\left(\sigma^{\delta,j}_t - \theta_t\right) + \sum_{k:t_k\geq t}^{K}E\left[\int_t^{t_k}D_t\left((\mu^{\delta,j}_s - r_s - \sigma^{\delta,j}_s\cdot\theta_s)\frac{Z^\theta_s}{B_s}\delta^j_s\right)|\mathcal{F}_t\right]ds. \quad (128)$$

Since

$$D_t\left(\frac{S^j_t Z^\theta_t}{B_t}\right) = (\sigma^j_{S,t} - \theta_t)\left(\frac{S^j_t Z^\theta_t}{B_t}\right)$$

$$= (\sigma^j_{S,t} - \theta_t)\Bigg(\sum_{k:t_k\geq t}^{K}\frac{Z^\theta_t}{B_t}\delta^j_t + \sum_{k:t_k\geq t}^{K}E[\int_t^{t_k}(\mu^{\delta,j}_s - r_s - \sigma^{\delta,j}_s\cdot\theta_s)\frac{Z^\theta_s}{B_s}\delta^j_s|\mathcal{F}_t]ds\Bigg), \quad (129)$$

where we used (127) in the second equality, comparing (128) and (129), we obtain

$$\left(\sum_{k:t_k\geq t}^{K}\frac{Z^\theta_t}{B_t}\delta^j_t\right)(\sigma^{\delta,j}_t - \theta_t) + \sum_{k:t_k\geq t}^{K}E\left[\int_t^{t_k}D_t\left((\mu^{\delta,j}_s - r_s - \sigma^{\delta,j}_s\cdot\theta_s)\frac{Z^\theta_s}{B_s}\delta^j_s\right)|\mathcal{F}_t\right]ds$$

$$= (\sigma^j_{S,t} - \theta_t)\Bigg(\sum_{k:t_k\geq t}^{K}\frac{Z^\theta_t}{B_t}\delta^j_t + \sum_{k:t_k\geq t}^{K}E[\int_t^{t_k}(\mu^{\delta,j}_s - r_s - \sigma^{\delta,j}_s\cdot\theta_s)\frac{Z^\theta_s}{B_s}\delta^j_s|\mathcal{F}_t]ds\Bigg). \quad (130)$$

Therefore

$$\begin{aligned}
\sigma^{j\top}_{S,t} &= \theta_t + \frac{\sum_{k:t_k\geq t}^{K}\frac{Z^\theta_t}{B_t}\delta^j_t}{\sum_{k:t_k\geq t}^{K}\frac{Z^\theta_t}{B_t}\delta^j_t + \sum_{k:t_k\geq t}^{K}E[\int_t^{t_k}(\mu^{\delta,j}_s - r_s - \sigma^{\delta,j}_s\cdot\theta_s)\frac{Z^\theta_s}{B_s}\delta^j_s|\mathcal{F}_t]ds}(\sigma^{\delta,j}_t - \theta_t)\\
&\quad + \frac{\sum_{k:t_k\geq t}^{K}E[\int_t^{t_k}D_t\left((\mu^{\delta,j}_s - r_s - \sigma^{\delta,j}_s\cdot\theta_s)\frac{Z^\theta_s}{B_s}\delta^j_s\right)|\mathcal{F}_t]}{\sum_{k:t_k\geq t}^{K}\frac{Z^\theta_t}{B_t}\delta^j_t + \sum_{k:t_k\geq t}^{K}E[\int_t^{t_k}(\mu^{\delta,j}_s - r_s - \sigma^{\delta,j}_s\cdot\theta_s)\frac{Z^\theta_s}{B_s}\delta^j_s|\mathcal{F}_t]ds}\\
&= \frac{\sum_{k:t_k>t}\delta^j_t H_t}{\sum_{k:t_k>t}\delta^j_t H_t + \sum_{k:t_k>t}\int_t^{t_k}E[(\mu^{\delta,j}_s - r_s - \sigma^{\delta,j}_s\cdot\theta_s)H_s\delta^j_s|\mathcal{F}_t]ds}\sigma^{\delta,j}_t\\
&\quad + \frac{\sum_{k:t_k>t}\int_t^{t_k}E[(\mu^{\delta,j}_s - r_s - \sigma^{\delta,j}_s\cdot\theta_s)H_s\delta^j_s|\mathcal{F}_t]ds}{\sum_{k:t_k>t}\delta^j_t H_t + \sum_{k:t_k>t}\int_t^{t_k}E[(\mu^{\delta,j}_s - r_s - \sigma^{\delta,j}_s\cdot\theta_s)H_s\delta^j_s|\mathcal{F}_t]ds}\theta_t\\
&\quad + \frac{\sum_{k:t_k>t}\int_t^{t_k}E[D_t\left((\mu^{\delta,j}_s - r_s - \sigma^{\delta,j}_s\cdot\theta_s)H_s\delta^j_s\right)|\mathcal{F}_t]ds}{\sum_{k:t_k>t}\delta^j_t H_t + \sum_{k:t_k>t}\int_t^{t_k}E[(\mu^{\delta,j}_s - r_s - \sigma^{\delta,j}_s\cdot\theta_s)H_s\delta^j_s|\mathcal{F}_t]ds},
\end{aligned} \quad (131)$$

where $H_t = \frac{Z^\theta_t}{B_t}$. □

3.2 Effect of outright purchasing securities: interpretation of the model

This section interprets the effects on the expected return of the market outstanding values and the yield curve obtained from the implied zero coupon bond prices in equilibrium when the central bank and the government control the amount of outright purchasing and the new issuance.

As we observe in the market, when the market expects that the central bank will decrease the amount of purchasing, with the expectation of an increasing supply of government bonds in the secondary market, the bond price goes down, and the yield rises. Conversely, when the supply of government bonds in the secondary market becomes scarce due to aggressive outright purchasing of the central bank, the government bond prices go up, and the yield goes down. These effects can be observed and understood through the model as follows.

Firstly, when the central bank purchases fewer government bonds upfront, the dividend cash flow into the secondary market, δ_t^j, at the time of purchase in the front end decreases due to reduced payments made by the central bank into the institutional investors' representative shareholders' current accounts. Conversely, δ_t^j in the long end, following the time of purchase, increases due to higher coupon payments and redemptions in the future.

More concretely, with the example of the dividend processes as the realization of the continuous stochastic processes following an SDE (103), the effect is observed as follows.

3.2.1 Effect on the market price of risk θ and the expected return μ_S^j in equilibrium

First of all, as we observed in Remark 8, when the bonds are less purchased by the central bank, i.e., the dividend process in the front end decreases while that of the long end increases, particularly when the drift μ^δ of the total dividend processes increase, by the expression of r in (108), which is

$$r_t = \mu_t^\delta + \beta_t - |\sigma_t^\delta|^2 + \sigma_t^\delta \cdot \hat{\lambda}_t, \tag{132}$$

the interest rate r in equilibrium increases in the long run. Also, given $\sigma_t^{\delta,j} > 0,\ j = 1, \ldots, N+1$ and thus $\sigma_t^\delta > 0$, since the third term and the fourth term are combined as $-|\sigma_t^\delta|^2 + \sigma_t^\delta \cdot \hat{\lambda}_t = -\sigma_t^\delta(\sigma_t^\delta - \hat{\lambda}_t)$, unless the aggregate view $\hat{\lambda}$ is excessively aggressive, these terms affect negatively the short rate r in equilibrium.

Next, in the expression of μ_S^j in (117), with the market price of risk in equilibrium θ is given by (110),

$$\mu_{S,t}^j = r_t + \sigma_{S,t}^j \theta_t. \tag{133}$$

This implies that if the inner product $\sigma_{S,t}^j \theta_t$, where $\sigma_{S,t}^j \theta_t = \sigma_{S,t}^j(\sigma_t^\delta - \hat{\lambda}_t)$ is positive, the term $\sigma_{S,t}^j \theta_t$ works as a positive term premium of the expected return μ_S^j over the interest rate r.

More in detail, by the expression of σ_S^j in (114), $\sigma_{S,t}^j \theta_t$ is expressed as

$$\begin{aligned}
\sigma_{S,t}^j \theta_t = &\frac{\sum_{k:t_k>t} \delta_t^j H_t}{\sum_{k:t_k>t} \delta_t^j H_t + \sum_{k:t_k>t} \int_t^{t_k} E[(\mu_s^{\delta,j} - r_s - \sigma_s^{\delta,j} \cdot \theta_s)\delta_s^j H_s|\mathcal{F}_t]ds} \sigma_t^{\delta,j}\theta_t \\
&+ \frac{\sum_{k:t_k>t} \int_t^{t_k} E[(\mu_s^{\delta,j} - r_s - \sigma_s^{\delta,j} \cdot \theta_s)\delta_s^j H_s|\mathcal{F}_t]ds}{\sum_{k:t_k>t} \delta_t^j H_t + \sum_{k:t_k>t} \int_t^{t_k} E[(\mu_s^{\delta,j} - r_s - \sigma_s^{\delta,j} \cdot \theta_s)\delta_s^j H_s|\mathcal{F}_t]ds} |\theta_t|^2 \\
&+ \frac{\sum_{k:t_k>t} \int_t^{t_k} E[D_t\left((\mu_s^{\delta,j} - r_s - \sigma_s^{\delta,j} \cdot \theta_s)\delta_s^j H_s\right)|\mathcal{F}_t]ds}{\sum_{k:t_k>t} \delta_t^j H_t + \sum_{k:t_k>t} \int_t^{t_k} E[(\mu_s^{\delta,j} - r_s - \sigma_s^{\delta,j} \cdot \theta_s)\delta_s^j H_s|\mathcal{F}_t]ds} \theta_t.
\end{aligned} \tag{134}$$

For the leading terms, the first term and the second term, the first term indicates that if the inner product between $\sigma_t^{\delta,j}$, the volatility of the dividend process for the j-th security, and θ_t, the market price of risk satisfying $\theta_t = \sigma_t^\delta - \hat{\lambda}_t$ in equilibrium, is positive, it affects the term premium positively.

The second term describes that since

$$\begin{aligned}
r_t &= \mu_t^\delta + \beta_t - |\sigma_t^\delta|^2 + \sigma_t^\delta \cdot \hat{\lambda}_t, \\
&= \mu_t^\delta + \beta_t - \sigma_t^\delta \cdot (\sigma_t^\delta - \hat{\lambda}_t) \\
&= \mu_t^\delta + \beta_t - \sigma_t^\delta \cdot \theta_t,
\end{aligned} \tag{135}$$

and

$$\mu_s^{\delta,j} - r_s - \sigma_s^{\delta,j} \cdot \theta_s = (\mu_s^{\delta,j} - \sigma_s^{\delta,j} \cdot \theta_s) - (\mu_s^\delta - \sigma_s^\delta \cdot \theta_s) - \beta_s,$$

if the expected return of the j-th dividend process $\mu^{\delta,j}$ deviates more positively from $\sigma^{\delta,j} \cdot \theta$, the inner product between the volatility of the j-th security's dividend and the market price of risk, than the expected return of the total dividend process μ^δ deviates from $\sigma^\delta \cdot \theta$, the inner product between the total dividend process and the market price of risk, and the deviation is greater than the aggregate market preference β, then the second term positively affects the term premium.

3.2.2 Effect on the term structure of interest rates for implied government bond prices

Moreover, although we consider the baskets of government bonds categorized by the remaining time to maturity as the tradable assets and do not directly model the evolution of the zero coupon bond price $P(0, \mathcal{T})$ corresponding to a certain maturity $\mathcal{T}$, the yield $\mathcal{Y}(0,\mathcal{T})$ for the implied zero coupon bond price by a state price density $\frac{Z_t^\theta}{B_t}$,

$$\frac{Z_t^\theta}{B_t} = \frac{1}{\delta_t}\sum_{i=1}^{I}\frac{\alpha_t^i x_0^i Z_t^i}{\sum_{k=1}^{K}\alpha_{t_k}^i},\ 0 \le t \le T, \tag{136}$$

by (109), can be obtained as follows.

$$\mathcal{Y}(0,\mathcal{T}) = -\frac{1}{\mathcal{T}}\log E\left[\frac{Z_{\mathcal{T}}^\theta}{B_{\mathcal{T}}}\right] = -\frac{1}{\mathcal{T}}\log E\left[\frac{1}{\delta_{\mathcal{T}}}\sum_{i=1}^{I}\frac{\alpha_{\mathcal{T}}^i x_0^i Z_{\mathcal{T}}^i}{\sum_{k=1}^{K}\alpha_{t_k}^i}\right], \tag{137}$$

where

$$d\delta_t = \delta_t\left[\left(\sum_{j=1}^{N+1}\frac{\delta_t^j}{\delta_t}\mu_t^{\delta,j}\right)dt + \left(\sum_{j=1}^{N+1}\frac{\delta_t^j}{\delta_t}\sigma_t^{\delta,j}\right)\cdot dW_t\right] \tag{138}$$

$$= \delta_t[\mu_t^\delta dt + \sigma_t^\delta \cdot dW_t], \tag{139}$$

$$\mu_t^\delta := \frac{1}{\delta_t}\left(\sum_{j=1}^{N+1}\delta_t^j\mu_t^{\delta,j}\right), \sigma_t^\delta := \frac{1}{\delta_t}\left(\sum_{j=1}^{N+1}\delta_t^j\sigma_t^{\delta,j}\right). \tag{140}$$

As (137), including the minus sign upfront and $\frac{1}{\delta_{\mathcal{T}}}$ in the expectation, indicates, since the implied zero coupon bond yield $\mathcal{Y}(0,\mathcal{T})$ is mainly affected by the drift of the total dividend δ, which is the drift of the each dividend process with their weight among the total dividend δ, thus when the drift of the total dividend process μ_t^δ increases as time passes, the yield curve steepens.

4 Numerical Examples

This section presents numerical examples of our equilibrium model developed in the previous sections, which derives super-long discount rates for insurance products and assesses the impact of policy changes on insurance pricing through these rates. Specifically, we provide equilibrium super-long discount rates and the pricing of death benefits and annuities, especially in relation to changes in the super-long discount rates.

Assuming an annual mortality rate λ_t^m that depends on two factors, namely, unhedgeable economic and public health deterioration factors denoted by Y^1 and Y^2, respectively which will be explained below, we can price a death benefit that pays V_τ upon death at exogenously given random time τ, and a life annuity that pays v_t annually until time τ. Particularly, we follow the approach of chapter 8 in Bieleck and Rutkowski [25] with regarding their default time τ as death time, and re-express the second equation in their Proposition 8.2.1 under the physical measure P: We first note that their so called spot martingale (risk-neutral) measure Q^* explained at the beginning of Section 8.1.1 (Risk-Neutral Valuation Formula) corresponds to a measure under which $(S^j + D^j)/B$, $j = 1,\cdots,N+1$ are martingales in our framework. Then, for simplicity we specifically set a density process for probability measure transformation from P to Q^* by Z^θ obtained as (72) in market equilibrium. Hence, using the associated state price density process $H = \frac{Z^\theta}{B}$ in (136), the initial (time-0) values of death benefit (insurance payable at death) and life annuity are given as follows:

Death benefit (insurance payable at death) value:

$$E[V_\tau H_\tau] = E\left[\int_0^\infty V_s\lambda_s^m \exp\left(-\int_0^s \lambda_u^m du\right) H_s\, ds\right]. \tag{141}$$

Here, we set $t = 0$, $\Gamma. = \int_0^{\cdot} \lambda_u^m du$, $Z = V$, and $A = X \equiv 0$ in the second equation in Proposition 8.2.1 of Bieleck and Rutkowski [25].

Life annuity value:

$$E\left[\int_0^\infty v_s 1_{\{s<\tau\}} H_s\, ds\right] = E\left[\int_0^\infty v_s \exp\left(-\int_0^s \lambda_u^m du\right) H_s\, ds\right], \tag{142}$$

where we set $t = 0$, $\Gamma. = \int_0^{\cdot} \lambda_u^m du$, $A. = \int_0^{\cdot} v_u du$, and $Z = X \equiv 0$ in the second equation in Proposition 8.2.1 of Bieleck and Rutkowski [25].

We present numerical examples to analyze the impact of reducing the central bank's outright purchase amount of government bonds on insurance pricing through the term structure of interest rates. Specifically, we set $V = v \equiv 1$ in pricing death benefit and life annuity for simplicity.

The terminal time is set to $T = 100$ years, which is sufficient for our analysis, and $N = 5$, i.e., $N+1 = 6$. Specifically, we categorize government bonds into five maturity sectors: 1-3 years, 3-5 years, 5-10 years, 10-25 years, and beyond 25 years, alongside one representative stock price index. We denote these sectors by $j = 1, 2, 3, 4, 5$ and the stock price index by $j = 6$.

We set $I = 3$ with an initial wealth $x_0^i = 100$, representing three distinct types of institutional investor shareholders. For instance, these types can be interpreted as follows: the first type represents life-cycle investment owners from the younger generation, the second type represents the older generation, and the third type represents shareholders of banks and insurance companies. Alternatively, these types may also be considered as representing shareholders of domestic private investors, institutional investors, and foreign institutional investors.

Furthermore, we consider 100 discrete time points $t_k = k$, $k = 1, \ldots, 100$, for dividend and consumption timings. We assume the dimension d of the Brownian motion W to be $d = 8$, which exceeds the number of securities $N + 1 = 6$. The drift of the dividend processes is a function of an economic factor Y^1, driven by a Brownian motion component W_7. Additionally, the mortality rate, which will be defined below, is influenced by both the economic factor Y^1 and the public health deterioration factor Y^2, driven by W_8, as follows:

We let $W = (W_1, \ldots, W_8)$, with the economic factor Y^1, public health deterioration factor Y^2, and dividend processes $\delta^j, j = 1, \ldots, 6$ satisfying the following SDEs:

$$\begin{aligned} dY_t^1 &= (lY_t^2 + \mu_{Y,1,t})\,dt + \sigma_{Y,1}\,dW_{7,t} \\ dY_t^2 &= \mu_{Y,2,t}\,dt + \sigma_{Y,2}\,dW_{8,t}. \end{aligned} \tag{143}$$

We assume that Y^1 and Y^2 are Gaussian processes with a positive initial value and drift, representing an economic growth factor and a public health deterioration factor, respectively. The public health deterioration factor also negatively affects the drift of Y^1 through $l < 0$ in the drift of Y^1 in (143), which implies that as public health worsens, it negatively affects the economy. We note that although the process may yield negative values with a small probability, we have carefully set the parameters $\mu_{Y,1}$, $\mu_{Y,2}$, $\sigma_{Y,1}$, and $\sigma_{Y,2}$ to ensure that Y^1 and Y^2 remain positive in nearly all simulated sample paths for calculation convenience. Both factors affect the mortality rate λ^m such that as the economy grows, the mortality rate decreases, and as disasters occur, the mortality rate increases, which defines the survival rate for life insurance pricing as follows:

$$\lambda_t^m = (1 + k_1 Y_t^1 + k_2 Y_t^2)\bar{\lambda}_t^m. \tag{144}$$

Specifically, for insurance products for individuals of age t_0, we consider the base mortality rate $\bar{\lambda}_t^m$ corresponding to that of a population aged $t_0 + t$. In the following numerical examples, we assume $t_0 = 20$, covering the mortality rate for ages 20-120.

The dividend processes of δ_t^j for $j = 1, \ldots, 6$ are characterized by the drift and diffusion coefficients, $\mu_t^{\delta,j}, \bar{\sigma}^{\delta,j}$:

$$d\delta_t^j = \delta_t^j[\mu^{\delta,j}(t, Y_t)\,dt + \bar{\sigma}^{\delta,j}(\rho_j\,dW_{7,t} + \sqrt{1 - \rho_j^2}\,dW_{j,t})], \tag{145}$$

where

$$\mu^{\delta,j}(t, Y_t) = \beta_{Y,j,t} + \alpha_{Y,j,t} Y_t^1, \tag{146}$$

$\bar{\sigma}^{\delta,j}$ represents the absolute value of the volatility $\sigma^{\delta,j}$ for the dividend process δ^j, and ρ_j describes the correlation between the Brownian motions driving the economic factor and the j-th dividend process δ^j. This indicates that the dividend processes, corresponding to government bond coupon payments and redemptions from secondary market outstanding bonds or stock index dividends, have drift components linked to the economic factor and diffusion components correlated with it.

Assuming constant drift and volatility for the Gaussian processes Y^1 and Y^2, we set $Y_0^1 = Y_0^2 = 1$, $l = -0.003$ $\mu_{Y,1} = \mu_{Y,2} = 0.1$, and $\sigma_{Y,1} = \sigma_{Y,2} = 0.1$. For the log-normal dividend processes, supposing that the dividend processes are for the market outstanding values of the government bonds for different time to maturity for $j = 1, \ldots, 5$ and of the stock price index for $j = 6$ as mentioned earlier in this section, we assume constant drift and volatility, defined as $\alpha_{Y,j} = 0$ $(j = 1, \ldots, 5), \alpha_{Y,6} = 0.02$, $\beta_{Y,j} = 0.01$ $(j = 1, \ldots, 5), \beta_{Y,6} = 0.03$, $\delta_0^j = \frac{1}{10}\delta_0$ $(j = 1, \ldots, 5)$, $\delta_0^6 = \frac{1}{2}\delta_0$, $\bar{\sigma}^{\delta,j} = 0.1$ $(j = 1, \ldots, 5), \bar{\sigma}^{\delta,6} = 0.3$, and $\rho^j = 0.5$ for $j = 1, \ldots, 6$, where δ_0 is the initial value of the total dividend defined by (107). Here, we set $\alpha_{Y,j} = 0$ $(j = 1, \ldots, 5)$ since we consider the impact of the economic factor on the drift of the dividend processes of the government bonds has both positive and negative sides. In detail, in a good economy, tightening monetary policy increases the market outstanding in the secondary market, while the increase in tax revenue leads to less issuance of government bonds.

Three agents, denoted by $i = 1, 2, 3$, are considered, each with an initial wealth of $x_0^i = 100$ and time preference parameters $\beta^1 = 0.01, \beta^2 = 0.02, \beta^3 = 0.03$. The parameters representing their views on fundamental risks, a^i for $\hat{\lambda}^i$ in (118), are set to $a^1 = 1.0$, $a^2 = 0$, and $a^3 = -1.0$. For example, the first agent holds aggressive views with a lower time preference, i.e., placing more emphasis on future spending, the second agent has neutral views with moderate time preference, and the third agent possesses conservative views with a high time preference, i.e., prioritizing near-term spending.

We define the base mortality rate $\bar{\lambda}^m$ as follows: 0.04% ($0 \le t < 10$), 0.06% ($10 \le t < 20$), 0.10% ($20 < t < 30$), 0.24% ($30 \le t < 40$), 0.63% ($40 \le t < 50$), 1.7% ($50 < t \le 60$), 4.8% ($60 < t \le 70$), 15% ($70 < t \le 80$), 40% ($80 < t \le 100$) which approximately corresponds to the mortality rate for ages 20 to over 100 for males in Japan (Ministry of Health Life and Welfare, Japan [8]). We set k_1 and k_2 to define λ^m in (144) as $k_1 = -0.01$ and $k_2 = 0.01$, implying that the economic factor has a decreasing effect on the mortality rate while the public health deterioration factor has an increasing effect on the mortality rate.

In summary, the parameters used for the base case are as follows.

Parameter	Value
Initial values of the economic and public health deterioration factors Y_0^1, Y_0^2	1
Drift of the economic and public health deterioration factors $\mu_{Y,1}, l, \mu_{Y,2}$	0.1, -0.003, 0.1
Volatility of the economic and public health deterioration factors $\sigma_{Y,1}$, $\sigma_{Y,2}$	0.1
Drift coefficients (factor proportional part) of the dividend processes $\alpha_{Y,j}(j = 1, \ldots, 5)$	0.00
Drift coefficients (factor proportional part) of the dividend processes $\alpha_{Y,j}(j = 6)$	0.02
Drift coefficients (constant part) of the dividend processes $\beta_{Y,j}(j = 1, \ldots, 5)$	0.010
Drift coefficients (constant part) of the dividend processes $\beta_{Y,j}(j = 6)$	0.030
Initial values of the dividend processes $\delta_0^j(j = 1, \ldots, 5)$	$0.1\delta_0$
Initial values of the dividend processes $\delta_0^j(j = 6)$	$0.5\delta_0$
Volatility of the dividend processes $\bar{\sigma}^{\delta,j}(j = 1, \ldots, 5)$	0.1
Volatility of the dividend processes $\bar{\sigma}^{\delta,j}(j = 6)$	0.3
Instantaneous correlation with the economic factor of the dividend processes $\rho^j(j = 1, \ldots, 6)$	0.5
Initial wealth of the agents x_0^1, x_0^2, x_0^3	100
Time preference parameters of the agents β^1, β^2, β^3	0.01, 0.02, 0.03
Parameters for views on fundamental risks of the agents a^1, a^2, a^3	1.0, 0, -1.0
Base mortality rate for age 20-30 $\bar{\lambda}^m(0 \le t < 10)$	0.04%
Base mortality rate for age 30-40 $\bar{\lambda}^m(10 \le t < 20)$	0.06%
Base mortality rate for age 40-50 $\bar{\lambda}^m(20 < t < 30)$	0.10%
Base mortality rate for age 50-60 $\bar{\lambda}^m(30 \le t < 40)$	0.24%
Base mortality rate for age 60-70 $\bar{\lambda}^m(40 \le t < 50)$	0.63%
Base mortality rate for age 70-80 $\bar{\lambda}^m(50 < t \le 60)$	1.7%
Base mortality rate for age 80-90 $\bar{\lambda}^m(60 < t \le 70)$	4.8%
Base mortality rate for age 90-100 $\bar{\lambda}^m(70 < t \le 80)$	15%
Base mortality rate for age 100 and above $\bar{\lambda}^m(80 < t \le 100)$	40%
Effect parameters of the economic and public health deterioration factors on the mortality rate k_1, k_2	-0.01, 0.01

Table 1
Summary of parameters for the base case.

Next, by shifting parameters in the model, we examine the impact of changes in bond supply in the secondary market, market expectations of agents on the market outstanding securities, and public health conditions on the super-long discount rate and insurance pricing.

After investigating the parameter shifts corresponding to the observed impact in the following historical events, we calculate 100-year discount rates and the insurance prices with the shifted parameters.

(1) **Bond Purchase in the Japanese Market under the Unconventional Monetary Easing:** First of all, we examine the impacts on insurance pricing of monetary easing and tightening accompanying supply changes of government bonds in the secondary market by the central bank and the government by shifting the parameters $\beta_{Y,3}$, $\beta_{Y,4}$, and $\beta_{Y,5}$, the constant part of the drift in the dividend processes δ^3, δ^4, and δ^5.

Under the unconventional monetary policy conducted by the former BOJ governor Kuroda, in addition to the minus rate policy, large amounts of government bonds were purchased for monetary easing, and the market outstanding in the secondary market decreased. We observed a 1.4% decrease in the 30-year discount rate from April 2013, when the monetary easing began, to July 2016, before the yield curve control, in which unlimited amount of bond purchase was attempted to keep the ten year yield at zero, started (Ministry of Finance, Japan [26]). This yield change corresponds to the shifts in the parameters $\beta_{Y,3}$, $\beta_{Y,4}$, and $\beta_{Y,5}$, the constant part of the drift in the dividend processes δ^3, δ^4, and δ^5, by -0.045.

	Base case	Easing case	Tightening case
Death benefit	0.082	0.148	0.026
Life annuity	25.4	32.9	17.6
Discount rate for 100 years	4.8%	3.7%	7.6%

Table 2
Insurance pricing in the base, the easing, and the tightening cases. $\beta_{Y,3}, \beta_{Y,4}, \beta_{Y,5} = 0.010$ for the base case, $\beta_{Y,3}, \beta_{Y,4}, \beta_{Y,5} = -0.035$ for the easing case, and $\beta_{Y,3}, \beta_{Y,4}, \beta_{Y,5} = 0.055$ for the tightening case.

Firstly, in Table 2, if $\beta_{Y,3}$, $\beta_{Y,4}$, and $\beta_{Y,5}$, the drift of the dividend processes δ^3, δ^4, δ^5 that affect the market outstanding of the securities, shift from the base case of 0.010 to -0.035, which is the same magnitude as the yield decrease resulting from the decrease in the secondary market outstanding caused by the government bond purchasing for monetary easing during the aforementioned period, the supply of bonds to the secondary market decreases, as observed in the central bank's bond purchasing operation, leading to lower yields. As a result, the death benefit price increases from 0.082 to 0.148 and from 25.4 to 32.9 for the life annuity price.

On the contrary, if $\beta_{Y,3}$, $\beta_{Y,4}$, and $\beta_{Y,5}$ shift from 0.010 to 0.055, corresponding to an increase in bond supply and thus monetary tightening resulting in higher yields, the death benefit price decreases from 0.082 to 0.026 and from 25.4 to 17.6 for the life annuity price.

These insurance price changes are mainly due to the shift in the discount rate. In detail, in this scenario, the super-long discount rate for 100 years is estimated at 4.8%, changing to 3.7% for easing and 7.6% for tightening.

Specifically, as Figure 1 illustrates, the discount rate increases with tightening monetary policy, whereas it decreases with easing monetary policy. Moreover, the yield curves are steeper than the base case in both easing and tightening scenarios. This steepness arises from the difference between the two parameter sets $(\beta_{Y,3}, \beta_{Y,4}, \beta_{Y,5})$ and $(\beta_{Y,1}, \beta_{Y,2}, \beta_{Y,6})$ in $\delta^j,\ j = 1, \dots, 6$ in (146). In detail, these values are all 0.01 in the base case, i.e., $\beta_{Y,1}, \dots, \beta_{Y,6} = 0.01$, while in the tightening case, $\beta_{Y,3}, \beta_{Y,4}, \beta_{Y,5} = 0.055$ and in the easing case, $\beta_{Y,3}, \beta_{Y,4}, \beta_{Y,5} = -0.035$. As a result, when we consider the total dividend δ deriving the yield $\mathcal{Y}(0, \mathcal{T})$ in (137), the parameters of greater values, i.e., $\beta_{Y,3}, \beta_{Y,4}, \beta_{Y,5} = 0.04$ in the tightening case, and $\beta_{Y,1}, \beta_{Y,2}, \beta_{Y,6} = 0.01$ in the easing case, are dominant as time passes in the total dividend δ due to the effect of those parameters in the exponential form of δ^j in (146). As a result, both easing and tightening scenarios display steeper yield curves.

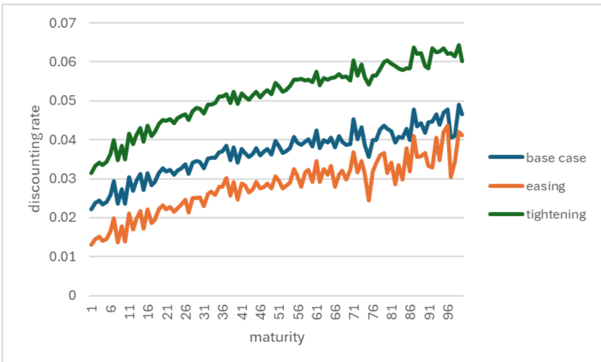

Fig. 1. Discount rate for the base, the easing, and the tightening cases. $\beta_{Y,3}, \beta_{Y,4}, \beta_{Y,5} = 0.010$ for the base case, $\beta_{Y,3}, \beta_{Y,4}, \beta_{Y,5} = -0.035$ for the easing case, and $\beta_{Y,3}, \beta_{Y,4}, \beta_{Y,5} = 0.055$ for the tightening case.

(2) **Market Expectation Change by the Announcement for Monetary Tightening by the Bank of Japan:** Next, we investigate the impact on the insurance pricing of the market expectation changes on the secondary market outstanding values of the assets by shifting a^1, a^2, and a^3, the parameters for the views on fundamental risks for Agents 1-3 defined as $\hat{\lambda}^i_t = a^i_t \sigma^\delta_t$ in (118).

Following the Bank of Japan's announcement of its exit from yield curve control in March 2024, market expectations of a future reduction in the BOJ's government bond purchases intensified. This expectation continued until the official announcement of the reduction in government bond purchases in July 2024. The increased market expectations resulted in lower bond prices and higher yields.

Specifically, we observed the rise in long-term interest rates, a 0.4% increase in the 30-year yield between March 2024, when the Bank of Japan exited the yield curve control policy, and July 2024, when the BOJ announced reduced bond purchases (Ministry of Finance, Japan [26]). We note that an increase in the market outstanding correlates with an increasing dividend process in our model, reflecting increased coupon/dividend payments. This 0.4% increase in the 30-year yield corresponds to shifts of a^1, a^2, and a^3 in (118), market expectations on the return of the secondary market outstanding values, by 0.1 in our model.

Then, Table 3 shows the impact on insurance pricing via the discount rate of the view change on the risks, i.e., the market expectation change on the expected return of the secondary market outstanding of Agents 1-3.

If we set $a^1 = 1.1$, $a^2 = 0.1$, $a^3 = -0.9$, i.e., the agents are more aggressive, that is, the agents expect the secondary market outstanding values to increase in this example, the discount rate is higher, and the death benefit price is lower. This change in the discount rate agrees with the result in equation (108) in Proposition 4: when a^1, a^2, and a^3 in $\hat{\lambda}^1$, $\hat{\lambda}^2$, $\hat{\lambda}^3$ are higher, due to the inner product $\hat{\lambda} \cdot \sigma_\delta$, the interest rate becomes high.

	Base case	Aggressive case
Death benefit	0.082	0.074
Life annuity	25.4	24.4
Discount rate for 100 years	4.8%	5.0%

Table 3
Impact of aggressive agents on the 100 year discount rate and insurance prices. $a^1 = 1.0$, $a^2 = 0$, $a^3 = -1.0$ for the base case and $a^1 = 1.1$, $a^2 = 0.1$, $a^3 = -0.9$ for the aggressive case.

(3) **Mortality Rate Changes after the Tohoku Earthquake:** Finally, we investigate the impact on insurance pricing of public health deterioration leading to an economic downturn by shifting $\mu_{Y,2}$, the drift of the public health deterioration factor Y^2.

After the 2011 Tohoku earthquake, the mortality rate in Iwate Prefecture, which was the area most severely affected, relatively increased by approximately 7% over the following decade compared to the pre-earthquake mortality rate (Iwate Prefecture, Japan [27]). This 7% relative increase in mortality rate over the ten-year period corresponds to a 0.7 shift in the public health deterioration factor drift because of $k_2 = 0.01$.

Moreover, we set l in (143), the parameter indicating impact of the public health deterioration factor Y^2 on the drift of the economic factor Y^1, as -0.003 so that with the shift of $\mu_{Y,2}$ above, the 30-year yield changes by -0.30%, which corresponds to the 30-year yield drop after the Tohoku earthquake from March 2011 to September 2011. In detail, with the shift of $\mu_{Y,2}$ by 0.7, through the drift of the economic factor Y^1 negatively impacted by the public health deterioration factor Y^2, the economic factor lowers, and accordingly, the yield curve lowers.

Table 4 presents the insurance prices in the case of public health deterioration, especially when the economy is led to a downturn and the central bank purchases government bonds, decreasing the secondary market supply of government bonds, for monetary easing. In this example, we set $\mu_{Y,2}$, the drift of the public health deterioration factor Y^2, as 0.8 by shifting 0.7 from the base case in the high mortality rate case. We also set $\beta_{Y,3}$, $\beta_{Y,4}$, and $\beta_{Y,5}$ to -0.035 as in the monetary easing case investigated in the first example.

By the shift in the public health deterioration factor, which directly affects mortality such as pandemic, earthquake, or deterioration of public health, the mortality rate increases and, at the same time, negatively affects the economic factor. This negative impact on the economic factor magnifies the lowering effect on the discount rate in the easing case.

	Base case	Easing case with high mortality
Death benefit	0.082	0.248
Life annuity	25.4	35.6
Discount rate for 100 years	4.8%	1.9%

Table 4
Insurance prices with high mortality rates during monetary easing. $\mu_{Y,2} = 0.1, 0.8$ for the base case and the high mortality rate case, and $\beta_{Y,3}, \beta_{Y,4}.\beta_{Y,5} = -0.035$ for the monetary easing situation.

Then, Table 4 indicates that in the high mortality case with the monetary easing scenario, where the central bank purchases the market outstanding bonds, and the supply of bonds in the secondary market decreases, insurance companies need more reserves for the insurance payment due to the effect on the discount rate of the public health deterioration factor.

Specifically, the result indicates that higher reserves are required for both products due to the lowering of discount rates amplified by public health deterioration in the monetary easing situation. Thus, it is essential for insurance companies to consider the impact of public health deterioration on the necessary reserves, taking into account the economic downturn and the resulting monetary easing situation.

5 Conclusion

We have effectively solved the multi-agent equilibrium problem in an incomplete market with discrete-time redemption, issuance, and coupon payments for groups of government bonds categorized by time to maturity and intermediate consumption. Our model specifically targets the optimal portfolio and consumption decisions of institutional investors' representative shareholders, with a particular emphasis on the market outstanding value of government bonds within the same maturity zone.

Moreover, we have derived equilibrium interest rates and yields for government bonds across different maturity sectors. By considering the supply control of these bonds by central banks and governments, particularly observed post-global financial crisis, our model incorporates changes in the market supply of government bonds. This allows the model to quantify the impact on the term structure of interest rates.

Additionally, we have examined the effects of net supply changes of government bonds in the secondary market, influenced by central bank and government policy changes, on insurance pricing such as death benefits and life annuities and their value at risk via the yield curve.

Furthermore, we have presented numerical examples demonstrating the impact of monetary policy changes, sentiment transitions, and mortality rate changes during periods of monetary easing on insurance prices, analyzed through shifts in super-long discount rates. Our model has proven to be valuable for estimating long-term discount rates, which are essential for insurance companies' claim pricing, and for comprehending the effects of central bank government bond purchases on insurance pricing.

This study is novel in investigating the impact of net supply changes of government bonds in the secondary market, driven by central bank and government policy changes, on insurance pricing through the term structure of interest rates using an endogenous approach within an incomplete market equilibrium model.

Future research will involve using empirical data to estimate our incomplete market equilibrium model and to assess its impact on the term structure of interest rates.

Declaration of competing interest

There is no competing interest.

Data availability

Data will be made available on request.

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