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An Incomplete Multi-Currency Equilibrium Model with Heterogeneous Time Preferences and Subjective Beliefs

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Abstract

For global multi-asset fund managers, reflecting their macroeconomic views in the prediction of expected interest rates across countries, exchange rates, and equity prices in a manner consistent with economic theory is challenging. The existing literature has yet to provide an established multi-currency model that is flexible enough to incorporate such views into the prediction of future asset price dynamics.

To address this problem, this paper proposes a novel multi-currency incomplete market model in which agents in each country have logarithmic utility but differ in their time preferences and subjective beliefs, within a market equilibrium framework, namely, supply and demand equilibrium.

With only a few exogenous inputs, such as each country's output process and agents' preference parameters, the model endogenously determines equilibrium interest rates, exchange rates, and stock prices, along with optimal consumption and portfolios. Thus, the model enables us to (i) flexibly incorporate cross-country differences in investors' time preferences and macroeconomic outlooks, and (ii) examine how these differences affect equilibrium interest rates and asset prices, including stock prices and exchange rates.

Moreover, by applying the particle filtering method within a state-space framework based on the two-country, two-currency version of the model to Japanese and U.S. market data (equity index futures, short-term interest rates, and the exchange rate), the model not only fits the observed dynamics of equity indices, short rates, and the exchange rate, but also effectively estimates the dynamics of home-country biases and latent economic factors, which can be utilized in making investment decisions in asset management practice.

Keywords: multi-currency equilibrium model, incomplete market, subjective beliefs, multi-asset investment, state-space model

1. Introduction and Related Work

1.1. Introduction

The globalization of financial markets has led investors to increasingly allocate capital across borders, seeking diversification and enhanced returns. This trend has fueled the rapid growth of multi-asset investment funds, which invest in multiple assets across multiple countries and now play a significant role in both institutional and individual portfolios worldwide. For example, in the U.S., target-date multi-asset funds are frequently used as default investment options in 401(k) retirement plans.

However, when operating such funds, even if the manager holds macroeconomic views, mapping them into predictions for each country's expected interest rates, exchange rates, and equity prices is challenging: these variables cannot be assigned arbitrarily but must be specified in a manner consistent with economic theory. Unfortunately, to the best of our knowledge, the existing literature has not provided an established multi-currency asset allocation framework that is sufficiently flexible to incorporate such a manager's macroeconomic views. As a result, multi-asset fund managers sometimes abandon rational asset allocation and resort to ad hoc methods (e.g., a naïve 25%-25%-25%-25% allocation across domestic equities, foreign equities, domestic bonds, and foreign bonds).

To address this problem, we develop a novel multi-currency incomplete market model in which agents in each country have logarithmic utility but may differ in time preferences and subjective beliefs within a market equilibrium framework, namely, supply and demand equilibrium. This framework uses only a few exogenous inputs, such as each country's output process and agents' preference parameters, and endogenously delivers equilibrium interest rates, exchange rates, and stock prices, along with optimal consumption and portfolios. With the model, we can flexibly capture cross-country differences in investors' time preferences and macroeconomic outlooks, and analyze how those differences influence equilibrium macro variables. Thus, the model enables practitioners to examine how investors' time preferences and macroeconomic views affect equilibrium asset prices in a manner consistent with the equilibrium framework, in which supply matches demand.

The main contributions of this study are as follows: (i) First, we extend an incomplete equilibrium model that incorporates heterogeneous time preferences and subjective beliefs to a multi-currency environment. (ii) Second, we obtain closed-form expressions for optimal consumption, investment strategies, interest rates, market prices of risk, and stock prices in equilibrium. (iii) Third, we empirically estimate transitions in agents' subjective beliefs and latent factors, which are reasonably explained by market reactions to past economic events, and demonstrate the flexibility of our model by fitting it to real-market data using state-space modeling and particle filtering.

The remainder of the paper is organized as follows. Subsection 1.2 reviews the related literature. Section 2 introduces the multi-currency equilibrium model in an incomplete market setting, allowing for agent heterogeneity in time preferences and subjective beliefs. Section 3 specializes the multi-currency model to a two-currency case and conducts an empirical study using Japanese and U.S. market data, demonstrating the model's applicability to actual market data. Section 4 concludes this paper and discusses future research directions.

1.2. Related Work

The general equilibrium model is a major topic in mathematical finance. Early foundational work includes Cox et al. (1985), who developed a continuous-time general equilibrium model for a simple but complete economy and analyzed asset price dynamics. Epstein & Miao (2003) examined a pure-exchange, continuous-time economy with two heterogeneous agents and complete markets under Knightian uncertainty. Žitković (2012) established existence and uniqueness results for stochastic equilibria in a class of incomplete continuous-time financial markets, where agents are exponential utility maximizers with heterogeneous risk aversion and general Markovian random endowments. Christensen & Larsen (2014) derive closed-form solutions for the equilibrium interest rate and market price of risk processes in an incomplete continuous-time market which has a finite number of heterogeneous exponential utility investors. Larsen & Sae-Sue (2016) construct continuous-time equilibrium models based on a finite number of exponential utility investors. For comprehensive textbook treatments, see Karatzas & Shreve (1998) for rigorous analysis of complete markets driven by Brownian motions, and Jarrow (2018) for discussion of incomplete markets and trading constraints in a semimartingale framework.

Studies focusing on the individual optimal investment problem include the following. Temocin et al. (2018) considered the optimal portfolio problem with minimum guarantee protection in a defined contribution pension scheme using a classical stochastic control approach. Chen et al. (2021) studied optimal investment problems for a pool of investors demanding minimum guarantees under two financial market settings. Ieda (2022) investigated a portfolio optimization problem with the following features: (i) a no-short selling constraint; (ii) a leverage constraint; and (iii) a performance criterion based on the lower mean square error between the investor's wealth and a predetermined target wealth level. Chen et al. (2025) studied the optimal investment and consumption strategy for an agent who has the addictive habit formation preference. Hu et al. (2025) researched optimal investment and consumption problems with various utilities, in a regime switching market with random coefficients and possibly subject to non-convex constraints. As a practitioner-authored paper from T. Rowe Price, Aboagye et al. (2024) extended the standard optimal investment problem for target date funds and estimated that their enhancements deliver an additional 5-6% per year in risk-adjusted spending.

In recent years, the following papers have addressed multi-agent settings and the mathematical techniques required in such contexts. Saito & Takahashi (2021) addressed sup-inf problems arising from agents' best- and worst-case scenario choices of probability measures, a situation frequently encountered in incomplete equilibrium models. Saito & Takahashi (2022) introduced a new portfolio optimization problem for a single agent facing model uncertainty, employing Malliavin calculus to handle uncertainties in fundamental risks. Kizaki et al. (2024a) proposed an equilibrium-based multi-agent model in a complete market, allowing agents to have heterogeneous sentiments regarding fundamental risks modeled by Brownian motion. Kizaki et al. (2024b) extended this line of research to incomplete markets, developing a multi-agent equilibrium model with heterogeneous risk preferences and income/payout profiles. Maggistro et al. (2025) considered a multi-agent portfolio optimization model with life insurance for two players with random lifetime under a dynamic game approach. Saito & Takahashi (2025) contributed by presenting a multi-agent equilibrium model in an incomplete market, incorporating a super-long discount rate for insurance companies and explicitly modeling government financing and central bank operations.

Building on the incomplete multi-agent equilibrium frameworks developed by Kizaki et al. (2024b) and Saito & Takahashi (2025), we extend the model to a multi-currency environment. The proposed framework is highly general and flexible, allowing for a realistic representation of international investment environments. This enables a systematic analysis of how cross-country differences in preferences and beliefs influence key financial variables such as equity prices, interest rates, and exchange rates.

2. Multi-Currency Model

This section develops an incomplete multi-currency general equilibrium model, allowing for heterogeneous agents with distinct time preferences and subjective beliefs. Each country is populated by many agents and endowed with a financial market consisting of many stocks and a single money market account, together with a goods market for a single domestic good. Also, we consider a finite time horizon $[0,T]$ ($T > 0$) and work on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, P)$ that satisfies the usual conditions. The uncertainty in the economy is modeled by a m-dimensional Brownian motion W_t defined on this space, with the filtration $\{\mathcal{F}_t\}_{0\le t\le T}$ generated by W_t.

2.1. Model Setting

We consider N countries, where each country i ($i = 1, \dots, N$) has L^i agents, K^i stocks, and one money market account. Thus, an agent in country i can invest in $\sum_{j=1}^N K^j$ stocks (including foreign stocks), $(N-1)$ foreign money market accounts and one riskless money market account. Here, to consider an incomplete market, we assume that m is greater than the number of risky assets $\sum_{j=1}^N K^j + N - 1$. Also, we consider exogenously given dividend processes for the k-th stock in country i $\delta_t^{i,k}$, driven by a latent factor process Y_t, are given by:

$$d\delta_t^{i,k} = \delta_t^{i,k} \left\{ \mu_\delta^{i,k}(Y_t,t)dt + \sigma_\delta^{i,k}(Y_t,t) \cdot dW_t \right\}, \tag{1}$$

$$dY_t = \mu_y(Y_t,t)dt + \sigma_y(Y_t,t) \cdot dW_t, \tag{2}$$

where $\mu_\delta^{i,k}(Y_t,t)$ and $\mu_y(Y_t,t)$ denote the drift terms and $\sigma_\delta^{i,k}(Y_t,t)$ and $\sigma_y(Y_t,t)$ represent $m \times 1$-dimensional volatility vectors, respectively. In addition, we introduce the aggregate dividend process

δ_t^i for country i:

$$\delta_t^i = \sum_{k=1}^{K^i} \delta_t^{i,k}, \tag{3}$$

$$\mu_\delta^i(Y_t, t) = \sum_{k=1}^{K^i} \frac{\delta_t^{i,k}}{\delta_t^i} \mu_\delta^{i,k}(Y_t, t), \tag{4}$$

$$\sigma_\delta^i(Y_t, t) = \sum_{k=1}^{K^i} \frac{\delta_t^{i,k}}{\delta_t^i} \sigma_\delta^{i,k}(Y_t, t), \tag{5}$$

$$d\delta_t^i = \delta_t^i \left\{ \mu_\delta^i(Y_t, t)dt + \sigma_\delta^i(Y_t, t) \cdot dW_t \right\}, \tag{6}$$

where $\mu_\delta^i(Y_t, t)$ and $\sigma_\delta^i(Y_t, t)$ are the drift and $m \times 1$-dimensional volatility of the aggregate dividend process, respectively.

The stock price process for the k-th stock in country i and aggregate stock price process S_t^i are specified as:

$$dS_t^{i,k} = S_t^{i,k} \left\{ \mu_S^{i,k}(Y_t, t)dt + \sigma_S^{i,k}(Y_t, t)dW_t \right\} - \delta_t^{i,k}dt; \; S_T^{i,k} = 0, \tag{7}$$

$$S_t^i = \sum_{k=1}^{K^i} S_t^{i,k}, \tag{8}$$

$$dS_t^i = S_t^i \left\{ \mu_S^i(Y_t, t)dt + \sigma_S^i(Y_t, t)dW_t \right\} - \delta_t^i dt; \; S_T^i = 0, \tag{9}$$

where $\mu_S^{i,k}(Y_t, t)$, $\mu_S^i(Y_t, t)$, $\sigma_S^{i,k}(Y_t, t)$, and $\sigma_S^i(Y_t, t)$ are the drift and $1 \times m$-dimensional volatility terms. Moreover, the money market account in country i, denoted by B_t^i, is given by:

$$dB_t^i = r_t^i B_t^i dt, \tag{10}$$

where r_t^i is the interest rate in country i.

Next, we consider the exchange rate process $q_t^{i,j}$ between countries i and j, where $q_t^{i,j}$ denotes the relative price of one unit of country j's consumption good in terms of country i's consumption good, evolving according to the following stochastic differential equation (SDE):

$$dq_t^{i,j} = q_t^{i,j} \left\{ \mu_{q,t}^{i,j} dt + \sigma_{q,t}^{i,j} dW_t \right\} \tag{11}$$

$$= q_t^{i,j} \left\{ \left(r_t^i - r_t^j \right) dt + \left(\theta_t^i - \theta_t^j \right) \cdot \left(dW_t + \theta_t^i dt \right) \right\}, \tag{12}$$

where $\mu_{q,t}^{i,j}$ is the drift term, $\sigma_{q,t}^{i,j}$ is the $1 \times m$-dimensional volatility term, and θ_t^i and θ_t^j are the $m \times 1$-dimensional market price of risk processes in countries i and j, respectively. This follows from no-arbitrage condition, which requires that the drift of the exchange rate $\mu_{q,t}^{i,j}$ is given by the difference in interest rates between the two countries, $r_t^i - r_t^j$, and the volatility is given by the difference in the market prices of risk, $\theta_t^i - \theta_t^j$. When $i = j$, we have $q_t^{i,i} = 1$. Note that, if $(\sigma_t^i \sigma_t^{i\top})^{-1}$ exists, θ_t^i are defined as follows:

$$\theta_t^i = \sigma_t^{i\top} (\sigma_t^i \sigma_t^{i\top})^{-1} \left(\mu_t^i - r_t^i \mathbf{1} \right), \tag{13}$$

where σ_t^i is a $(\sum_{i=1}^N K^i + N - 1) \times m$ matrix given by $\begin{pmatrix} \sigma_{S,t}^{1,1} + \sigma_{q,t}^{i,1} \\ \cdots \\ \sigma_{S,t}^{N,K^N} + \sigma_{q,t}^{i,N} \\ \sigma_{q,t}^{i,1} \\ \cdots \\ \sigma_{q,t}^{i,N} \end{pmatrix}$, where each row corresponds to a risky asset and the row associated with $\sigma_{q,t}^{i,i}$ (the country i's money market account) is omitted. Similarly, μ_t^i is a $(\sum_{i=1}^N K^i + N - 1) \times 1$ vector given by $\begin{pmatrix} \mu_{S,t}^{1,1} + \mu_{q,t}^{i,1} + \sigma_{S,t}^{1,1}(\sigma_{q,t}^{i,1})^\top \\ \cdots \\ \mu_{S,t}^{N,K^N} + \mu_{q,t}^{i,N} + \sigma_{S,t}^{N,K^N}(\sigma_{q,t}^{i,N})^\top \\ r_t^1 + \mu_{q,t}^{i,1} \\ \cdots \\ r_t^N + \mu_{q,t}^{i,N} \end{pmatrix}$. $\mathbf{1}$ is the $(\sum_{i=1}^N K^i + N - 1) \times 1$ vector of ones.

2.2. Optimal Consumption Problem

This subsection formulates and solves each agent's optimal consumption problem in the multi-currency model. We introduce following notations about portfolio position for the l-th agent in country i (all values are denominated by one unit of country i's consumption good):

- $\pi_t^{i,l,(j,k)}$: investment value in stock k of country j,
- $\pi_t^{i,l,j} := \sum_{k=1}^{K^j} \pi_t^{i,l,(j,k)}$: total investment value in country j's stock market,
- $\pi_t^{i,l,N+j}$: investment value in country j's money market account,
- $\pi_t^{i,l} = (\pi_t^{i,l,(1,1)}, \ldots, \pi_t^{i,l,(N,K^N)}, \pi_t^{i,l,N+1}, \ldots, \pi_t^{i,l,2N})^\top$: vector of all risky positions except the domestic (country i) money market account,

Note that the pair $(\pi_t^{i,l}, \pi_t^{i,l,N+i})$ represents the agent's investment strategy for all securities. Also, while $\pi^{i,l,j}$ $(j = 1, \ldots, N)$ and $\pi^{i,l,N+j}$ $(j = 1, \ldots, N)$ denote the amounts invested in country j's stocks and money market account, we sometimes collectively denote them by $\pi^{i,l,j}$ for $j = 1, \ldots, 2N$.

Then, the agent's optimization problem is formulated as follows:

$$\max_{\pi_t^{i,l}, \pi_t^{i,l,N+i}, \{c_t^{i,l,j}\}_{j=1,\ldots,N}} \mathbf{E}\left[\int_0^T u_t^{i,l}(c^{i,l,1}, ..., c^{i,l,N}) dt\right], \tag{14}$$

$$s.t.\ dX_t^{i,l} = \sum_{j=1}^N \sum_{k=1}^{K^j} \pi_t^{i,l,(j,k)} \frac{d(q_t^{i,j} S_t^{j,k}) + q_t^{i,j} \delta_t^{j,k} dt}{q_t^{i,j} S_t^{j,k}} + \sum_{j=1}^N \pi_t^{i,l,N+j} \frac{d(q_t^{i,j} B_t^j)}{q_t^{i,j} B_t^j}$$

$$- \left(\sum_{j=1}^N q_t^{i,j} c_t^{i,l,j}\right) dt;\ X_0^{i,l} = x_0^{i,l} > 0, \tag{15}$$

$$X_t^{i,l} \geq 0;\ c_t^{i,l,j} \geq 0;\ \forall t \in [0, T], \tag{16}$$

where $c_t^{i,l,j}$ is the consumption of the agent for a good in country j, $X_t^{i,l}$ is the wealth process of the agent, and $x_0^{i,l}$ is the initial wealth of the agent. Here, $X_t^{i,l} = \sum_{j=1}^{2N} \pi_t^{i,l,j}$.

The utility function $u_t^{i,l}(c^{i,l,1}, ..., c^{i,l,N})$ in (14) is given by:

$$u_t^{i,l}(c^{i,l,1}, ..., c^{i,l,N}) = \eta_t^{i,l} \alpha_t^{i,l} \left(\sum_{j=1}^{N} \gamma^{i,l,j} \log c_t^{i,l,j} \right); \ \gamma^{i,l,j} \in [0, 1], \tag{17}$$

$$\alpha_t^{i,l} = e^{-\beta^{i,l} t}, \tag{18}$$

where $\eta_t^{i,l}$ is the subjective belief, $\gamma^{i,l,j}$ is the preference parameter for goods with $\sum_{j=1}^{N} \gamma^{i,l,j} = 1$ and $\beta^{i,l}$ is the time preference parameter of the agent. The subjective belief $\eta_t^{i,l}$ is defined as:

$$\eta_t^{i,l} = \exp\left(\int_0^t \lambda_s^{i,l} \cdot dW_s - \frac{1}{2} \int_0^t |\lambda_s^{i,l}|^2 ds \right); \ \lambda_s^{i,l} = \lambda^{i,l}(Y_s, s), \tag{19}$$

where $\lambda^{i,l}(Y_s, s)$ is the subjective view of agent. Here, $\lambda^{i,l}$ represents the subjective views of the agent l in country i on the Brownian motion W_t. Namely, for the probability measure $P^{i,l}$ defined as

$$\frac{dP^{i,l}}{dP} = \eta_T^{i,l}$$

by Girsanov's theorem, $W^{P^{i,l}}$ defined as $dW_t^{P^{i,l}} = dW_t^P - \lambda_t^{i,l} dt$ is a $P^{i,l}$-Brownian motion and $\lambda^{i,l}$ indicates the agent l's bias on the Brownian motion under the physical measure P.

Constraint (15) represents the wealth evolution of the agent, which consists of the returns from investments in risky assets and money market accounts, minus the consumption. Constraint (16) represents the budget constraint and non-negative condition for wealth and consumption, respectively.

To solve the optimization problem, defined by (14), (15), and (16), we first rewrite the constraints as follows:

$$dX_t^{i,l} = \sum_{j=1}^{N} \sum_{k=1}^{K^j} \pi_t^{i,l,(j,k)} \frac{d(q_t^{i,j} S_t^{j,k}) + q_t^{i,j} \delta_t^{j,k} dt}{q_t^{i,j} S_t^{j,k}} + \sum_{j=1}^{N} \pi_t^{i,l,N+j} \frac{d(q_t^{i,j} B_t^j)}{q_t^{i,j} B_t^j} - \left(\sum_{j=1}^{N} q_t^{i,j} c_t^{i,l,j} \right) dt \tag{20}$$

$$= r_t^i X_t^{i,l} dt + \pi_t^{i,l\top} \sigma_t^i (dW_t + \theta_t^i dt) - \left(\sum_{j=1}^{N} q_t^{i,j} c_t^{i,l,j} \right) dt. \tag{21}$$

Next, we consider the admissibility of the consumption process. The admissibility derives from the condition:

$$X_t^{i,l} \geq 0; \forall t \in [0, T]. \tag{22}$$

This means that the total wealth of the agent must be non-negative at all times, which is a standard assumption in financial models to prevent bankruptcy. Here, we consider the (subjective) state price density process $H_t^{i,l}$:

$$H_t^{i,l} = Z_t^{\theta^{i,l}} / B_t^i, \tag{23}$$

$$Z_t^{\theta^{i,l}} = \exp\left\{ -\frac{1}{2} \int_0^t |\theta_s^i + \nu_s^{i,l}|^2 ds - \int_0^t (\theta_s^i + \nu_s^{i,l}) \cdot dW_s \right\}. \tag{24}$$

Here, $Z_t^{\theta^{i,l}}$ is the Radon-Nikodym derivative induced by the agent's beliefs, which consists of common part θ_t^i and the agent specific part $\nu_t^{i,l}$. The $\nu_t^{i,l}$ is an orthogonal vector of the risky assets' volatility vector space $range\left(\sigma_t^{i\top}\right)$, i.e., $\sigma_t^i \nu_t^{i,l} = 0$. When the market is complete, we have $\nu_t^{i,l} = 0$. Then, we obtain the following relationship:

$$d(H_t^{i,l} X_t^{i,l}) = -H_t^{i,l}(\sum_{j=1}^{N} q_t^{i,j} c_t^{i,l,j})dt + H_t^{i,l}\left\{\pi_t^{i,l\top}\sigma_t^i - X_t^{i,l}(\theta_s^i + \nu_s^{i,l})^\top\right\} dW_t. \tag{25}$$

Then, we have:

$$H_T^{i,l} X_T^{i,l} + \int_0^T H_t^{i,l}(\sum_{j=1}^{N} q_t^{i,j} c_t^{i,l,j})dt = x_0^{i,l} + \int_0^T H_t^{i,l}\left\{\pi_t^{i,l\top}\sigma_t^i - X_t^{i,l}(\theta_s^i + \nu_s^{i,l})^\top\right\} dW_t. \tag{26}$$

Here, since wealth and consumption are non-negative, the left-hand side of (26) is lower bounded. Thus, local martingale on the right-hand side of (26) must be a supermartingale. Taking expectations on both sides, we have:

$$\mathbf{E}\left[\int_0^T H_t^{i,l}\left(\sum_{j=1}^{N} q_t^{i,j} c_t^{i,l,j}\right) dt\right] \leq x_0^{i,l}. \tag{27}$$

Then, we consider the following admissible set of consumption processes:

$$\mathcal{A}^{i,l} = \left\{\left(c_t^{i,l,j}\right)_{j=1,\ldots,N} \mid c_t^{i,l,j} \geq 0, \mathbf{E}\left[\int_0^T H_t^{i,l}\left(\sum_{j=1}^{N} q_t^{i,j} c_t^{i,l,j}\right) dt\right] \leq x_0^{i,l}, \forall H_t^{i,l}\right\}, \tag{28}$$

where the non-negative condition for consumption in $\mathcal{A}^{i,l}$ is automatically satisfied, since we adopt the log utility function.

Summarizing the above discussion, directly addressing the original optimization is difficult, thus, we instead consider the following consumption-only optimization problem:

$$\max_{c_t^{i,l,j}} \mathbf{E}\left[\int_0^T \eta_t^{i,l}\alpha_t^{i,l}\left(\sum_{j=1}^{N} \gamma^{i,l,j} \log c_t^{i,l,j}\right) dt\right], \tag{29}$$

$$s.t.\ \mathbf{E}\left[\int_0^T H_t^{i,l}\left(\sum_{j=1}^{N} q_t^{i,j} c_t^{i,l,j}\right) dt\right] \leq x_0^{i,l},\ \forall\, H_t^{i,l} \text{ satisfying } \sigma_t^i \nu_t^{i,l} = 0. \tag{30}$$

After we find the optimal solution $c_t^{i,l,j,*}$ for (29) under (30), we confirm that there exists an investment strategy $(\pi^{i,l,*}, \pi_t^{i,l,N+i,*})$ achieving the optimal consumption because, unlike the complete market case, the existence of the investment strategy is not guaranteed in incomplete markets. By this confirmation, we ensure that the pair $(c_t^{i,l,j,*}, (\pi^{i,l,*}, \pi_t^{i,l,N+i,*}))$ constitutes a solution to the original problem (14), (15), and (16).

To proceed, we define the Lagrangian as follows:

$$\mathcal{L} = \mathbf{E}\left[\int_0^T \eta_t^{i,l}\alpha_t^{i,l}(\sum_{j=1}^{N} \gamma^{i,l,j} \log c_t^{i,l,j})dt\right] + y^{i,l}\left(x_0^{i,l} - \mathbf{E}\left[\int_0^T H_t^{i,l}(\sum_{j=1}^{N} q_t^{i,j} c_t^{i,l,j})dt\right]\right), \tag{31}$$

where $y^{i,l}$ is the Lagrange multiplier. Then, the agent's optimization problem can be rewritten as following sup-inf problem:

$$\sup_{c_t^{i,l,j}} \inf_{y^{i,l}, \nu_t^{i,l}, \sigma_t^i \nu_t^{i,l}=0} \mathcal{L}(c_t^{i,l,j}, y^{i,l}, \nu_t^{i,l}). \tag{32}$$

Since the market is incomplete, there exist multiple risk-neutral measures. Thus, each agent chooses the consumption process under the worst case scenario among all measures $\nu_t^{i,l}$, which is represented by the infimum over $\nu_t^{i,l}$ in the above problem.

To solve the sup-inf problem, we reformulate it as the following dual (inf-sup) problem:

$$\inf_{y^{i,l}, \nu_t^{i,l}, \sigma_t^i \nu_t^{i,l}=0} \sup_{c_t^{i,l,j}} \mathcal{L}(c_t^{i,l,j}, y^{i,l}, \nu_t^{i,l}). \tag{33}$$

The solution to the dual problem (33) can be obtained by Proposition 1.

Proposition 1. *$c_t^{i,l,j,*}$, $\nu_t^{i,l}$, and $y^{i,l}$ set as follows solve the inf-sup dual problem (33).*

$$c_t^{i,l,j,*} = \frac{\gamma^{i,l,j} \alpha_t^{i,l} B_t^i Z_t^{i,l}}{y^{i,l} Z_t^{\theta^i} q_t^{i,j}}, \tag{34}$$

$$\nu_t^{i,l} = -\hat{\lambda}_t^{i,l,\perp}, \tag{35}$$

$$y^{i,l} = \frac{1 - e^{-\beta^{i,l} T}}{\beta^{i,l} x_0^{i,l}}. \tag{36}$$

Here, $Z_t^{\theta^i}$ and $Z_t^{i,l}$ are defined as follows:

$$Z_t^{\theta^i} = \exp\left(-\int_0^t \theta_s^i \cdot dW_s - \frac{1}{2} \int_0^t |\theta_s^i|^2 ds \right); Z_0^{\theta^i} = 1, \tag{37}$$

$$Z_t^{i,l} = \exp\left(\int_0^t \hat{\lambda}_s^{i,l} \cdot dW_s - \frac{1}{2} \int_0^t |\hat{\lambda}_s^{i,l}|^2 ds \right); Z_0^{i,l} = 1, \tag{38}$$

and $\hat{\lambda}_t^{i,l,\perp}$ is the projection of $\lambda_t^{i,l}$ onto the orthogonal space of range $\left(\sigma_t^{i\top}\right)$. Here, we assume that $\lambda_t^{i,l}$ admits the following decomposition: $\lambda_t^{i,l} = \hat{\lambda}_t^{i,l} \oplus \hat{\lambda}_t^{i,l,\perp}$, where $\hat{\lambda}_t^{i,l} \in range(\sigma_t^{i\top})$ and $\hat{\lambda}_t^{i,l,\perp}$ lies in its orthogonal space.

Proof. See Appendix A. □

We next confirm that the dual solution also solves the original optimization problem, which is guaranteed by the following Theorem 1.

Theorem 1. *The solution of the dual problem (33), $c_t^{i,l,j,*}$, $\nu_t^{i,l}$, and $y^{i,l}$, is also a solution of the primal problem (32).*

$$c_t^{i,l,j,*} = \frac{\gamma^{i,l,j} \alpha_t^{i,l} B_t^i Z_t^{i,l}}{y^{i,l} Z_t^{\theta^i} q_t^{i,j}}, \tag{39}$$

$$\nu_t^{i,l} = -\hat{\lambda}_t^{i,l,\perp}, \tag{40}$$

$$y^{i,l} = \frac{1 - e^{-\beta^{i,l} T}}{\beta^{i,l} x_0^{i,l}}. \tag{41}$$

Proof. See Appendix B. ☐

Finally, the following Theorem 2 shows that the investment strategy $(\pi_t^{i,l,*}, \pi_t^{i,l,N+i,*})$ achieves the optimal consumption $c_t^{i,l,j,*}$, which means that $c_t^{i,l,j,*}$ and $(\pi_t^{i,l,*}, \pi_t^{i,l,N+i,*})$ are the optimal solution for the original consumption problem defined in (14), (15), and (16).

Theorem 2. *Under the assumption that rank* $\left(\sigma_t^i\right) = \sum_1^N K^i + N - 1$, *i.e.* $(\sigma_t^i \sigma_t^{i\top})^{-1}$ *exists, the optimal wealth* $X_t^{i,l,*}$ *and investment strategy* $(\pi_t^{i,l,*}, \pi_t^{i,l,N+i,*})$ *achieving the optimal consumption* $c_t^{i,l,j,*}$ *is given by:*

$$X_t^{i,l,*} = \frac{e^{-\beta^{i,l} t} - e^{-\beta^{i,l} T}}{1 - e^{-\beta^{i,l} T}} \left(\frac{x_0^{i,l} B_t^i Z_t^{i,l}}{Z_t^{\theta^i}} \right), \tag{42}$$

$$\pi_t^{i,l,*} = X_t^{i,l,*} (\sigma_t^i \sigma_t^{i\top})^{-1} \sigma_t^i (\theta_t^i + \hat{\lambda}_t^{i,l}), \tag{43}$$

$$\pi_t^{i,l,N+i,*} = X_t^{i,l,*} - \pi_t^{i,l} \cdot 1. \tag{44}$$

Proof. See Appendix C. ☐

Having established the explicit solutions for each agent's optimal consumption and investment strategies in the multi-currency setting, we now turn to the considerations of equilibrium asset prices and market clearing. In the following Section, we derive the equilibrium interest rates and market prices of risk by imposing goods market clearing conditions.

2.3. Interest Rate, Market Price of Risk, and Goods Market Clearing

This subsection introduces Theorem 3, which determines the equilibrium interest rate r_t^i and market price of risk θ_t^i based on the goods market clearing conditions given by:

$$\sum_{j=1}^{N} \sum_{l=1}^{L^j} c_t^{j,l,i} = \delta_t^i; \ \forall i = 1, ..., N. \tag{45}$$

Theorem 3. *Equilibrium interest rate* r_t^i *and market price of risk* θ_t^i *are given by:*

$$\begin{aligned} r_t^i = \left(\mu_{\delta,t}^i - |\sigma_{\delta,t}^i|^2\right) + \sum_{j=1}^{N} \sum_{l=1}^{L^j} \frac{\left(\frac{\alpha_t^{j,l} \gamma^{j,l,i}}{y^{j,l} q_0^{j,i}}\right) Z_t^{j,l}}{\sum_{f=1}^{N} \sum_{g=1}^{L^f} \left(\frac{\alpha_t^{f,g} \gamma^{f,g,i}}{y^{f,g} q_0^{f,i}}\right) Z_t^{f,g}} \beta_t^{j,l} \\ + \sigma_{\delta,t}^i \cdot \sum_{j=1}^{N} \sum_{l=1}^{L^j} \frac{\left(\frac{\alpha_t^{j,l} \gamma^{j,l,i}}{y^{j,l} q_0^{j,i}}\right) Z_t^{j,l}}{\sum_{f=1}^{N} \sum_{g=1}^{L^f} \left(\frac{\alpha_t^{f,g} \gamma^{f,g,i}}{y^{f,g} q_0^{f,i}}\right) Z_t^{f,g}} \hat{\lambda}_t^{j,l}, \end{aligned} \tag{46}$$

$$\theta_t^i = \sigma_{\delta,t}^i - \sum_{j=1}^{N} \sum_{l=1}^{L^j} \frac{\left(\frac{\alpha_t^{j,l} \gamma^{j,l,i}}{y^{j,l} q_0^{j,i}}\right) Z_t^{j,l}}{\sum_{f=1}^{N} \sum_{g=1}^{L^f} \left(\frac{\alpha_t^{f,g} \gamma^{f,g,i}}{y^{f,g} q_0^{f,i}}\right) Z_t^{f,g}} \hat{\lambda}_t^{j,l}, \tag{47}$$

Proof. The relationship between Z^{θ^i}/B_t^i and Z^{θ^j}/B_t^j is expressed as:

$$q_t^{i,j} = q_0^{i,j} \exp\left[\int_0^t \left(r_s^i - r_s^j + \frac{1}{2}|\theta_s^i|^2 - \frac{1}{2}|\theta_s^j|^2\right) ds + \int_0^t (\theta_s^i - \theta_s^j) \cdot dW_s\right] \tag{48}$$

$$= q_0^{i,j} \frac{B_t^i}{Z_t^{\theta^i}} \frac{Z_t^{\theta^j}}{B_t^j}. \tag{49}$$

By substituting $c^{i,l,j,*}$ and (49) into the goods market clearing conditions (45), we have:

$$\frac{Z_t^{\theta^i}}{B_t^i} = \frac{1}{\delta_t^i}\sum_{j=1}^{N}\sum_{l=1}^{L^j}\frac{\gamma^{j,l,i}\alpha_t^{j,l}Z_t^{j,l}}{y^{j,l}q_0^{j,i}};\ dZ^{\theta^i} = -Z_t^{\theta^i}\theta_t^i\cdot dW_t;\ Z_0^{\theta^i} = 1, \tag{50}$$

$$dZ_t^{j,l} = Z_t^{j,l}\hat{\lambda}_t^{j,l}\cdot dW_t;\ Z_0^{j,l} = 1;\ \lambda_t^{j,l} = \hat{\lambda}_t^{j,l}\oplus\hat{\lambda}_t^{j,l,\perp},\ \hat{\lambda}_t^{j,l}\in range(\sigma_t^{j\top}). \tag{51}$$

By differentiating both sides of equation (50) and comparing the coefficients, we can derive the expressions for the interest rate and market price of risk.

$$\begin{aligned} r_t^i &= \left(\mu_{\delta,t}^i - |\sigma_{\delta,t}^i|^2\right) + \sum_{j=1}^{N}\sum_{l=1}^{L^j}\frac{\left(\frac{\alpha_t^{j,l}\gamma^{j,l,i}}{y^{j,l}q_0^{j,i}}\right)Z_t^{j,l}}{\sum_{f=1}^{N}\sum_{g=1}^{L^f}\left(\frac{\alpha_t^{f,g}\gamma^{f,g,i}}{y^{f,g}q_0^{f,i}}\right)Z_t^{f,g}}\beta_t^{j,l} \\ &+\sigma_{\delta,t}^i\cdot\sum_{j=1}^{N}\sum_{l=1}^{L^j}\frac{\left(\frac{\alpha_t^{j,l}\gamma^{j,l,i}}{y^{j,l}q_0^{j,i}}\right)Z_t^{j,l}}{\sum_{f=1}^{N}\sum_{g=1}^{L^f}\left(\frac{\alpha_t^{f,g}\gamma^{f,g,i}}{y^{f,g}q_0^{f,i}}\right)Z_t^{f,g}}\hat{\lambda}_t^{j,l}, \end{aligned} \tag{52}$$

$$\theta_t^i = \sigma_{\delta,t}^i - \sum_{j=1}^{N}\sum_{l=1}^{L^j}\frac{\left(\frac{\alpha_t^{j,l}\gamma^{j,l,i}}{y^{j,l}q_0^{j,i}}\right)Z_t^{j,l}}{\sum_{f=1}^{N}\sum_{g=1}^{L^f}\left(\frac{\alpha_t^{f,g}\gamma^{f,g,i}}{y^{f,g}q_0^{f,i}}\right)Z_t^{f,g}}\hat{\lambda}_t^{j,l}. \tag{53}$$

□

Moreover, we obtain the following expression for the volatility term of $q_t^{i,j}$:

$$\begin{aligned} \sigma_{q,t}^{i,j\top} &= \theta_t^i - \theta_t^j = \sigma_{\delta,t}^i - \sigma_{\delta,t}^j - \sum_{j=1}^{N}\sum_{l=1}^{L^j}\frac{\left(\frac{\alpha_t^{j,l}\gamma^{j,l,i}}{y^{j,l}q_0^{j,i}}\right)Z_t^{j,l}}{\sum_{f=1}^{N}\sum_{g=1}^{L^f}\left(\frac{\alpha_t^{f,g}\gamma^{f,g,i}}{y^{f,g}q_0^{f,i}}\right)Z_t^{f,g}}\hat{\lambda}_t^{j,l} \\ &+\sum_{i=1}^{N}\sum_{l=1}^{L^i}\frac{\left(\frac{\alpha_t^{i,l}\gamma^{i,l,j}}{y^{i,l}q_0^{i,j}}\right)Z_t^{i,l}}{\sum_{f=1}^{N}\sum_{g=1}^{L^f}\left(\frac{\alpha_t^{f,g}\gamma^{f,g,j}}{y^{f,g}q_0^{f,j}}\right)Z_t^{f,g}}\hat{\lambda}_t^{i,l}. \end{aligned} \tag{54}$$

2.4. Stock Valuation and Financial Market Equilibrium

This subsection focuses on the stock valuation and financial market clearing. The aggregate and individual stock prices S_t^i and $S_t^{i,k}$ are given by the following Theorem 4.

Theorem 4. *Equilibrium aggregate and individual stock prices are given by:*

$$S_t^i = \frac{\delta_t^i}{\sum_{j=1}^{N}\sum_{l=1}^{L^j}\frac{\gamma^{j,l,i}e^{-\beta^{j,l}t}Z_t^{j,l}x_0^{j,l}\beta^{j,l}}{q_0^{j,i}(1-e^{-\beta^{j,l}T})}}\sum_{j=1}^{N}\sum_{l=1}^{L^j}\frac{\gamma^{j,l,i}Z_t^{j,l}(e^{-\beta^{j,l}t}-e^{-\beta^{j,l}T})x_0^{j,l}}{q_0^{j,i}(1-e^{-\beta^{j,l}T})}, \tag{55}$$

$$S_t^{i,k} = \frac{\delta_t^i}{\sum_{j=1}^{N}\sum_{l=1}^{L^j}\frac{\gamma^{j,l,i}\alpha_t^{j,l}Z_t^{j,l}}{y^{j,l}q_0^{j,i}}}\mathbf{E}_t\left[\int_t^T\frac{\delta_s^{i,k}}{\delta_s^i}\sum_{j=1}^{N}\sum_{l=1}^{L^j}\frac{\gamma^{j,l,i}\alpha_s^{j,l}Z_s^{j,l}}{y^{j,l}q_0^{j,i}}ds\right]. \tag{56}$$

Proof. See Appendix D. □

Next, we confirm the market clearing in the financial market by Proposition 2.

Proposition 2. *Under the assumption that* $\left(\sigma_{S,t}^{j,k}+\sigma_{q,t}^{1,j}\right)$ *and* $\sigma_{q,t}^{1,j}$ *are independent volatility vectors, equilibrium stock price* $S_t^{i,k}$ *and optimal investment amounts* ($\pi_t^{j,l,(i,k),*}$ *and* $\pi_t^{j,l,N+i,*}$) *satisfy the following market clearing conditions in financial market based on* 1*-st country currency.*

$$\sum_{j=1}^{N}\sum_{l=1}^{L^j} q_t^{1,j}\pi_t^{j,l,(i,k),*}=q_t^{1,i}S_t^{i,k};\ \forall i=1,...,N;\ \forall k=1,...,K^i, \tag{57}$$

$$\sum_{j=1}^{N}\sum_{l=1}^{L^j} q_t^{1,j}\pi_t^{j,l,N+i,*}=0;\ \forall i=1,...,N. \tag{58}$$

Proof. See Appendix E. □

Here, we remark on the recursive structure in Theorem 4 that the stock prices $S_t^{i,k}$ are determined by $Z_t^{i,l}$ $\left((i,l)=(1,1),\dots,(N,L^N)\right)$, which includes $\hat{\lambda}_t^{i,l}$. Since $\hat{\lambda}_t^{i,l}$ is the projection of the exogenously given subjective view $\lambda_t^{i,l}$ onto the risky assets' volatility vector space *range* $\left(\sigma_t^{i\top}\right)$, $\hat{\lambda}_t^{i,l}$ also depends on the volatility of $S_t^{i,k}$. Thus, when we simulate the model, we need to properly find $\hat{\lambda}_t^{i,l}$ so that the recursive structure in Theorem 4 is satisfied.

This recursive structure arises from the incompleteness of the market, which is generated by the incorporation of different subjective views of agents on fundamental risks represented by Brownian motions or equivalently on the expected return of the stock prices $S_t^{i,k}$. Here, $\lambda_t^{i,l}$ is associated with some factors that cannot be hedged with the investable assets in the market, which lead to different state price density processes $H_t^{i,l}=Z_t^{\theta^i}/B_t^i\cdot\eta_t^{i,l}/Z_t^{i,l}$ in their individual optimization problems.

To specify $\hat{\lambda}_t^{i,l}$, we need to consider the equilibrium volatility vectors $\sigma_{S,t}^{i,k}$ provided in Proposition 3.

Proposition 3. *The volatility vector* $\sigma_{S,t}^{i,k}$ *is obtained as follows:*

$$\sigma_{S,t}^{i,k\top}=\sigma_{\delta,t}^{i,k}+$$
$$\sum_{j=1}^{N}\sum_{l=1}^{L^j}\left(\frac{\mathbf{E}_t\left[\int_t^T\frac{\delta_s^{i,k}}{\delta_s^i}\left(\frac{\gamma^{j,l,i}\alpha_s^{j,l}}{y^{j,l}q_0^{j,i}}\right)Z_s^{j,l}ds\right]}{\sum_{f=1}^{N}\sum_{g=1}^{L^f}\mathbf{E}_t\left[\int_t^T\frac{\delta_s^{i,k}}{\delta_s^i}\left(\frac{\gamma^{f,g,i}\alpha_s^{f,g}}{y^{f,g}q_0^{f,i}}\right)Z_s^{f,g}ds\right]}-\frac{\left(\frac{\gamma^{j,l,i}\alpha_t^{j,l}}{y^{j,l}q_0^{j,i}}\right)Z_t^{j,l}}{\sum_{f=1}^{N}\sum_{g=1}^{L^f}\left(\frac{\gamma^{f,g,i}\alpha_t^{f,g}}{y^{f,g}q_0^{f,i}}\right)Z_t^{f,g}}\right)\hat{\lambda}_t^{j,l}. \tag{59}$$

Proof. See Appendix F. □

Here, $\hat{\lambda}_t^{i,l}$ is the projections onto the space spanned by the stock volatility vectors $\sigma_{S,t}^{j,k}$ and exchange rate volatility vectors $\sigma_{q,t}^{i,j}$, which means that $\hat{\lambda}_t^{i,l}$ is a linear combination of these vectors:

$$\hat{\lambda}_t^{i,l}=\sum_{j=1}^{N}\sum_{k=1}^{K^j}a_t^{j,k}\sigma_{S,t}^{j,k\top}+\sum_{j\neq i}^{N}b_t^j\sigma_{q,t}^{i,j\top}. \tag{60}$$

On the other hand, from the Proposition 3, $\sigma_{S,t}^{j,k}$ can be expressed as linear combinations of $\sigma_{\delta,t}^{j,k}$ and $\hat{\lambda}_t^{j,l}$ as shown in equation (61). Similarly, from (54), $\sigma_{q,t}^{i,j}$ is expressed as a linear combination of $\sigma_{\delta,t}^{i}$,

$\sigma^j_{\delta,t}$ and $\hat{\lambda}^{j,l}_t$. Since $\sigma^i_{\delta,t}$ is also a linear combinations of the volatility vectors $\sigma^{i,k}_{\delta,t}$ $(k = 1, ..., K^i)$ from (5), $\sigma^{i,j}_{q,t}$ is expressed as (62).

$$\sigma^{j,k\top}_{S,t} = \sigma^{j,k}_{\delta,t} + \sum_{j=1}^{N}\sum_{l=1}^{L^j} c^{j,l}_t \hat{\lambda}^{j,l}_t, \tag{61}$$

$$\sigma^{i,j\top}_{q,t} = \sum_{k=1}^{K^i} d^{i,k}_t \sigma^{i,k}_{\delta,t} + \sum_{k=1}^{K^j} e^{j,k}_t \sigma^{j,k}_{\delta,t} + \sum_{g=1}^{N}\sum_{l=1}^{L^g} f^{g,l}_t \hat{\lambda}^{g,l}_t. \tag{62}$$

Therefore, by substituting (61) and (62) into (60) and rearranging terms, it follows that $\hat{\lambda}^{j,l}_t$ must be constructed as a linear combination of the output process volatility vectors $\sigma^{i,k}_{\delta,t}$.

Concretely, we discuss the simple two-country two-currency model ($N = 2$, $K^i = 1$, and $L^i = 1$ for $i = 1, 2$) in Section 3.1, where the country subscripts d (domestic) and f (foreign) are used instead of $i = 1, 2$, respectively. In this case, (60), (61) and (62) become:

$$\hat{\lambda}^d_t = A^{1,1}_t \sigma^{d\top}_{S,t} + A^{1,2}_t \sigma^{f\top}_{S,t} + B^1_t \sigma^{\top}_{q,t}, \tag{63}$$

$$\hat{\lambda}^f_t = A^{2,1}_t \sigma^{d\top}_{S,t} + A^{2,2}_t \sigma^{f\top}_{S,t} + B^2_t \sigma^{\top}_{q,t}, \tag{64}$$

$$\sigma^{d\top}_{S,t} = \sigma^d_{\delta,t} + C^{1,1}_t \hat{\lambda}^d_t + C^{1,2}_t \hat{\lambda}^f_t, \tag{65}$$

$$\sigma^{f\top}_{S,t} = \sigma^f_{\delta,t} + C^{2,1}_t \hat{\lambda}^d_t + C^{2,2}_t \hat{\lambda}^f_t, \tag{66}$$

$$\sigma^{\top}_{q,t} = \sigma^d_{\delta,t} - \sigma^f_{\delta,t} + D^1_t \hat{\lambda}^d_t + D^2_t \hat{\lambda}^f_t, \tag{67}$$

where $A^{i,j}_t$, B^i_t, $C^{i,j}_t$ and D^i_t $(i, j = 1, 2)$ are the coefficients. If this system of equations has a solution, $\hat{\lambda}^d_t$, $\hat{\lambda}^f_t$, $\sigma^{d\top}_{S,t}$, $\sigma^{f\top}_{S,t}$ and $\sigma^{\top}_{q,t}$ need to be represented as a linear combination of the $\sigma^d_{\delta,t}$ and $\sigma^f_{\delta,t}$. In particular, an example case for $\hat{\lambda}^d_t$ and $\hat{\lambda}^f_t$ is provided in equations (100) and (101). Note that, although Section 3.2 omits a detailed discussion of λ^d_t for simplicity, once $\hat{\lambda}^d_t$ is specified, the full subjective belief λ^d_t can be constructed by adding any vector $\hat{\lambda}^{d\perp}_t$ orthogonal to the span of $\sigma^d_{\delta,t}$ and $\sigma^f_{\delta,t}$: $\lambda^d_t = \hat{\lambda}^d_t \oplus \hat{\lambda}^{d\perp}_t$.

Based on the above discussion, once we specify $\hat{\lambda}^{j,l}_t$ that satisfies the recursive structure, we confirm that the optimal consumption, investment strategy, interest rates, market prices of risk, and stock prices determined by Theorems 1, 2, 3, and 4 ensure that both the goods and financial markets are in equilibrium. This establishes the internal consistency and validity of the equilibrium framework developed in this paper.

Finally, while the discussion so far has been based on real values under the neutrality of money, the next subsection briefly confirms that these arguments can be easily extended to the nominal case by introducing country specific price level processes.

2.5. Nominal Case

This subsection concisely discusses the equilibrium asset pricing in nominal situation. Let p^i_t denote the (exogenous) price level in country i, evolving as:

$$dp^i_t = p^i_t \left\{ \mu^i_p(Y_t, t)dt + \sigma^i_p(Y_t, t) \cdot dW_t \right\}; p^i_0 > 0. \tag{68}$$

Then, the nominal exchange rate between countries i and j (one unit of currency j equals $q_t^{n,i,j}$ units of currency i) can be expressed as:

$$q_t^{n,i,j} = q_t^{i,j} \frac{p_t^i}{p_t^j}. \tag{69}$$

Thus, its dynamics follow:

$$\frac{dq_t^{n,i,j}}{q_t^{n,i,j}} = \left\{ r_t^i - r_t^j + (\theta_t^i - \theta_t^j) \cdot \theta_t^i + \mu_{p,t}^i - \mu_{p,t}^j + |\sigma_{p,t}^j|^2 - \sigma_{p,t}^i \cdot \sigma_{p,t}^j + (\theta_t^i - \theta_t^j) \cdot (\sigma_{p,t}^i - \sigma_{p,t}^j) \right\} dt$$
$$+(\theta_t^i - \theta_t^j + \sigma_{p,t}^i - \sigma_{p,t}^j) \cdot dW_t. \tag{70}$$

Also, nominal interest rate and market price of risk are obtained by differentiating the nominal state price density (as seen in the proof of Theorem 3). The nominal state price density satisfies:

$$\frac{Z_t^{\theta^{n,i}}}{B_t^{n,i}} = \frac{p_0^i}{p_t^i} \frac{Z_t^{\theta^i}}{B_t^i}, \tag{71}$$

$$Z_t^{\theta^{n,i}} = \exp\left(- \int_0^t \theta_s^{n,i} \cdot dW_s - \frac{1}{2} \int_0^t |\theta_s^{n,i}|^2 ds \right), \tag{72}$$

$$B_t^{n,i} = \exp\left(\int_0^t r_s^{n,i} ds \right), \tag{73}$$

where $Z_t^{\theta^{n,i}}$, $B_t^{n,i}$, $r_t^{n,i}$ and $\theta_t^{n,i}$ are the nominal counterparts of $Z_t^{\theta^{i,l}}$, B_t^i, r_t^i and θ_t^i, respectively. Here, following equation holds:

$$d\left(\frac{Z_t^{\theta^i}}{B_t^i} \right) = \left(\frac{Z_t^{\theta^i}}{B_t^i} \right) (-r_t^i dt - \theta_t^i \cdot dW_t), \tag{74}$$

with r_t^i and θ_t^i given in Theorem 3. Hence:

$$r_t^{n,i} = r_t^i + \mu_{p,t}^i - |\sigma_{p,t}^i|^2 - \theta_t^i \cdot \sigma_{p,t}^i, \tag{75}$$

$$\theta_t^{n,i} = \theta_t^i + \sigma_{p,t}^i, \tag{76}$$

Finally, the nominal stock price $S_t^{n,i,k}$ can be written by multiplying the price level and the real stock price, and is therefore expressed as follows:

$$S_t^{n,i,k} = p_t^i S_t^{i,k}. \tag{77}$$

In addition, from (D.7) and (71), $S_t^{n,i,k}$ is as follows:

$$S_t^{n,i,k} = p_t^i \frac{B_t^i}{Z_t^{\theta^i}} \mathbf{E}_t \left[\int_t^T \delta_s^{i,k} \frac{Z_s^{\theta^i}}{B_s^i} ds \right] \tag{78}$$

$$= \frac{B_t^{n,i}}{Z_t^{\theta^{n,i}}} \mathbf{E}_t \left[\int_t^T \left(p_s^i \delta_s^{i,k} \right) \frac{Z_s^{\theta^{n,i}}}{B_s^{n,i}} ds \right]. \tag{79}$$

That is, the nominal stock price equals the present value of the nominal dividend (output) stream.

3. Empirical Study

This section demonstrates that the model introduced in Section 2 can be calibrated to actual economic variables in the two-currency setting of Japan and the U.S. In particular, after specifying the two-currency model, a special case of the multi-currency model presented in Section 2, in Section 3.1, we describe the state-space model used to estimate transitions in the latent economic factors and subjective beliefs of the two countries in Section 3.2. Moreover, in Section 3.3, we show that the transitions of the estimated state variables are consistent with changes in market environments and are well explained by key economic events that occurred during the period, which may be useful for making investment decisions in international asset management practices. Also, in the calibration, nominal data are deflated by the price indices and we examine the real-terms version of our model.

3.1. Two-Currency Model

This subsection describes the two-currency model, which is a special case of the multi-currency model presented in Section 2, involving two countries, each with one agent and one stock index. The model will serve as the basis for the state-space model introduced in the following subsection, which is used to estimate latent economic factors and subjective beliefs. The two-currency model consists of a domestic country (denoted by d) and a foreign country (denoted by f). Each country is assumed to have a single agent, one stock, one money market account and its own currency. Also, this subsection focuses on country d; the foreign case is analogous by replacing d with f.

Exogenously given output processes for the domestic and foreign countries, δ_t^d and δ_t^f, driven by a factor process Y_t are described by the following stochastic differential equations:

$$d\delta_t^d = \delta_t^d \left[\mu_{\delta,t}^d(Y,t)dt + \sigma_{\delta,t}^d(Y,t)\cdot dW_t\right], \tag{80}$$

$$d\delta_t^f = \delta_t^f \left[\mu_{\delta,t}^f(Y,t)dt + \sigma_{\delta,t}^f(Y,t)\cdot dW_t\right], \tag{81}$$

$$dY_t = \mu_y(Y,t)dt + \sigma_y(Y,t)\cdot dW_t. \tag{82}$$

Here, we assume that m (dimension of Brownian motion) > 3, which means that the market is incomplete.

The foreign exchange rates between the two countries, denoted by q_t (domestic/foreign) and $1/q_t$ (foreign/domestic), evolve according to the following SDEs:

$$dq_t = q_t\left[\mu_t^q dt + \sigma_t^q dW_t\right] = q_t\left[(r_t^d - r_t^f)dt + \sigma_t^q(dW_t + \theta_t^d dt)\right], \tag{83}$$

$$d(1/q_t) = (1/q_t)\left[(-\mu_t^q + |\sigma_t^q|^2)dt - \sigma_t^q dW_t\right] = (1/q_t)\left[(r_t^f - r_t^d)dt - \sigma_t^q(dW + \theta_t^f dt)\right]. \tag{84}$$

The r_t^i $(i = d, f)$ represents the risk-free interest rate in country i.

We assume an agent in country d invests in three risky assets (S_t^d, $q_tS_t^f$, $q_tB_t^f$) and a domestic money market account (B_t^d), which follow these SDEs:

$$dS_t^d = \left(r_t^d S_t - \delta_t^d\right) dt + S_t^d \sigma_{S,t}^d (dW_t + \theta_t^d dt);\ S_T^d = 0, \tag{85}$$

$$d(q_t S_t^f) = \left(r_t^d q_t S_t^f - q_t \delta_t^f\right) dt + q_t S_t^f \left(\sigma_{S,t}^f + \sigma_t^q\right) (dW_t + \theta_t^d dt);\ S_T^f = 0, \tag{86}$$

$$dB_t^d = r_t^d B_t^d dt, \tag{87}$$

$$d(q_t B_t^f) = r_t^d q_t B_t^f dt + q_t B_t^f \sigma_t^q (dW_t + \theta_t^d dt). \tag{88}$$

Also, as seen in Section 2.2, the agent's optimization problem involving only consumption is as follows:

$$\max_{c_t^{d,d}, c_t^{d,f}} \mathbf{E}\left[\int_0^T u_t^d(c_t^{d,d}, c_t^{d,f}) dt\right] \quad s.t.\ \mathbf{E}\left[\int_0^T H_t^d(c_t^{d,d} + q_t c_t^{d,f}) dt\right] \le x_0^d,\ \forall H_t^d, \tag{89}$$

where H_t^d is the state price density process for currency d and the utility function $u_t^d(c_t^{d,d}, c_t^{d,f})$ is given by:

$$u_t^d(c_t^{d,d}, c_t^{d,f}) = \eta_t^d \alpha_t^d \left[\gamma^d \log c_t^{d,d} + (1-\gamma^d) \log c_t^{d,f}\right];\ \gamma^d \in [0,1], \tag{90}$$

$$\alpha_t^d = e^{-\beta^d t}, \tag{91}$$

$$\eta_t^d = \exp\left(\int_0^t \lambda_s^d \cdot dW_s - \frac{1}{2}\int_0^t |\lambda_s^d|^2 ds\right);\ \lambda_s^d = \lambda^d(Y_s, s). \tag{92}$$

Here η_t^d is the subjective belief, β^d is the time preference, γ^d is the preference of the goods between domestic and foreign countries, $\lambda^d(Y_s, s)$ is the subjective belief process.

Then, we obtain the following equilibrium variables needed for the empirical analysis.

$$r_t^d = (\mu_{\delta,t}^d - |\sigma_{\delta,t}^d|^2) + \sum_{j=d,f} \frac{\left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j}\right) Z_t^j}{\sum_{j=d,f}\left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j}\right) Z_t^j} \beta_t^j + \sigma_{\delta,t}^d \cdot \sum_{j=d,f} \frac{\left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j}\right) Z_t^j}{\sum_{j=d,f}\left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j}\right) Z_t^j} \hat{\lambda}_{j,t}, \tag{93}$$

$$\theta_t^d = \sigma_{\delta,t}^d - \sum_{j=d,f} \frac{\left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j}\right) Z_t^j}{\sum_{j=d,f}\left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j}\right) Z_t^j} \hat{\lambda}_{j,t}, \tag{94}$$

$$S_t^d = \delta_t^d \frac{\frac{(1-e^{-\beta^d(T-t)})}{\beta^d} K_t^{d,d} Z_t^d + \frac{(1-e^{-\beta^f(T-t)})}{\beta^f} K_t^{d,f} Z_t^f}{K_t^{d,d} Z_t^d + K_t^{d,f} Z_t^f}. \tag{95}$$

where $q_j^d = 1$ ($q_j^d = q_0$) when $j = d$ ($j = f$) , and $K_t^{d,d}$ and $K_t^{d,f}$ are defined as follows:

$$K_t^{d,d} = \frac{x_0^d \gamma^d \beta^d e^{-\beta^d t}}{(1 - e^{-\beta^d T})}, \tag{96}$$

$$K_t^{d,f} = \frac{q_0 x_0^f (1-\gamma^f) \beta^f e^{-\beta^f t}}{(1 - e^{-\beta^f T})}. \tag{97}$$

Since these derivations are analogous to those in Section 2, we do not repeat them; see Section Appendix G for details.

3.2. State-Space Model for Particle Filtering

This subsection presents a state-space model for particle filtering, used to estimate latent economic factors and home biases, based on the two-currency model in Section 3.1, applied to the specific case of Japan (domestic, d) and the U.S. (foreign, f).

First, we present the following state equations:

$$Y_{l,t+1} = Y_{l,t} - \mu_{y,l} Y_{l,t} \Delta t + \sigma_{y,l} Y_{l,t} \Delta W_{l,t};\ Y_{l,0} = 0,\ l = 1, \ldots, 4, \tag{98}$$

$$\delta^i_{t+1} = \delta^i_t + \delta^i_t \left[\mu^i_\delta \left(\sum_{l=1}^{4} a^i_l Y_{l,t} \right) \Delta t + \sigma^i_\delta \cdot \Delta W_t \right],\ i = d, f \tag{99}$$

$$\hat{\lambda}^d_{t+1} = a^d(Y_{1,t}, Y_{2,t}, Y_{3,t}, Y_{4,t}) \sigma^d_\delta + b^d(Y_{1,t}, Y_{2,t}, Y_{3,t}, Y_{4,t}) \sigma^f_\delta, \tag{100}$$

$$\hat{\lambda}^f_{t+1} = a^f(Y_{1,t}, Y_{2,t}, Y_{3,t}, Y_{4,t}) \sigma^d_\delta + b^f(Y_{1,t}, Y_{2,t}, Y_{3,t}, Y_{4,t}) \sigma^f_\delta, \tag{101}$$

$$Z^i_{t+1} = Z^i_t + Z^i_t \hat{\lambda}^i_t \cdot \Delta W_t;\ Z^i_0 = 1,\ i = d, f \tag{102}$$

where $\sigma^d_\delta = \sigma^1 \left(\rho^d, \bar{\rho}^d, \hat{\rho}^d, \sqrt{1 - (\rho^d)^2 - (\bar{\rho}^d)^2 - (\hat{\rho}^d)^2} \right)^\top$, $\sigma^f_\delta = \sigma^2 \left(\rho^f, \bar{\rho}^f, \hat{\rho}^f, \sqrt{1 - (\rho^f)^2 - (\bar{\rho}^f)^2 - (\hat{\rho}^f)^2} \right)^\top$, $a^d(Y_{1,t}, Y_{2,t}, Y_{3,t}, Y_{4,t}) = \max\{Y_{1,t+1}, 0\} - \min\{Y_{2,t+1}, 0\} + \max\{Y_{3,t+1}, 0\}$, $b^d(Y_{1,t}, Y_{2,t}, Y_{3,t}, Y_{4,t}) = \min\{Y_{1,t+1}, 0\} + \min\{Y_{2,t+1}, 0\} + \min\{Y_{4,t+1}, 0\}$, $a^f(Y_{1,t}, Y_{2,t}, Y_{3,t}, Y_{4,t}) = \min\{Y_{1,t+1}, 0\} - \max\{Y_{2,t+1}, 0\} + \min\{Y_{3,t+1}, 0\}$, $b^f(Y_{1,t}, Y_{2,t}, Y_{3,t}, Y_{4,t}) = \max\{Y_{1,t+1}, 0\} + \max\{Y_{2,t+1}, 0\} + \max\{Y_{4,t+1}, 0\}$ and δ^d_0 and δ^f_0 are chosen to satisfy the following equations:

$$\delta^d_0 = \left[x^d_0 \frac{\beta^d \gamma^d}{(1 - e^{-\beta^d T})} + x^f_0 \frac{\beta^f (1 - \gamma^f) q_0}{(1 - e^{-\beta^f T})} \right] B^d_0, \tag{103}$$

$$\delta^f_0 = \left[x^d_0 \frac{\beta^d (1 - \gamma^d)}{(1 - e^{-\beta^d T}) q_0} + x^f_0 \frac{\beta^f \gamma^f}{(1 - e^{-\beta^f T})} \right] B^f_0. \tag{104}$$

For more detailed information about the necessity of these initial conditions for δ^d_0 and δ^f_0, see the proof of Corollary 2 in Appendix G.

Since the calibration is conducted at a monthly frequency, we set $\Delta t = 1/12$ and $\Delta W_{i,t} \sim N(0, \Delta t)$ for $i = 1, ..., 4$, assumed to be independent. Although λ^i_t $(i = d, f)$ can be introduced as a state variable, the equilibrium is characterized by its projection $\hat{\lambda}^i_t$ $(i = d, f)$, thus we adopt $\hat{\lambda}^i_t$ as the state variable. Also, as discussed toward the end of Section 2.4, $\hat{\lambda}^d_t$ and $\hat{\lambda}^f_t$ can be represented as linear combinations of σ^d_δ and σ^f_δ for the following reasons.

- Since $\hat{\lambda}^d_t$ and $\hat{\lambda}^f_t$ are the projections onto the space spanned by σ^d_S, σ^f_S, and σ_q, they can be written as linear combinations of those three vectors.

- Conversely, since σ^d_S, σ^f_S, and σ_q themselves can be expressed as linear combinations of σ^d_δ, σ^f_δ, $\hat{\lambda}^d_t$, and $\hat{\lambda}^f_t$. For the explicit forms, see equations (G.63), (G.64), (G.46) in Appendix G.

Thus, $\hat{\lambda}^d_t$ and $\hat{\lambda}^f_t$ can be represented as linear combinations of σ^d_δ and σ^f_δ unless exceptional conditions arise.

We assume the following latent economic factors: Y_1, Y_2, Y_3, Y_4.

Factor	Interpretation
$Y_{1,t}$	Macroeconomic factor influencing both Japan and U.S.; proxy for global business-cycle conditions.
$Y_{2,t}$	Common commodity price factor; detrimental to resource-poor Japan, beneficial to resource-rich U.S.
$Y_{3,t}$	Country-specific factor for Japan.
$Y_{4,t}$	Country-specific factor for the U.S.

Table 1: Interpretation of latent factors in the model

Each $Y_{i,t}$ is modeled as a Vasicek-type stochastic differential equation that mean-reverts to zero.

Also, δ_t^d, which represents the domestic output, is influenced by the factors $Y_{1,t}$, $Y_{2,t}$, $Y_{3,t}$, and $Y_{4,t}$. The drift terms are specified to be influenced by all of $Y_{1,t}$, $Y_{2,t}$, $Y_{3,t}$, and $Y_{4,t}$. In subsequent parameter estimation, if any of these factors are irrelevant, their corresponding coefficients are expected to become zero. The volatility terms are also modeled to be influenced by all elements of $\Delta W_t = (\Delta W_{1,t}, \Delta W_{2,t}, \Delta W_{3,t}, \Delta W_{4,t})^\top$. The same modeling approach is applied to δ_t^f.

The state variables $\hat{\lambda}_t^d$ and $\hat{\lambda}_t^f$ are specified as linear combinations of σ_δ^d and σ_δ^f, and are interpreted as representing the home-country bias in each agent's subjective view. Home-country bias refers to the empirical tendency for agents to be more optimistic about their own country and less optimistic about foreign countries. For example, under the domestic agent's subjective measure η_t^d, the drifts of δ_t^d and δ_t^f are shifted to $\mu_\delta^d + \sigma_\delta^d \cdot \hat{\lambda}_t^d$ and $\mu_\delta^f + \sigma_\delta^f \cdot \hat{\lambda}_t^d$, respectively. To capture home-country bias, it is expected that $\sigma_\delta^d \cdot \hat{\lambda}_t^d$ is positive (reflecting optimism toward domestic output), while $\sigma_\delta^f \cdot \hat{\lambda}_t^d$ is negative (reflecting pessimism toward foreign output). Therefore, by specifying the functional form so that the coefficient of σ_δ^d is positive and that of σ_δ^f is negative as in (100), and by appropriately setting the parameters described later, the model tends to capture the home-country bias.

Next, we introduce the following observation equations:

$$
\begin{aligned}
S_t^d &= \frac{\frac{1-\exp\left(-\beta^d(T-t)\right)}{\beta^d} K_t^{d,d} Z_t^d + \frac{1-\exp\left(-\beta^f(T-t)\right)}{\beta^f} K_t^{d,f} Z_t^f}{K_t^{d,d} Z_t^d + K_t^{d,f} Z_t^f} \delta_t^d + \epsilon_{1,t}, && (105) \\
S_t^f &= \frac{\frac{1-\exp\left(-\beta^d(T-t)\right)}{\beta^d} K_t^{f,d} Z_t^d + \frac{1-\exp\left(-\beta^f(T-t)\right)}{\beta^f} K_t^{f,f} Z_t^f}{K_t^{f,d} Z_t^d + K_t^{f,f} Z_t^f} \delta_t^f + \epsilon_{2,t}, && (106) \\
r_t^d &= \mu_\delta^d - |\sigma_\delta^d|^2 \\
&+ \sum_{j=d,f} \frac{\left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j}\right) Z_t^j}{\sum_{j=d,f} \left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j}\right) Z_t^j} \beta_t^j + \sigma_\delta^d \cdot \sum_{j=d,f} \frac{\left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j}\right) Z_t^j}{\sum_{j=d,f} \left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j}\right) Z_t^j} \hat{\lambda}_{j,t} + \epsilon_{3,t}, && (107) \\
r_t^f &= \mu_\delta^f - |\sigma_\delta^f|^2 \\
&+ \sum_{j=d,f} \frac{\left(\frac{\alpha_t^j \gamma_j^f q_j^f}{y^j}\right) Z_t^j}{\sum_{j=d,f} \left(\frac{\alpha_t^j \gamma_j^f q_j^f}{y^j}\right) Z_t^j} \beta_t^j + \sigma_\delta^f \cdot \sum_{j=d,f} \frac{\left(\frac{\alpha_t^j \gamma_j^f q_j^f}{y^j}\right) Z_t^j}{\sum_{j=d,f} \left(\frac{\alpha_t^j \gamma_j^f q_j^f}{y^j}\right) Z_t^j} \hat{\lambda}_{j,t} + \epsilon_{4,t}, && (108) \\
q_{t+1} &= q_t + q_t\{(r_t^d - r_t^f)dt + (\theta_t^d - \theta_t^f) \cdot (dW_t + \theta_t^d dt)\} + \epsilon_{5,t}, && (109)
\end{aligned}
$$

where $\epsilon_{1,t}, \epsilon_{2,t}, \epsilon_{3,t}, \epsilon_{4,t}$, and $\epsilon_{5,t}$ are independent Gaussian noise terms with mean zero and variances $\sigma_{S,1}^2, \sigma_{S,2}^2, \sigma_{r,1}^2, \sigma_{r,2}^2, \sigma_q^2$, respectively, and $K_t^{d,d}$, $K_t^{d,f}$, $K_t^{f,d}$, $K_t^{f,f}$, θ_t^d, and θ_t^f are calculated as in Section 3.1.

In this study, we calibrate the model in real-term settings. Then, S_t^d is proxied by the inflation-adjusted TOPIX[1] futures price series, while S_t^f is proxied by the inflation-adjusted S&P 500 futures price series. Also, following Fisher equation[2], r_t^d and r_t^f are defined as the nominal short-term interest rates of Japan and the U.S., respectively, minus the corresponding inflation rates, while q_t denotes the price-deflated USD/JPY exchange rate. In summary, the variables are defined as follows.

- $S_t^d = \frac{TOPIX_t}{CPI_t^d}$: TOPIX futures prices/Japan price index
- $S_t^f = \frac{S\&P500_t}{CPI_t^f}$: S&P 500 futures prices/U.S. price index
- $r_t^d =$ JP short rate $- \Delta CPI_t^d$: Japan nominal short rate - monthly inflation rate
- $r_t^f =$ US short rate $- \Delta CPI_t^f$: U.S. nominal short rates - monthly inflation rate
- $q_t = USDJPY_t * \frac{CPI_t^f}{CPI_t^d}$: USD/JPY exchange rates * U.S. price index/Japan price index

[1] The Tokyo Stock Price Index, commonly known as TOPIX, is one of the most widely used stock market indices in Japan. It is a broad-based, capitalization-weighted benchmark that tracks all listed companies on the Prime Market of the Tokyo Stock Exchange.

[2] The Fisher equation states that the nominal interest rate is approximately equal to the real interest rate plus the expected inflation rate.

Here, the stock-price, interest-rate, and exchange-rate data are retrieved from Bloomberg, with tickers TP1 Index (TOPIX futures), SP1 Index (S&P 500 futures), MUTSCALM Index (Japan nominal short rate), FEDL01 and SOFRRATE Index[3] (U.S. nominal short rate), and USDJPY Curncy, respectively. Regarding price index data, we obtain the Japanese series from the Statistics Bureau of Japan[4] and the U.S. series from the Federal Reserve Bank of St. Louis[5].

The data spans from January 2000 to December 2021. Also, to remove the effect of scale heterogeneity across the series in the particle-filter estimation, the raw data were rescaled so that, over the entire sample period, each series has a mean of 1 and a standard deviation of 1. All estimations were then performed on these standardized series.

The model thus contains the following approximately 40 parameters: $\mu_{y,1}$, $\sigma_{y,1}$, $\mu_{y,2}$, $\sigma_{y,2}$, $\mu_{y,3}$, $\sigma_{y,3}$, $\mu_{y,4}$, $\sigma_{y,4}$, μ_δ^d, μ_δ^f, σ^1, ρ^d, $\bar{\rho}^d$, $\hat{\rho}^d$, σ^2, ρ^f, $\bar{\rho}^f$, $\hat{\rho}^f$, σ_δ^d, σ_δ^f, a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, $\sigma_{S,1}^2$, $\sigma_{S,2}^2$, $\sigma_{r,1}^2$, $\sigma_{r,2}^2$, σ_q^2, β^d, β^f, x_0^d, x_0^f, $\gamma^{d,d}$, $\gamma^{d,f}$, $\gamma^{f,d}$, and $\gamma^{f,f}$. Given the high dimensionality of the parameter space, a full grid search or exhaustive optimization is computationally infeasible. Therefore, several parameters are fixed exogenously based on empirical considerations.

First, we set $\sigma^1 = \sigma^2 = 0.2$, reflecting the fact that the process δ is intended to capture equity market dynamics, which typically exhibit annualized volatility around 20%. Second, for the volatility loading vectors, we set $\rho^d = 0.6$, $\bar{\rho}^d = -0.6$, and $\hat{\rho}^d = 0.5$, resulting in $\sigma_\delta^d = 0.2(0.6, -0.6, 0.5, 0)^\top$, so that the weights on the common factors Y_1, Y_2, and the domestic-specific factor Y_3 are approximately balanced. Similarly, we set $\rho^f = 0.6$, $\bar{\rho}^f = 0.6$, and $\hat{\rho}^f = 0$, yielding $\sigma_\delta^f = 0.2(0.6, 0.6, 0, 0.5)^\top$.

Third, we set the time preference parameters to $\beta^d = 0.05$ and $\beta^f = 0.1$, consistent with the empirical observation that Japanese investors tend to have longer investment horizons, while U.S. investors are generally more short-term oriented. Fourth, we set the preference parameters as $\gamma^{d,d} = 0.7$, $\gamma^{d,f} = 0.3$, $\gamma^{f,d} = 0.5$, and $\gamma^{f,f} = 0.5$, indicating that the domestic agent (interpreted as a Japanese investor) exhibits a stronger preference for domestic goods relative to foreign goods. Additionally, we set $x_0^d = x_0^f = 100$, and the observation noise variances as $\sigma_{S,1}^2 = \sigma_{S,2}^2 = 0.01$, $\sigma_{r,1}^2 = \sigma_{r,2}^2 = 0.02$, and $\sigma_q^2 = 0.01$.

Based on the above settings, we perform a grid search over the remaining parameters based on the loglikelihood loss function. Using the parameter set that maximizes the likelihood, we further adjust several parameters manually to enhance interpretability, while ensuring that the likelihood is not significantly compromised. The final parameter values are chosen to balance empirical fit and economic plausibility: $\mu_{y,1} = 1, \sigma_{y,1} = 0.1, \mu_{y,2} = 0.5, \sigma_{y,2} = 0.1, \mu_{y,3} = 0.1, \sigma_{y,3} = 0.1, \mu_{y,4} = 0.1, \sigma_{y,4} = 0.5, \mu_\delta^d = 0.1, \mu_\delta^f = 0.1, a_1^d = 0.5, a_2^d = 1, a_3^d = 0.5, a_4^d = -0.5, a_1^f = -1, a_2^f = -0.5, a_3^f = 1,$

[3] Since the SOFRRATE Index is available only from April 2018, we proxy the earlier period with the FRDL01 Index.

[4] https://www.e-stat.go.jp/stat-search/files?page=1&layout=datalist&toukei=00200573&tstat=000001150147&cycle=1&year=20250&month=12040605&tclass1=000001150149 (in Japanese)

[5] https://fred.stlouisfed.org/series/CPIAUCSL

and $a_4^f = 0.5$. The empirical results presented in Section 3.3 are based on this parameter set.

Parameter	**Description**	**Values**
Drift and Volatility of Y Process		
$\mu_{y,1}$, $\sigma_{y,1}$	Drift and volatility of $Y_{1,t}$ (global macro factor)	1, 0.1
$\mu_{y,2}$, $\sigma_{y,2}$	Drift and volatility of $Y_{2,t}$ (commodity factor)	0.5, 0.1
$\mu_{y,3}$, $\sigma_{y,3}$	Drift and volatility of $Y_{3,t}$ (Japan-specific)	0.1, 0.1
$\mu_{y,4}$, $\sigma_{y,4}$	Drift and volatility of $Y_{4,t}$ (U.S.-specific)	0.1, 0.5
Drift and Volatility of δ Process		
μ_δ^d, μ_δ^f	Components of drift for δ_t^d and δ_t^f	0.1, 0.1
a_1^d, a_2^d, a_3^d, a_4^d	Components of drift for δ_t^d	0.5, 1, 0.5, -0.5
a_1^f, a_2^f, a_3^f, a_4^f	Components of drift for δ_t^f	-1, -0.5, 1, 0.5
σ_δ^d	Volatility vector for δ_t^d	$0.2(0.6, -0.6, 0.5, 0)^\top$
σ_δ^f	Volatility vector for δ_t^f	$0.2(0.6, 0.6, 0, 0.5)^\top$
Preference and Initial Condition		
β^d, β^f	Time preference parameters	0.05, 0.1
$\gamma^{d,d}$, $\gamma^{d,f}$	Domestic agent's preference for goods	0.7, 0.3
$\gamma^{f,d}$, $\gamma^{f,f}$	Foreign agent's preference for goods	0.5, 0.5
x_0^d, x_0^f	Initial wealth levels	100, 100
Observation Noise Variance		
$\sigma_{S,1}^2$, $\sigma_{S,2}^2$	Noise in stock prices	0.01, 0.01
$\sigma_{r,1}^2$, $\sigma_{r,2}^2$	Noise in interest rates	0.02, 0.02
σ_q^2	Noise in exchange rate	0.01

Table 2: List of model parameters, their roles, and assigned values

3.3. Estimation Results for State-space Model

This subsection examines the estimation results of the state-space model, including the latent economic factors and the home biases of Japan and the U.S. In particular, we demonstrate that the transitions of the estimated factor processes and the biases are consistent and can be explained by the changes in the economic environments of the two countries.

First, the estimation results for the unobservable state variable Y_i $(i = 1, ..., 4)$ are shown in Figures 1.

Figure 1: Estimated state variables Y_i $(i = 1, 2, 3, 4)$.

Regarding $Y_{1,t}$, which we assume to be the global business-cycle factor, the following dynamics are observed over the sample period:

Period	**Transition of $Y_{1,t}$ and Key Events**
Early 2000s	Declined due to the collapse of the global IT bubble.
2003-2007	Increased steadily, supported by BRICS expansion and the U.S. housing boom; entered positive territory.
2008	Dropped sharply due to the Lehman shock and global financial crisis.
2010-2015	Recovered gradually, driven by quantitative easing; exceeded pre-crisis levels by the mid-2010s.
2015-2016	Suppressed by the China shock, oil-price collapse, and U.S.–China trade tensions; remained moderately positive.
2020-2021	Temporarily declined during the Covid-19 pandemic; remained above zero due to global monetary easing.

Table 3: Transition of $Y_{1,t}$ in response to global macroeconomic events

Turning to $Y_{2,t}$—interpreted as the commodity price factor—its trajectory over the sample horizon can be summarized as follows:

Period	**Transition of $Y_{2,t}$ and Key Events**
2000-2010	Fluctuated around zero.
2012-2013	Declined significantly due to the U.S. shale gas revolution and global energy oversupply.
2014-	Returned to near zero and remained relatively stable.

Table 4: Transition of $Y_{2,t}$ in response to global commodity market dynamics

As for $Y_{3,t}$ interpreted as the Japan specific factor, its evolution can be characterized as follows:

Period	Transition of $Y_{3,t}$ and Key Events
Early 2000s	Declined due to Japan's economic stagnation following the collapse of the IT bubble.
2003-2011	Experienced a cyclical upswing under Koizumi's financial reforms;
	remained negative due to the 2008 global financial crisis and the 2011 Tōhoku earthquake.
2013-	Rebounded under Abenomics;
	aggressive monetary easing and yen depreciation enhanced corporate earnings and $Y_{3,t}$ recoverd to near zero.

Table 5: Transition of $Y_{3,t}$ in response to Japan-specific economic events

Regarding $Y_{4,t}$, which we assume to be the U.S. specific factor, the following dynamics are observed over the sample period:

Period	Transition of $Y_{4,t}$ and Key Events
Early 2000s	Declined following the collapse of the IT bubble.
2003-2007	Increased due to the housing boom and strong consumer spending; entered positive territory.
2008	Dropped sharply during the Lehman shock.
2009-2015	Expanded significantly with the Federal Reserve's quantitative easing,
	the shale gas boom, and growth in the technology sector.
2015-	Remained elevated despite the China shock, oil price decline, trade tensions, and the Covid-19 pandemic.

Table 6: Transition of $Y_{4,t}$ in response to U.S.-specific events

Second, the estimation results for state variables δ_t^d and δ_t^f are shown in Figure 2.

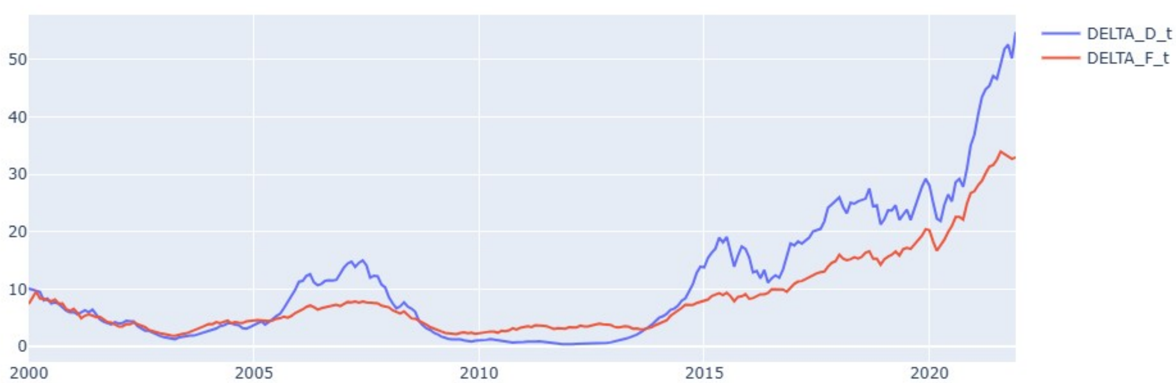

Figure 2: Estimated δ_t^d and δ_t^f.

- Both δ_t^d and δ_t^f display similar paths to the corresponding equity indices, respectively.

Since δ_t^d and δ_t^f are the most important variables for S_t^d and S_t^f as seen in (105) and (106), it is natural that they follow a similar path as the equity indices.

Third, the estimated results for $\hat{\lambda}_t^d$ are shown in Figure 3, where a_t^d denotes the coefficient of σ_δ^d, $(\max\{Y_{1,t+1},0\} - \min\{Y_{2,t+1},0\} + \max\{Y_{3,t+1},0\})$, and b_t^d denotes the coefficient of σ_δ^f, $(\min\{Y_{1,t+1},0\} + \min\{Y_{2,t+1},0\} + \min\{Y_{4,t+1},0\})$. As discussed in Subsection 3.2, $\hat{\lambda}_t^d$ is specified as a linear combination of the volatility vectors: $\hat{\lambda}_t^d = a_t^d \sigma_\delta^d + b_t^d \sigma_\delta^f$. In this formulation, the degree of positive (negative) bias toward domestic (foreign) output is captured by the inner product $\sigma_\delta^d \cdot \hat{\lambda}_t^d = a_t^d |\sigma_\delta^d|^2 \left(\sigma_\delta^f \cdot \hat{\lambda}_t^d = b_t^d |\sigma_\delta^f|^2 \right)$, where the cross term vanishes due to $\sigma_\delta^d \cdot \sigma_\delta^f = 0$. Thus, the coefficients a_t^d and b_t^d quantify the strength of the positive home bias and the negative foreign bias, respectively.

Figure 3: Estimated a_t^d and b_t^d.

The following biases are estimated in Japan.

Period	**Transition of Home Bias in Japan**
Early 2000s	No positive bias observed before/after the IT bubble collapse.
2003-2007	Gradual increase in positive home bias under Koizumi's financial reforms.
2008	Positive bias erased by Lehman shock; negative bias toward U.S. intensified.
2013-2015	Abenomics improved domestic conditions; positive bias rose again.
2016-2019	Positive bias peaked and then retreated due to limited impact of fiscal and structural reforms.
2020-	Covid-19 and yen depreciation boosted corporate earnings; positive bias increased again.

Table 7: Transition of Japan's estimated home bias parameter

Similarly, the estimated results for $\hat{\lambda}_t^f$ are shown in Figure 4, where a_t^f denotes the coefficient of σ_δ^d, $(\min\{Y_{1,t+1},0\} - \max\{Y_{2,t+1},0\} + \min\{Y_{3,t+1},0\})$, and b_t^f denotes the coefficient of σ_δ^f, $(\max\{Y_{1,t+1},0\} + \max\{Y_{2,t+1},0\} + \max\{Y_{4,t+1},0\})$. Thus, the coefficients a_t^f and b_t^f quantify the strength of the negative foreign bias and the positive home bias from the view of the foreign country, respectively.

Figure 4: Estimated a_t^f and b_t^f.

The following biases are estimated in the U.S.

Period	**Transition of Home Bias in the U.S.**
Early 2000s	After dot-com bubble burst, positive home bias collapsed.
2003-2007	Housing boom generated positive bias.
2008	Lehman shock eliminated positive home bias and generated negative foreign bias.
2010-2014	Positive bias recovered with Fed's quantitative easing and subsequent recovery of the U.S. economy.
2014-2016	Further boosted by shale-gas revolution and tech sector growth (GAFAM).
2015-2016	Temporary correction from China shock, oil price crash, and U.S.-China trade tensions.
2020-	Despite the Covid-19 shock, extensive fiscal and monetary stimulus enabled the economy to survive the downturn and this also prevented the positive bias from dropping.

Table 8: Transition of U.S. estimated home bias parameter

Finally, we examine the filtering results for observation variables S_t^d, S_t^f, r_t^d, r_t^f, and q_t in Figures 5, 6, 7, 8, and 9, respectively. Since the initial values of the raw data and filtered values are not necessarily at the same level due to estimation settings, we plot the data after 2002 to avoid confusion.

Figure 5: Raw data and its filtered value for TOPIX futures price S_t^d.

Figure 6: Raw data and its filtered value for S&P 500 futures price S_t^f.

Figure 7: Raw data and its filtered value for Japan short rate r_t^d.

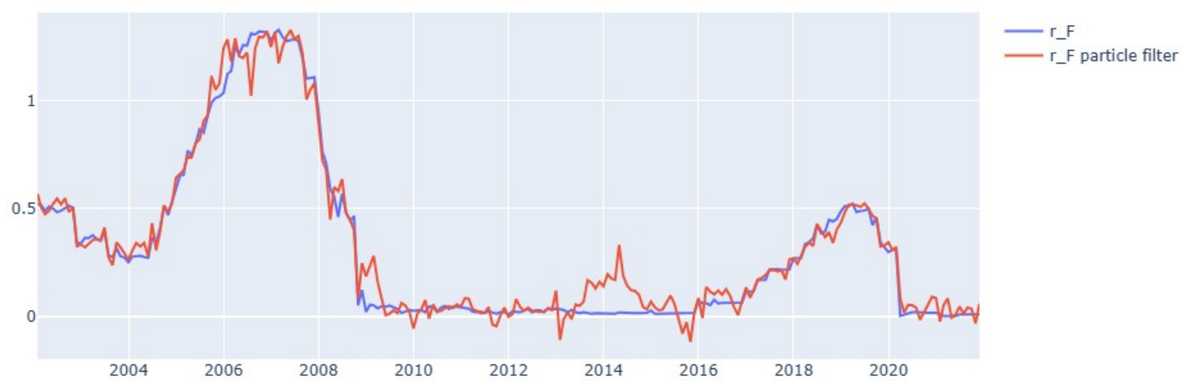

Figure 8: Raw data and its filtered value for U.S. short rate r_t^f.

Figure 9: Raw data and its filtered value for exchange rate q_t.

- All filtered values shown in Figures 5, 6, 7, 8, and 9 closely track or at least follow the trend of the raw data.

4. Conclusion

As financial markets have become increasingly globalized, multi-asset investment funds have assumed a more prominent role. However, the effective operation of such multi-asset funds remains challenging. Even if a fund manager has an outlook on macroeconomic conditions, mapping these views into implications for each country's expected interest rates, exchange rates, and equity prices is difficult. These variables cannot be assigned arbitrarily but must be specified in a manner consistent with economic theory. Unfortunately, the existing literature does not provide an established multi-currency asset allocation framework that is sufficiently flexible to incorporate such practical needs. As a result, multi-asset fund managers sometimes abandon rational asset allocation and resort to ad hoc methods (e.g., a naïve 25%-25%-25%-25% allocation across domestic equities, foreign equities, domestic bonds, and foreign bonds).

To address this issue, this paper develops a novel multi-currency incomplete market equilibrium model with agents who have logarithmic utility and heterogeneous time preferences and subjective beliefs, within a market equilibrium framework based on supply and demand. Despite relying on only a few exogenous inputs (e.g., each country's output process and agents' preference parameters), the model endogenously generates equilibrium interest rates, exchange rates, stock prices, and optimal consumption and portfolios. From a practical perspective, the model offers (i) the flexibility to capture cross-country differences in investors' time preferences and macroeconomic outlooks, and (ii) the tractability to examine how these differences affect equilibrium interest rates and asset prices, including stock prices and exchange rates. This enables practitioners to evaluate how investors' time preferences and macroeconomic views affect equilibrium asset prices in a manner consistent with the equilibrium framework.

As an application of the proposed model, we calibrate it to actual market data, specifically equity indices, short-term interest rates, and exchange rates for Japan and the United States, using state-space modeling and particle filtering techniques. The calibration is performed under the assumption of home-country bias, reflecting the empirical tendency of investors to be more optimistic about their domestic markets and more pessimistic about foreign markets. The estimated results not only replicate the observed dynamics of equity indices, short-term interest rates, and exchange rates, but also capture transitions in time-varying home-country biases and latent economic factors, which may be useful for practical investment decision-making. Future research may explore applications in risk management, portfolio optimization, and the development of investment strategies that achieve high risk-adjusted returns.

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Appendix A. Proof of Proposition 1

Proof. First, we consider the following supremum problem:

$$\sup_{c_t^{i,l,j}} \eta_t^{i,l}\alpha_t^{i,l}\left(\sum_{j=1}^{N}\gamma^{i,l,j}\log c_t^{i,l,j}\right) - y^{i,l}H_t^{i,l}\left(\sum_{j=1}^{N} q_t^{i,j}c_t^{i,l,j}\right), \tag{A.1}$$

and obtain the first-order condition:

$$c_t^{i,l,j,*} = \frac{\gamma^{i,l,j}\eta_t^{i,l}\alpha_t^{i,l}}{y^{i,l}H_t^{i,l}q_t^{i,j}}. \tag{A.2}$$

Second, setting

$$\tilde{U}(y^{i,l}H_t^{i,l},t) = \eta_t^{i,l}\alpha_t^{i,l}\left(\sum_{j=1}^{N}\gamma^{i,l,j}\log c_t^{i,l,j,*}\right) - y^{i,l}H_t^{i,l}\left(\sum_{j=1}^{N} q_t^{i,j}c_t^{i,l,j,*}\right), \tag{A.3}$$

we address the following infimum problem:

$$\inf_{\nu_t^{i,l},\sigma_t^i\nu_t^{i,l}=0} \mathbf{E}\left[\int_0^T \tilde{U}(y^{i,l}H_t^{i,l},t)dt\right]. \tag{A.4}$$

Here, the expectation term can be rewritten as:

$$\mathbf{E}\left[\int_0^T \tilde{U}(y^{i,l}H_t^{i,l},t)dt\right] \tag{A.5}$$

$$= \mathbf{E}\left[\int_0^T \eta_t^{i,l}\alpha_t^{i,l}\left(\sum_{j=1}^{N}\gamma^{i,l,j}\log c_t^{i,l,j,*}\right)dt\right] - y^{i,l}\mathbf{E}\left[\int_0^T H_t^{i,l}\left(\sum_{j=1}^{N} q_t^{i,j}c_t^{i,l,j,*}\right)dt\right] \tag{A.6}$$

$$= \mathbf{E}\left[\int_0^T \eta_t^{i,l}\alpha_t^{i,l}\left\{\sum_{j=1}^{N}\gamma^{i,l,j}(\log\gamma^{i,l,j} + \log\eta_t^{i,l} + \log\alpha_t^{i,l} - \log y^{i,l} - \log H_t^{i,l} - \log q_t^{i,j})\right\}dt\right]$$

$$-\mathbf{E}\left[\int_0^T \left(\sum_{j=1}^{N}\gamma^{i,l,j}\eta_t^{i,l}\alpha_t^{i,l}\right)dt\right]. \tag{A.7}$$

Thus, to solve the infimum problem on $\nu_t^{i,l}$, we only need to consider the following problem:

$$\inf_{\nu_t^{i,l},\sigma_t^i\nu_t^{i,l}=0} \mathbf{E}\left[\int_0^T -\eta_t^{i,l}\alpha_t^{i,l}\log H_t^{i,l}dt\right]. \tag{A.8}$$

In addition, the objective for the infimum can be rewritten as:

$$\mathbf{E}\left[\int_0^T -\eta_t^{i,l}\alpha_t^{i,l}\log H_t^{i,l}dt\right] \tag{A.9}$$

$$= \mathbf{E}\left[\int_0^T \eta_t^{i,l}\alpha_t^{i,l}\left(\int_0^t r_s^i ds + \frac{1}{2}\int_0^t |\theta_s^i + \nu_s^{i,l}|^2 ds + \int_0^t (\theta_s^i + \nu_s^{i,l})\cdot dW_s\right)dt\right] \tag{A.10}$$

$$= \mathbf{E}^{i,l}\left[\int_0^T \alpha_t^{i,l}\left(\int_0^t r_s^i ds + \frac{1}{2}\int_0^t |\theta_s^i + \nu_s^{i,l}|^2 ds + \int_0^t (\theta_s^i + \nu_s^{i,l})\cdot(\lambda_s^{i,l}ds + dW_s^{i,l})\right)dt\right], \tag{A.11}$$

where $\mathbf{E}^{i,l}$ denotes the expectation under the subjective belief of agent l in country i. Since $\nu_s^{i,l} \perp \theta_s^i$, we can consider the term relating to $\nu_s^{i,l}$:

$$\inf_{\nu_t^{i,l}, \sigma_t^i \nu_t^{i,l}=0} \mathbf{E}^{i,l} \left[\int_0^T \left(\frac{1}{2} |\nu_s^{i,l}|^2 + \nu_s^{i,l} \cdot \lambda_s^{i,l} \right) dt \right]. \tag{A.12}$$

The infimum is attained at

$$\nu_t^{i,l} = -\hat{\lambda}_t^{i,l,\perp}. \tag{A.13}$$

This means that $\eta_t^{i,l}/H_t^{i,l} = (B_t^i/Z_t^{\theta^i})Z_t^{i,l}$. Here, $Z_t^{\theta^i}$ represents the common part of the risk neutral measures, and $Z_t^{i,l}$ is the specific part related to each agent.

Also, we rewrite the consumption process $c_t^{i,l,j,*}$ by substituting $\eta_t^{i,l}/H_t^{i,l}$ as follows:

$$c_t^{i,l,j,*} = \frac{\gamma^{i,l,j} \eta_t^{i,l} \alpha_t^{i,l}}{y^{i,l} H_t^{i,l} q_t^{i,j}} = \frac{\gamma^{i,l,j} \alpha_t^{i,l} B_t^i Z_t^{i,l}}{y^{i,l} Z_t^{\theta^i} q_t^{i,j}}. \tag{A.14}$$

Third, since the constraint must be satisfied, we have:

$$y^{i,l} = \frac{1 - e^{-\beta^{i,l} T}}{\beta^{i,l} x_0^{i,l}}. \tag{A.15}$$

□

Appendix B. Proof of Theorem 1

Proof. We show this by a convex duality technique. Noting that for $y_i (i = 1, ..., N) > 0$, $u(x_1, ..., x_N)$ twice continuously differentiable and $\tilde{u}(y_1, ..., y_N) = \sup_{x_1,...,x_N} (u(x_1, ..., x_N) - \sum_{j=1}^N x_j y_j)$,

$$\tilde{u}(y_1, ..., y_N) = \sup_{x_1,...,x_N} \left(u(x_1, ..., x_N) - \sum_{j=1}^N x_j y_j \right) \geq u(x_1, ..., x_N) - \sum_{j=1}^N x_j y_j, \tag{B.1}$$

$$\tilde{u}(u'_1, ..., u'_N) = u(x_1, ..., x_N) - \sum_{j=1}^N x_j u'_j, \tag{B.2}$$

where u'_j is the derivative of $u(x_1, ..., x_N)$ with respect to x_j, respectively.

Here, we set:

$$u(x_1, ..., x_N) = \alpha_t^{i,l} \eta_t^{i,l} \sum_{j=1}^N \gamma^{i,l,j} \log x_j, \tag{B.3}$$

$$u'_j(x_j) = \frac{\gamma^{i,l,j} \alpha_t^{i,l} \eta_t^{i,l}}{x_j}. \tag{B.4}$$

For any $c_t^{i,l,j}$ satisfying the constraint with $y_j = y^{i,l} H_t^{i,l} q_t^{i,j}$, we have:

$$u(c_t^{i,l,1}, ..., c_t^{i,l,N}) = \alpha_t^{i,l} \eta_t^{i,l} \sum_{j=1}^N \gamma^{i,l,j} \log c_t^{i,l,j} \tag{B.5}$$

$$\leq \tilde{u}(y_1, ..., y_N) + \sum_{j=1}^N c_t^{i,l,j} y_j = \tilde{u}(y^{i,l} H_t^{i,l} q_t^{i,1}, ..., y^{i,l} H_t^{i,l} q_t^{i,N}) + \sum_{j=1}^N c^{i,l,j} y^{i,l} H_t^{i,l} q_t^{i,j}, \tag{B.6}$$

and, for $c^{i,l,j,*}$, we have:

$$\tilde{u}(u_1'(c^{i,l,1,*}),...,u_N'(c^{i,l,N,*})) = u(c^{i,l,1,*},...,c^{i,l,N,*}) - \sum_{j=1}^{N} c^{i,l,j,*} u_j'(c^{i,l,j,*}) \tag{B.7}$$

$$= \alpha_t^{i,l} \eta_t^{i,l} \sum_{j=1}^{N} \gamma^{i,l,j} \log c^{i,l,j,*} - \sum_{j=1}^{N} c^{i,l,j,*} y^{i,l} H_t^{i,l} q_t^{i,j}. \tag{B.8}$$

By budget constraint, we have:

$$\mathbf{E}\left[\int_0^T \left(\sum_{j=1}^{N} q_t^{i,j} c_t^{i,l,j}\right) y^{i,l} H_t^{i,l} dt\right] \le \mathbf{E}\left[\int_0^T \left(\sum_{j=1}^{N} q_t^{i,j} c_t^{i,l,j,*}\right) y^{i,l} H_t^{i,l} dt\right]. \tag{B.9}$$

Therefore, using (B.6), (B.8), and (B.9), we have:

$$\mathbf{E}\left[\int_0^T \eta_t^{i,l} \alpha_t^{i,l} \left(\sum_{j=1}^{N} \gamma^{i,l,j} \log c_t^{i,l,j}\right) dt\right] = \mathbf{E}\left[\int_0^T u(c_t^{i,l,1},...,c_t^{i,l,N}) dt\right] \tag{B.10}$$

$$\le \mathbf{E}\left[\int_0^T \tilde{u}(y^{i,l} H_t^{i,l} q_t^{i,1},...,y^{i,l} H_t^{i,l} q_t^{i,N}) dt\right] + \mathbf{E}\left[\int_0^T \sum_{j=1}^{N} c^{i,l,j} q_t^{i,j} y^{i,l} H_t^{i,l} dt\right] \tag{B.11}$$

$$\le \mathbf{E}\left[\int_0^T \tilde{u}(y^{i,l} H_t^{i,l} q_t^{i,1},...,y^{i,l} H_t^{i,l} q_t^{i,N}) dt\right] + \mathbf{E}\left[\int_0^T \sum_{j=1}^{N} c^{i,l,j,*} q_t^{i,j} y^{i,l} H_t^{i,l} dt\right] \tag{B.12}$$

$$= \mathbf{E}\left[\int_0^T \left\{\tilde{u}(u_1'(c_t^{i,l,1,*}),...,u_1'(c_t^{i,l,N,*})) + \sum_{j=1}^{N} c^{i,l,j,*} u_j'(c_t^{i,l,j,*})\right\} dt\right] \tag{B.13}$$

$$= \mathbf{E}\left[\int_0^T u(c_t^{i,l,1,*},...,c_t^{i,l,N,*}) dt\right]. \tag{B.14}$$

This means that the solution $c_t^{i,l,j,*}$ for the dual problem (33) is also a solution for the primal problem (32). □

Appendix C. Proof of Theorem 2

Proof. If we find the wealth process $X_t^{i,l,*}$ with an investment strategy $(\pi_t^{i,l,*}, \pi_t^{i,l,N+i,*})$ such that $X_t^{i,l,*} H_t^{i,l} + \int_0^t H_t^{i,l} \left(\sum_{j=1}^{N} q_s^{i,j} c_s^{i,l,j,*}\right) ds$ is a martingale and $X_T^{i,l,*} = 0$, then $c_t^{i,l,j,*}$ is in the admissible set $\mathcal{A}^{i,l}$. We can find such wealth process based on the following equation:

$$X_t^{i,l,*} H_t^{i,l} + \int_0^t H_s^{i,l} \left(\sum_{j=1}^{N} q_s^{i,j} c_s^{i,l,j,*}\right) ds = \mathbf{E}_t\left[X_T^{i,l,*} H_t^{i,l} + \int_0^T H_s^{i,l} \left(\sum_{j=1}^{N} q_s^{i,j} c_s^{i,l,j,*}\right) ds\right] \tag{C.1}$$

Then, we can express the wealth process $X_t^{i,l,*}$ as follows:

$$X_t^{i,l,*} = \frac{1}{H_t^{i,l}} \left\{ \mathbf{E}_t \left[\int_0^T H_s^{i,l} \left(\sum_{j=1}^N q_s^{i,j} c_s^{i,l,j,*} \right) ds \right] - \int_0^t H_s^{i,l} \left(\sum_{j=1}^N q_s^{i,j} c_s^{i,l,j,*} \right) ds \right\} \tag{C.2}$$

$$= \frac{1}{H_t^{i,l}} \mathbf{E}_t \left[\int_t^T H_s^{i,l} \left(\sum_{j=1}^N q_s^{i,j} c_s^{i,l,j,*} \right) ds \right] \tag{C.3}$$

$$= \frac{1}{y^{i,l} H_t^{i,l}} \mathbf{E}_t \left[\int_t^T \eta_s^{i,l} \alpha_s^{i,l} ds \right] \tag{C.4}$$

$$= \frac{\eta_t^{i,l}}{y^{i,l} H_t^{i,l}} \int_t^T \alpha_s^{i,l} ds \tag{C.5}$$

$$= \frac{e^{-\beta^{i,l} t} - e^{-\beta^{i,l} T}}{1 - e^{-\beta^{i,l} T}} \left(\frac{x_0^{i,l} B_t^i Z_t^{i,l}}{Z_t^{\theta^i}} \right). \tag{C.6}$$

Applying Ito's formula, we have:

$$dX_t^{i,l,*} = d \left[\frac{e^{-\beta^{i,l} t} - e^{-\beta^{i,l} T}}{1 - e^{-\beta^{i,l} T}} \left(\frac{x_0^{i,l} B_t^i Z_t^{i,l}}{Z_t^{\theta^i}} \right) \right] \tag{C.7}$$

$$= r_t^i X_t^{i,l,*} dt + X_t^{i,l,*} \left(\theta_t^i + \hat{\lambda}_t^{i,l} \right) \cdot \left(dW_t + \theta_t^i dt \right) - \left(\sum_{j=1}^N q_t^{i,j} c_t^{i,l,j,*} \right) dt. \tag{C.8}$$

Thus, we find that an investment strategy $(\pi_t^{i,l,*}, \pi_t^{i,l,N+i,*})$ achieving the following equation satisfies (C.8):

$$\pi_t^{i,l,*\top} \sigma_t^i = X_t^{i,l,*} (\theta_t^i + \hat{\lambda}_t^{i,l})^\top, \tag{C.9}$$

$$\pi_t^{i,l,N+i,*} = X_t^{i,l,*} - \pi_t^{i,l,*} \cdot 1. \tag{C.10}$$

If $(\sigma_t^i \sigma_t^{i\top})^{-1}$ exists, $\pi_t^{i,l,*} = X_t^{i,l,*} (\sigma_t^i \sigma_t^{i\top})^{-1} \sigma_t^i (\theta_t^i + \hat{\lambda}_t^{i,l})$. □

Appendix D. Proof of Theorem 4

Proof. Since $S_t^i Z_t^{\theta^i} / B_t^i + \int_0^t \delta_s^i Z_s^{\theta^i} / B_s^i ds$ is a martingale and $S_T^i = 0$, we obtain the following equation:

$$S_t^i \frac{Z_t^{\theta^i}}{B_t^i} + \int_0^t \delta_s^i \frac{Z_s^{\theta^i}}{B_s^i} ds = \mathbf{E}_t \left[S_T^i \frac{Z_T^{\theta^i}}{B_T^i} + \int_0^T \delta_s^i \frac{Z_s^{\theta^i}}{B_s^i} ds \right]. \tag{D.1}$$

By rearranging the above equation, we can express the stock price S_t^i as follows:

$$S_t^i = \frac{B_t^i}{Z_t^{\theta^i}} \mathbf{E}_t \left[\int_t^T \delta_s^i \frac{Z_s^{\theta^i}}{B_s^i} ds \right]. \tag{D.2}$$

This equation implies that the stock price is the present value of the future dividend stream. Here, substituting (50) into the above equation, we have:

$$S_t^i = \frac{\delta_t^i}{\sum_{j=1}^N \sum_{l=1}^{L^j} \frac{\gamma^{j,l,i}\alpha_t^{j,l} Z_t^{j,l}}{y^{j,l} q_0^{j,i}}} \mathbf{E}_t \left[\int_t^T \sum_{j=1}^N \sum_{l=1}^{L^j} \frac{\gamma^{j,l,i}\alpha_s^{j,l} Z_s^{j,l}}{y^{j,l} q_0^{j,i}} ds \right] \tag{D.3}$$

$$= \frac{\delta_t^i}{\sum_{j=1}^N \sum_{l=1}^{L^j} \frac{\gamma^{j,l,i}\alpha_t^{j,l} Z_t^{j,l}}{y^{j,l} q_0^{j,i}}} \int_t^T \sum_{j=1}^N \sum_{l=1}^{L^j} \frac{\gamma^{j,l,i}\alpha_s^{j,l} Z_t^{j,l}}{y^{j,l} q_0^{j,i}} ds \tag{D.4}$$

$$= \frac{\delta_t^i}{\sum_{j=1}^N \sum_{l=1}^{L^j} \frac{\gamma^{j,l,i}\alpha_t^{j,l} Z_t^{j,l}}{y^{j,l} q_0^{j,i}}} \left(\sum_{j=1}^N \sum_{l=1}^{L^j} \frac{\gamma^{j,l,i} Z_t^{j,l}}{y^{j,l} q_0^{j,i}} \frac{e^{-\beta^{j,l} t} - e^{-\beta^{j,l} T}}{\beta^{j,l}} \right) \tag{D.5}$$

$$= \frac{\delta_t^i}{\sum_{j=1}^N \sum_{l=1}^{L^j} \frac{\gamma^{j,l,i} e^{-\beta^{j,l} t} Z_t^{j,l} x_0^{j,l} \beta^{j,l}}{q_0^{j,i}(1-e^{-\beta^{j,l} T})}} \sum_{j=1}^N \sum_{l=1}^{L^j} \frac{\gamma^{j,l,i} Z_t^{j,l}(e^{-\beta^{j,l} t} - e^{-\beta^{j,l} T}) x_0^{j,l}}{q_0^{j,i}(1 - e^{-\beta^{j,l} T})}. \tag{D.6}$$

As for the individual stock price $S_t^{i,k}$, replacing δ_t^i with $\delta_t^{i,k}$ in the above discussion, we have:

$$S_t^{i,k} = \frac{B_t^i}{Z_t^{\theta^i}} \mathbf{E}_t \left[\int_t^T \delta_s^{i,k} \frac{Z_s^{\theta^i}}{B_s^i} ds \right] \tag{D.7}$$

$$= \frac{\delta_t^i}{\sum_{j=1}^N \sum_{l=1}^{L^j} \frac{\gamma^{j,l,i}\alpha_t^{j,l} Z_t^{j,l}}{y^{j,l} q_0^{j,i}}} \mathbf{E}_t \left[\int_t^T \frac{\delta_s^{i,k}}{\delta_s^i} \sum_{j=1}^N \sum_{l=1}^{L^j} \frac{\gamma^{j,l,i}\alpha_s^{j,l} Z_s^{j,l}}{y^{j,l} q_0^{j,i}} ds \right]. \tag{D.8}$$

□

Appendix E. Proof of Proposition 2

Proof. Before confirming the market clearing conditions, we check that the following relationship holds:

$$\sum_{i=1}^N \sum_{l=1}^{L^j} q_t^{1,i} X_t^{i,l} = \sum_{i=1}^N \sum_{k=1}^{K^i} q_t^{1,i} S_t^{i,k}. \tag{E.1}$$

From (12) and (C.8), we obtain the following expression for the wealth process:

$$d(q_t^{1,i} X_t^{i,l}) = q_t^{1,i} X_t^{i,l} \left[\left\{ r_t^1 + \theta_t^1 \cdot (\theta_t^1 + \hat{\lambda}^{i,l}) \right\} dt + (\theta_t^1 + \hat{\lambda}_t^{i,l}) \cdot dW_t \right] - \left(\sum_{j=1}^N q_t^{1,j} c_t^{i,l,j,*} \right) dt \tag{E.2}$$

$$= q_t^{1,i} X_t^{i,l} \left[r_t^1 dt + (\theta_t^1 + \hat{\lambda}_t^{i,l}) \cdot dW_t^{1,*} \right] - \left(\sum_{j=1}^N q_t^{1,j} c_t^{i,l,j,*} \right) dt, \tag{E.3}$$

where $dW_t^{1,*}$ is the Brownian motion under the risk-neutral measure associated with the first country:

$dW_t^{1,*} = dW_t + \theta_t^1$. Summing up the above equation, we have:

$$d\left(\sum_{i=1}^{N}\sum_{l=1}^{L^i} q_t^{1,i} X_t^{i,l}\right) = r_t^1 \sum_{i=1}^{N}\sum_{l=1}^{L^i} q_t^{1,i} X_t^{i,l} dt + \sum_{i=1}^{N}\sum_{l=1}^{L^i} q_t^{1,i} X_t^{i,l} \left(\theta_t^1 + \hat{\lambda}_t^{i,l}\right) \cdot dW_t^{1,*}$$

$$-\sum_{i=1}^{N}\sum_{l=1}^{L^i}\left(\sum_{j=1}^{N} q_t^{1,j} c_t^{i,l,j,*}\right) dt \tag{E.4}$$

$$= r_t^1 \sum_{i=1}^{N}\sum_{l=1}^{L^i} q_t^{1,i} X_t^{i,l} dt - \sum_{j=1}^{N} q^{1,j} \delta_t^j dt + \sum_{i=1}^{N}\sum_{l=1}^{L^i} q_t^{1,i} X_t^{i,l} \left(\theta_t^1 + \hat{\lambda}_t^{i,l}\right) \cdot dW_t^{1,*}. \tag{E.5}$$

Then, we have:

$$\sum_{i=1}^{N}\sum_{l=1}^{L^i} q_t^{1,i} X_t^{i,l} = \mathbf{E}^1\left[\int_t^T \frac{B_t^1}{B_s^1}\left(\sum_{j=1}^{N} q^{1,j}\delta_t^j\right) ds\right] = \sum_{i=1}^{N} q_t^{1,i} S_t^i = \sum_{i=1}^{N}\sum_{k=1}^{K^i} q_t^{1,i} S_t^{i,k}, \tag{E.6}$$

where $\mathbf{E}^1$ is the expectation under the risk-neutral measure associated with the first country. Thus, we obtain (E.1).

Then, we examine the market clearing conditions (57) and (58). According to equation (3.3) on page 11 of Karatzas & Shreve (1998), the Γ-financed (consumption-financed) strategy $(\pi_t^{i,l,*}, \pi_t^{i,l,N+i,*})$ ensures that $X^{i,l}$ satisfies the following equation:

$$d\left(\sum_{i=1}^{N}\sum_{l=1}^{L^i} q_t^{1,i} X_t^{i,l}\right) = r_t^1 \left(\sum_{i=1}^{N}\sum_{l=1}^{L^i} q^{1,i} X^{i,l}\right) dt - \sum_{j=1}^{N} q_t^{1,j}\delta_t^j dt$$

$$+\sum_{j=1}^{N}\sum_{i=1}^{N}\sum_{k=1}^{K^j}\sum_{l=1}^{L^i} q_t^{1,i}\pi_t^{i,l,(j,k),*}(\sigma_{S,t}^{j,k} + \sigma_{q,t}^{1,j}) dW_t^{1,*} + \sum_{j=N+1}^{2N}\sum_{i=1}^{N}\sum_{l=1}^{L^i} q_t^{1,i}\pi_t^{i,l,j,*}\sigma_{q,t}^{1,j} dW_t^{1,*}, \tag{E.7}$$

where $\sigma_{q,t}^{1,j} = \sigma_{q,t}^{1,j-N}$ when $j > N$, and $\sigma_{q,t}^{1,1} = \sigma_{q,t}^{1,N+1} = 0$. This equation means that the dynamics of wealth process $dX_t^{i,l}$ is driven by three components: (i) the risk-free rate part, (ii) the consumption part, and (iii) the investment on risky asset part.

On the other hand, S_t^i satisfies the following equation:

$$d\left(\sum_{i=1}^{N} q_t^{1,i} S_t^i\right) = r_t^1\left(\sum_{i=1}^{N} q_t^{1,i} S_t^i\right) dt - \sum_{i=1}^{N} q_t^{1,i}\delta_t^i dt + \sum_{j=1}^{N}\sum_{k=1}^{K^j} q_t^{1,j} S_t^{j,k}(\sigma_{S,t}^{j,k} + \sigma_{q,t}^{1,j}) dW_t^{*,1}. \tag{E.8}$$

As aggregate wealth equals aggregate stock price as in (E.1), the volatility terms in (E.7) and (E.8) must be equal:

$$\sum_{j=1}^{N}\sum_{i=1}^{N}\sum_{k=1}^{K^j}\sum_{l=1}^{L^i} q_t^{1,i}\pi_t^{i,l,(j,k),*}(\sigma_{S,t}^{j,k} + \sigma_{q,t}^{1,j}) + \sum_{j=N+1}^{2N}\sum_{i=1}^{N}\sum_{l=1}^{L^i} q_t^{1,i}\pi_t^{i,l,j,*}\sigma_{q,t}^{1,j}$$

$$= \sum_{j=1}^{N}\sum_{k=1}^{K^j} q_t^{1,j} S_t^{j,k}(\sigma_{S,t}^{j,k} + \sigma_{q,t}^{1,j}). \tag{E.9}$$

Since $\left(\sigma_{S,t}^{j,k} + \sigma_{q,t}^{1,j}\right)$ and $\sigma_{q,t}^{1,j}$ are $\sum_{i=1}^{N} K^i + N - 1$ independent vectors, we obtain the following market clearing conditions:

$$q_t^{1,j} S_t^{j,k} = \sum_{i=1}^{N} \sum_{l=1}^{L^i} q^{1,i} \pi^{i,l,(j,k),*}; \ j = 1, ..., N; k = 1, ..., K^j, \tag{E.10}$$

$$0 = \sum_{i=1}^{N} \sum_{l=1}^{L^i} q^{1,i} \pi^{i,l,j,*}; \ (j = N+2, ..., 2N), \tag{E.11}$$

which are equivalent to the market clearing conditions (57) and (58) except for the case of $j = N+1$.

For $j = N+1$, since $\sum_{j=1}^{N} \sum_{k=1}^{K^j} q_t^{1,i} \pi_t^{i,l,(j,k),*} + \sum_{j=N+1}^{2N} q_t^{1,i} \pi_t^{i,l,j,*} = q_t^{1,i} X_t^{i,l}$ and $\sum_{j=1}^{N} \sum_{l=1}^{L^j} q_t^{1,j} X_t^{j,l} = \sum_{j=1}^{N} \sum_{k=1}^{K^j} q_t^{1,j} S_t^{j,k}$, we have

$$\sum_{j=1}^{N} \sum_{k=1}^{K^j} \sum_{i=1}^{N} \sum_{l=1}^{L^i} q_t^{1,i} \pi_t^{i,l,(j,k),*} + \sum_{j=N+1}^{2N} \sum_{i=1}^{N} \sum_{l=1}^{L^i} q_t^{1,i} \pi_t^{i,l,j,*} = \sum_{i=1}^{N} \sum_{l=1}^{L^i} q_t^{1,i} X_t^{i,l} = \sum_{i=1}^{N} q_t^{1,i} S_t^i. \tag{E.12}$$

Thus, we have:

$$\sum_{j=1}^{N} \sum_{k=1}^{K^j} \sum_{i=1}^{N} \sum_{l=1}^{L^i} q_t^{1,i} \pi_t^{i,l,(j,k),*} + \sum_{i=1}^{N} \sum_{l=1}^{L^i} q_t^{1,i} \pi_t^{i,l,N+1,*} + \sum_{j=N+2}^{2N} \sum_{i=1}^{N} \sum_{l=1}^{L^i} q_t^{1,i} \pi_t^{i,l,j,*} = \sum_{i=1}^{N} q_t^{1,i} S_t^i. \tag{E.13}$$

By substituting the market clearing condition (57), we have:

$$\sum_{j=1}^{N} \sum_{k=1}^{K^j} q_t^{1,j} S_t^{j,k} + \sum_{i=1}^{N} \sum_{l=1}^{L^i} q_t^{1,i} \pi_t^{i,l,N+1,*} + 0 = \sum_{i=1}^{N} q_t^{1,i} S_t^i. \tag{E.14}$$

Then we obtain:

$$\sum_{i=1}^{N} \sum_{l=1}^{L^i} q_t^{1,i} \pi_t^{i,l,N+1,*} = 0. \tag{E.15}$$

Therefore, we can conclude that the market clearing conditions are satisfied. □

Appendix F. Proof of Proposition 3

Proof. The derivative of $S_t^{i,k} Z_t^{\theta^i} / B_t^i$ is given by:

$$d\left(\frac{S_t^{i,k} Z_t^{\theta^i}}{B_t^i}\right) = \left(\frac{S_t^{i,k} Z_t^{\theta^i}}{B_t^i}\right) \left\{ \left(\mu_{S,t}^{i,k} - r_t^i - \sigma_{S,t}^{i,k} \theta_t^i\right) dt + \left(\sigma_{S,t}^{i,k\top} - \theta_t^i\right) \cdot dW_t \right\}. \tag{F.1}$$

Thus, we obtain the following relationship:

$$D_t \left(\frac{S_t^{i,k} Z_t^{\theta^i}}{B_t^i}\right) = \left(\sigma_{S,t}^{i,k\top} - \theta_t^i\right) \left(\frac{S_t^{i,k} Z_t^{\theta^i}}{B_t^i}\right), \tag{F.2}$$

where D_t is the Malliavin derivative operator.

On the other hand, from (50) and (56), $S_t^{i,k}$ can be expressed as following:

$$S_t^{i,k} \frac{Z_t^{\theta^i}}{B_t^i} = \mathbf{E}_t \left[\int_t^T \frac{\delta_s^{i,k}}{\delta_s^i} \sum_{j=1}^{N} \sum_{l=1}^{L^j} A_s^{i,j,l} Z_s^{j,l} ds \right], \tag{F.3}$$

$$A_s^{i,j,l} = \frac{\gamma^{j,l,i} \alpha_s^{j,l}}{y^{j,l} q_0^{j,i}}. \tag{F.4}$$

By applying D_t to both sides, we have:

$$D_t\left(S_t^{i,k}\frac{Z_t^{\theta^i}}{B_t^i}\right)=\mathbf{E}_t\left[\int_t^T D_t\left(\frac{\delta_s^{i,k}}{\delta_s^i}\sum_{j=1}^{N}\sum_{l=1}^{L^j}A_s^{i,j,l}Z_s^{j,l}\right)ds\right]. \tag{F.5}$$

Here, when $t\leq s$, applying chain rule of D_t to exponential martingale $Z_s^{j,l}$, $\delta_s^{i,k}$ and δ_s^i, we have:

$$D_tZ_s^{j,l}=\hat{\lambda}_t^{j,l}Z_s^{j,l}, \tag{F.6}$$

$$D_t\delta_s^{i,k}=\sigma_{\delta,t}^{i,k}\delta_s^{i,k}, \tag{F.7}$$

$$D_t\left(\frac{1}{\delta_s^i}\right)=-\frac{1}{\delta_s^i}\sigma_{\delta,t}^{i}. \tag{F.8}$$

Using these relationship,

$$\mathbf{E}_t\left[\int_t^T D_t\left(\frac{\delta_s^{i,k}}{\delta_s^i}\sum_{j=1}^{N}\sum_{l=1}^{L^j}A_s^{i,j,l}Z_s^{j,l}\right)ds\right] \tag{F.9}$$

$$=\mathbf{E}_t\left[\int_t^T\left(\frac{D_t\delta_s^{i,k}}{\delta_s^i}+D_t\frac{1}{\delta_s^i}\cdot\delta_s^{i,k}\right)\sum_{j=1}^{N}\sum_{l=1}^{L^j}A_s^{i,j,l}Z_s^{j,l}+\frac{\delta_s^{i,k}}{\delta_s^i}\sum_{j=1}^{N}\sum_{l=1}^{L^j}A_s^{i,j,l}D_tZ_s^{j,l}ds\right] \tag{F.10}$$

$$=\mathbf{E}_t\left[\int_t^T\left(\sigma_{\delta,t}^{i,k}-\sigma_{\delta,t}^{i}\right)\frac{\delta_s^{i,k}}{\delta_s^i}\sum_{j=1}^{N}\sum_{l=1}^{L^j}A_s^{i,j,l}Z_s^{j,l}+\frac{\delta_s^{i,k}}{\delta_s^i}\sum_{j=1}^{N}\sum_{l=1}^{L^j}A_s^{i,j,l}Z_s^{j,l}\hat{\lambda}_t^{j,l}ds\right] \tag{F.11}$$

$$=\left(S_t^{i,k}\frac{Z_t^{\theta^i}}{B_t^i}\right)\left(\sigma_{\delta,t}^{i,k}-\sigma_{\delta,t}^{i}+\sum_{j=1}^{N}\sum_{l=1}^{L^j}\frac{\mathbf{E}_t\left[\int_t^T\frac{\delta_s^{i,k}}{\delta_s^i}A_s^{i,j,l}Z_s^{j,l}ds\right]}{\sum_{f=1}^{N}\sum_{g=1}^{L^f}\mathbf{E}_t\left[\int_t^T\frac{\delta_s^{i,k}}{\delta_s^i}A_s^{i,f,g}Z_s^{f,g}ds\right]}\hat{\lambda}_t^{j,l}\right). \tag{F.12}$$

The last equality follows from (F.3). Thus, we have:

$$\left(\sigma_{S,t}^{i,k\top}-\theta_t^i\right)=\left(\sigma_{\delta,t}^{i,k}-\sigma_{\delta,t}^{i}+\sum_{j=1}^{N}\sum_{l=1}^{L^j}\frac{\mathbf{E}_t\left[\int_t^T\frac{\delta_s^{i,k}}{\delta_s^i}A_s^{i,j,l}Z_s^{j,l}ds\right]}{\sum_{f=1}^{N}\sum_{g=1}^{L^f}\mathbf{E}_t\left[\int_t^T\frac{\delta_s^{i,k}}{\delta_s^i}A_s^{i,f,g}Z_s^{f,g}ds\right]}\hat{\lambda}_t^{j,l}\right). \tag{F.13}$$

Therefore, using (47), the volatility vector $\sigma_{S,t}^{i,k}$ is given by:

$$\sigma_{S,t}^{i,k\top}=\sigma_{\delta,t}^{i,k}-\sum_{j=1}^{N}\sum_{l=1}^{L^j}\frac{A_t^{i,j,l}Z_t^{j,l}}{\sum_{f=1}^{N}\sum_{g=1}^{L^f}A_t^{i,f,g}Z_t^{f,g}}\hat{\lambda}_t^{j,l} \tag{F.14}$$

$$+\sum_{j=1}^{N}\sum_{l=1}^{L^j}\frac{\mathbf{E}_t\left[\int_t^T\frac{\delta_s^{i,k}}{\delta_s^i}A_s^{i,j,l}Z_s^{j,l}ds\right]}{\sum_{f=1}^{N}\sum_{g=1}^{L^f}\mathbf{E}_t\left[\int_t^T\frac{\delta_s^{i,k}}{\delta_s^i}A_s^{i,f,g}Z_s^{f,g}ds\right]}\hat{\lambda}_t^{j,l}. \tag{F.15}$$

□

Appendix G. Detailed Information about Two-Currency Model in Section 3.1

Let π_t^d and $\pi_{0,t}^d$ be the investment values in the three risky assets and money market account held by the agent d, respectively. The investment value in risky assets is set to $\pi_t^d=(\pi_{1,t}^d,\pi_{2,t}^d,\pi_{3,t}^d)^\top$, where $\pi_{1,t}^d$ and $\pi_{2,t}^d$ are the investment value in domestic and foreign stocks, respectively, and $\pi_{3,t}^d$ is the investment value in the foreign money market account.

Then, the optimization problem of agent d is given by:

$$\max_{\pi_{0,t}^d, \pi_t^d, c_t^{d,d}, c_t^{d,f}} \mathbf{E}\left[\int_0^T u_t^d(c_t^{d,d}, c_t^{d,f})dt\right], \tag{G.1}$$

$$s.t.\ dX_t^d = \pi_{0,t}^d \frac{dB_t^d}{B_t^d} + \pi_{1,t}^d \frac{dS_t^d + \delta_t^d dt}{S_t^d} + \pi_{2,t}^d \frac{d(q_t S_t^f) + q_t \delta_t^f dt}{q_t S_t^f} + \pi_{3,t}^d \frac{d(q_t B_t^f)}{q_t B_t^f}$$
$$-(c_t^{d,d} + q_t c_t^{d,f})dt;\ X_0^d = x_0^d > 0, \tag{G.2}$$

$$X_t^d \geq 0;\ c_t^{d,d} \geq 0;\ c_t^{d,f} \geq 0;\ \forall t \in [0, T]. \tag{G.3}$$

As seen in Section 2.2, we can focus on the following optimization problem involving only consumption:

$$\max_{c_t^{d,d}, c_t^{d,f}} \mathbf{E}\left[\int_0^T u_t^d(c_t^{d,d}, c_t^{d,f})dt\right], \tag{G.4}$$

$$s.t.\ \mathbf{E}\left[\int_0^T H_t^d(c_t^{d,d} + q_t c_t^{d,f})dt\right] \leq x_0^d,\ \forall H_t^d, \tag{G.5}$$

where H_t^d is the state price density process for currency d.

Thus, to solve the optimization problem (G.4) and (G.5), define the Lagrangian as follows:

$$\mathcal{L} = E\left[\int_0^T \alpha_t^d \eta_t^d \left(\gamma^d \log c_t^{d,d} + (1-\gamma^d)\log c_t^{d,f}\right) dt\right]$$
$$+ y^d \left(x_0^d - E\left[\int_0^T H_t^d(c_t^{d,d} + q_t c_t^{d,f})dt\right]\right), \tag{G.6}$$

where y^d is the Lagrange multiplier. Then, the optimization problem can be rewritten as the following sup-inf problem:

$$\sup_{c_t^{d,d}, c_t^{d,f}} \inf_{y^d > 0, \nu_t^d, \sigma_t^d \nu_t^d = 0} \mathcal{L}(c_t^{d,d}, c_t^{d,f}, y^d, \nu^d), \tag{G.7}$$

and we focus on its dual problem:

$$\inf_{y^d > 0, \nu_t^d, \sigma_t^d \nu_t^d = 0} \sup_{c_t^{d,d}, c_t^{d,f}} \mathcal{L}(c_t^{d,d}, c_t^{d,f}, y^d, \nu^d). \tag{G.8}$$

The solution to the dual problem (G.8) is given by the following Corollary.

Corollary 1. *Following $c_t^{d,d,*}$, $c_t^{d,f,*}$, ν_t^d and y^d, attain the optimal solution of the inf-sup dual problem (G.8).*

$$c_t^{d,d,*} = \frac{Z_t^d B_t^d \alpha_t^d \gamma^d}{y^d Z_t^{\theta^d}}, \tag{G.9}$$

$$c_t^{d,f,*} = \frac{Z_t^d B_t^d \alpha_t^d (1-\gamma^d)}{y^d Z_t^{\theta^d} q_t}, \tag{G.10}$$

$$\nu_t^d = -\hat{\lambda}_t^{d,\perp}, \tag{G.11}$$

$$y^d = \frac{1 - e^{-\beta^d T}}{\beta^d x_0^d}. \tag{G.12}$$

Here, $Z_t^{\theta^d}$ *and* Z_t^d *are defined as follows:*

$$Z_t^{\theta^d} = \exp\left(-\int_0^t \theta_s^d \cdot dW_s - \frac{1}{2}\int_0^t |\theta_s^d|^2 ds\right);\ Z_0^{\theta^d} = 1, \tag{G.13}$$

$$Z_t^d = \exp\left(\int_0^t \hat{\lambda}_s^d \cdot dW_s - \frac{1}{2}\int_0^t |\hat{\lambda}_s^d|^2 ds\right);\ Z_0^d = 1. \tag{G.14}$$

Also, $\hat{\lambda}_t^{d,\perp}$ *is the projection of* λ_t^d *onto the orthogonal space of range* $\left(\sigma_t^{d\top}\right)$*, where* $\sigma_t^d = \begin{pmatrix} \sigma_{S,t}^d \\ \sigma_{S,t}^f + \sigma_t^q \\ \sigma_t^q \end{pmatrix}$

Proof. First, we consider the following supremum problem:

$$\sup_{c_t^{d,d},c_t^{d,f}} \alpha_t^d \eta_t^d \left(\gamma^d \log c_t^{d,d} + (1-\gamma^d)\log c_t^{d,f}\right) - y^d H_t^d (c_t^{d,d} + q_t c_t^{d,f}). \tag{G.15}$$

Then, we obtain the optimal consumption as follows:

$$c_t^{d,d,*} = \frac{\eta_t^d \alpha_t^d \gamma^d}{y^d H_t^d}, \tag{G.16}$$

$$c_t^{d,f,*} = \frac{\eta_t^d \alpha_t^d (1-\gamma^d)}{y^d H_t^d q_t}. \tag{G.17}$$

Next, setting

$$\tilde{U}(y^d H_t^d, t) = \alpha_t^d \eta_t^d \left\{\gamma^d \log c_t^{d,d,*} + (1-\gamma^d)\log c_t^{d,f,*}\right\} - y^d H_t^d (c_t^{d,d,*} + q_t c_t^{d,f,*}), \tag{G.18}$$

we address the following infimum problem:

$$\inf_{\nu_t^d, \sigma_t^d \nu_t^d = 0} \mathbf{E}\left[\int_0^T \tilde{U}(y^d H_t^d, t) dt\right]. \tag{G.19}$$

Here, the expectation term can be expressed as:

$$\begin{aligned}
&\mathbf{E}\left[\int_0^T \tilde{U}(y^d H_t^d, t) dt\right] \\
&= \mathbf{E}\left[\int_0^T \alpha_t^d \eta_t^d \gamma^d (\log \eta_t^d + \log \alpha_t^d + \log \gamma^d - \log y^d - \log H_t^d) dt\right] \\
&+ \mathbf{E}\left[\int_0^T \alpha_t^d \eta_t^d (1-\gamma^d)(\log \eta_t^d + \log \alpha_t^d + \log(1-\gamma^d) - \log y^d - \log H_t^d - \log q_t) dt\right] \\
&- \mathbf{E}\left[\int_0^T y^d H_t^d (c_t^{d,d,*} + q_t c_t^{d,f,*}) dt\right].
\end{aligned} \tag{G.20}$$

To solve the infimum problem on ν^d, we need to focus on:

$$\inf_{\nu_t^d, \sigma_t^d \nu_t^d = 0} \mathbf{E}\left[\int_0^T -\alpha_t^d \eta_t^d \log H_t^d dt\right] \tag{G.21}$$

$$= \inf_{\nu_t^d, \sigma_t^d \nu_t^d = 0} \mathbf{E}\left[\int_0^T \alpha_t^d \eta_t^d \left(\int_0^t r_s ds + \frac{1}{2}\int_0^t |\theta_s^d + \nu_s^d|^2 ds + \int_0^t (\theta_s^d + \nu_s^d) \cdot dW_s\right) dt\right] \tag{G.22}$$

$$= \inf_{\nu_t^d, \sigma_t^d \nu_T^d = 0} \mathbf{E}^d\left[\int_0^T \alpha_t^d (\int_0^t r_s ds + \frac{1}{2}\int_0^t |\theta_s^d + \nu_s^d|^2 ds + \int_0^t (\theta_s^d + \nu_s^d) \cdot \left(\lambda_s^d ds + dW_s^d\right)) dt\right], \tag{G.23}$$

where $\mathbf{E}^d$ denotes the expectation under the subjective view. Since $\nu_s^d \perp \theta_s$, we pick up the term relating to ν_s^d:

$$\inf_{\nu_t^d, \sigma_t^d \nu_t^d = 0} \mathbf{E}^d \left[\int_0^T \left(\frac{1}{2} |\nu_s^d|^2 + \nu_s^d \cdot \lambda_s^d \right) dt \right]. \tag{G.24}$$

The infimum is attained at

$$\nu_t^d = -\hat{\lambda}_t^{d,\perp}, \tag{G.25}$$

which implies that $\eta_t^d / H_t^d = (B_t / Z_t^{\theta^d}) Z_t^d$.

Moreover, substituting η_t^d / H_t^d into $c_t^{d,d,*}$ and $c_t^{d,f,*}$ reveals the following equations:

$$c_t^{d,d,*} = \frac{Z_t^d B_t^d \alpha_t^d \gamma^d}{y^d Z_t^{\theta^d}}, \tag{G.26}$$

$$c_t^{d,f,*} = \frac{Z_t^d B_t^d \alpha_t^d (1-\gamma^d)}{y^d Z_t^{\theta^d} q_t}, \tag{G.27}$$

As in the same way, consumption for the agent f is given by:

$$c_t^{f,f,*} = \frac{Z_t^f B_t^f \alpha_t^f \gamma^f}{y^f Z_t^{\theta^f}}, \tag{G.28}$$

$$c_t^{f,d,*} = \frac{Z_t^f B_t^f \alpha_t^f (1-\gamma^f) q_t}{y^f Z_t^{\theta^f}}, \tag{G.29}$$

where α_t^f and γ^f are the subjective belief and preference for domestic versus foreign goods for agent f, respectively.

Finally, since the constraint must be bind, y^d is determined by the following relationship:

$$y^d = \frac{1 - e^{-\beta^d T}}{\beta^d x_0^d}. \tag{G.30}$$

□

As seen in Section 2, the solution $c_t^{d,d,*}$, $c_t^{d,f,*}$, ν_t^d and y^d is also a solution of the original problem (G.1), (G.2), and (G.3). Also, the equilibrium interest rate r_t^d and market price of risk θ_t^d can be derived from the following Corollary 2.

Corollary 2. *Equilibrium interest rate r_t^d and market price of risk θ_t^d are given by:*

$$\begin{aligned} r_t^d &= (\mu_{\delta,t}^d - |\sigma_{\delta,t}^d|^2) \\ &+ \sum_{j=d,f} \frac{\left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j} \right) Z_t^j}{\sum_{j=d,f} \left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j} \right) Z_t^j} \beta_t^j + \sigma_{\delta,t}^d \cdot \sum_{j=d,f} \frac{\left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j} \right) Z_t^j}{\sum_{j=d,f} \left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j} \right) Z_t^j} \hat{\lambda}_{j,t}, \end{aligned} \tag{G.31}$$

$$\theta_t^d = \sigma_{\delta,t}^d - \sum_{j=d,f} \frac{\left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j} \right) Z_t^j}{\sum_{j=d,f} \left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j} \right) Z_t^j} \hat{\lambda}_{j,t}. \tag{G.32}$$

Proof. From market clearing conditions, we have:

$$c_t^{d,d} + c_t^{f,d} = \delta_t^d, \tag{G.33}$$

$$c_t^{d,f} + c_t^{f,f} = \delta_t^f. \tag{G.34}$$

The relationship between $Z_t^{\theta^i}/B_t^i$ in domestic ($i = d$) and foreign ($i = f$) countries is given by:

$$q_t = q_0 \exp\left[\int_0^t \left(r_s^d - r_s^f + (\theta_s^d - \theta_s^f)\cdot\theta_s^d - \frac{1}{2}|\theta_s^d - \theta_s^f|^2\right) ds + \int_0^t (\theta_s^d - \theta_s^f)\cdot dW_s\right] \tag{G.35}$$

$$= q_0 \frac{B_t^d}{Z_t^{\theta^d}} \frac{Z_t^{\theta^f}}{B_t^f}. \tag{G.36}$$

By substituting optimal consumption and (G.36) into the market-clearing conditions (G.33) and (G.34), we obtain the following relationship:

$$\frac{Z_t^{\theta^d}}{B_t^d} = \frac{1}{\delta_t^d}\left[\frac{\gamma^d \alpha_t^d Z_t^d}{y^d} + \frac{(1-\gamma^f) q_0^d \alpha_t^f Z_t^f}{y^f}\right];\ dZ_t^{\theta^d} = -Z_t^{\theta^d}\theta^d\cdot dW_t;\ Z_0^{\theta^d} = 1, \tag{G.37}$$

$$\frac{Z_t^{\theta^f}}{B_t^f} = \frac{1}{\delta_t^f}\left[\frac{\gamma^f \alpha_t^f Z_t^f}{y^f} + \frac{(1-\gamma^d) q_0^f \alpha_t^d Z_t^d}{y^d}\right];\ dZ_t^{\theta^f} = -Z_t^{\theta^f}\theta^f\cdot dW_t;\ Z_0^{\theta^f} = 1, \tag{G.38}$$

$$dZ_t^i = Z_t^i \hat{\lambda}_i \cdot dW_t;\ Z_0^i = 1;\ \lambda^i = \hat{\lambda}_i \oplus \hat{\lambda}_i^{\perp};\ \hat{\lambda}_i \in range(\sigma^{i\top});\ i = d, f, \tag{G.39}$$

where $q_0^d = q_0$ and $q_0^f = 1/q_0$. Since $Z_0^{\theta^i} = 1$ at $t = 0$, the following equations must be hold:

$$\delta_0^d = \left[x_0^d \frac{\beta^d \gamma^d}{(1 - e^{-\beta^d T})} + x_0^f \frac{\beta^f (1-\gamma^f) q_0}{(1 - e^{-\beta^f T})}\right] B_0^d, \tag{G.40}$$

$$\delta_0^f = \left[x_0^d \frac{\beta^d (1-\gamma^d)}{(1 - e^{-\beta^d T}) q_0} + x_0^f \frac{\beta^f \gamma^f}{(1 - e^{-\beta^f T})}\right] B_0^f. \tag{G.41}$$

Given δ_0^i, β^i, γ^i, x_0^i ($i = d, f$), and q_0, the values of B_0^d and B_0^f must satisfy these equations. Alternatively, if δ_0^i, β^i, γ^i, B_0^i ($i = d, f$), and q_0 are given, then x_0^d and x_0^f should satisfy these equations.

Differentiating the state price density processes in (G.37), we obtain:

$$d\left(\frac{Z_t^{\theta^d}}{B_t^d}\right) = \left(\frac{Z_t^{\theta^d}}{B_t^d}\right)[-r_t^d dt - \theta_t^d \cdot dW_t], \tag{G.42}$$

$$d\left(\frac{\gamma^d \alpha_t^d Z_t^d}{y^d \delta_t^d} + \frac{(1-\gamma^f) q_0^d \alpha_t^f Z_t^f}{y^f \delta_t^d}\right)$$

$$= \sum_{j=d,f}\left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j \delta_t^d}\right) Z_t^j [\{-\mu_{\delta,t}^d + |\sigma_{\delta,t}^d|^2 - \hat{\lambda}_{j,t}\cdot\sigma_{\delta,t}^d - \beta_t^j\} dt + \{\hat{\lambda}_{j,t} - \sigma_{\delta,t}^d\}\cdot dW_t], \tag{G.43}$$

where $\gamma_j^d = \gamma^d$ ($\gamma_j^d = 1-\gamma^f$) when $j = d$ ($j = f$), and $q_j^d = 1$ ($q_j^d = q_0^d$) when $j = d$ ($j = f$). Comparing the (G.42) and (G.43), we derive expressions for the equilibrium interest rate and market price of risk:

$$r_t^d = (\mu_{\delta,t}^d - |\sigma_{\delta,t}^d|^2) + \sum_{j=d,f} \frac{\left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j}\right) Z_t^j}{\sum_{j=d,f}\left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j}\right) Z_t^j} \beta_t^j + \sigma_{\delta,t}^d \cdot \sum_{j=d,f} \frac{\left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j}\right) Z_t^j}{\sum_{j=d,f}\left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j}\right) Z_t^j} \hat{\lambda}_{j,t}, \tag{G.44}$$

$$\theta_t^d = \sigma_{\delta,t}^d - \sum_{j=d,f} \frac{\left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j}\right) Z_t^j}{\sum_{j=d,f}\left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j}\right) Z_t^j} \hat{\lambda}_{j,t}. \tag{G.45}$$

□

Furthermore, we derive the equilibrium volatility of the foreign exchange rate:

$$\sigma_t^{q\top} = \sigma_{\delta,t}^d - \sigma_{\delta,t}^f - \left\{ \sum_{j=d,f} \frac{\left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j}\right) Z_t^j}{\sum_{j=d,f}\left(\frac{\alpha_t^j \gamma_j^d q_j^d}{y^j}\right) Z_t^j} \hat{\lambda}_{j,t} \right\} + \left\{ \sum_{j=d,f} \frac{\left(\frac{\alpha_t^j \gamma_j^f q_j^f}{y^j}\right) Z_t^j}{\sum_{j=d,f}\left(\frac{\alpha_t^j \gamma_j^f q_j^f}{y^j}\right) Z_t^j} \hat{\lambda}_{j,t} \right\}. \tag{G.46}$$

Finally, we obtain the equilibrium stock price S_t^d by the following Corollary 3.

Corollary 3. *The equilibrium stock price S_t^d is given by:*

$$S_t^d = \delta_t^d \frac{\frac{(1-e^{-\beta^d(T-t)})}{\beta^d} K_t^{d,d} Z_t^d + \frac{(1-e^{-\beta^f(T-t)})}{\beta^f} K_t^{d,f} Z_t^f}{K_t^{d,d} Z_t^d + K_t^{d,f} Z_t^f}, \tag{G.47}$$

where $K_t^{d,d}$ and $K_t^{d,f}$ are defined as follows:

$$K_t^{d,d} := \frac{x_0^d \gamma^d \beta^d e^{-\beta^d t}}{(1-e^{-\beta^d T})}, \tag{G.48}$$

$$K_t^{d,f} := \frac{q_0 x_0^f (1-\gamma^f) \beta^f e^{-\beta^f t}}{(1-e^{-\beta^f T})}. \tag{G.49}$$

Proof. Since $S_t^d Z_t^{\theta^d}/B_t^d + \int_0^t \delta_s^d Z_s^{\theta^d}/B_s^d ds$ is a martingale and $S_T^d = 0$, we obtain the following relationship:

$$S_t^d \frac{Z_t^{\theta^d}}{B_t^d} + \int_0^t \delta_s^d \frac{Z_s^{\theta^d}}{B_s^d} ds = \mathbf{E}_t \left[S_T^d \frac{Z_T^{\theta^d}}{B_T^d} + \int_0^T \delta_s^d \frac{Z_s^{\theta^d}}{B_s^d} ds \right]. \tag{G.50}$$

Thus, we can express the stock price S_t^d as follows:

$$S_t^d = \frac{B_t^d}{Z_t^{\theta^d}} \mathbf{E}_t \left[\int_t^T \frac{Z_s^{\theta^d}}{B_s^d} \delta_s^d ds \right]. \tag{G.51}$$

To calculate the stock price, we utilize the following relationships:

$$\frac{Z_t^{\theta^d}}{B_t^d} = \frac{1}{\delta_t^d} \left[\frac{\gamma^d \alpha_t^d Z_t^d}{y^d} + \frac{(1-\gamma^f) q_0^d \alpha_t^f Z_t^f}{y^f} \right] = \frac{1}{\delta_t^d} \frac{y^f(\gamma^d \alpha_t^d Z_t^d) + y^d((1-\gamma^f) q_0^d \alpha_t^f Z_t^f)}{y^d y^f}, \tag{G.52}$$

$$\frac{B_t^d}{Z_t^{\theta^d}} = \delta_t^d \frac{y^d y^f}{y^f(\gamma^d \alpha_t^d Z_t^d) + y^d((1-\gamma^f) q_0^d \alpha_t^f Z_t^f)}, \tag{G.53}$$

$$\mathbf{E}_t \left[Z_s^i \right] = Z_t^i; \ s \geq t, \ i = d, f. \tag{G.54}$$

By substituting (G.52) and (G.53) into (G.51), we obtain the expression of the stock price in equilibrium as follows:

$$S_t^d = \delta_t^d \frac{\mathbf{E}_t\left[\int_t^T y^f \gamma^d \alpha_s^d Z_s^d ds\right] + \mathbf{E}_t\left[\int_t^T y^d (1-\gamma^f) q_0^d \alpha_s^f Z_s^f ds\right]}{y^f(\gamma^d \alpha_t^d Z_t^d) + y^d((1-\gamma^f) q_0^d \alpha_t^f Z_t^f)}. \tag{G.55}$$

$$= \delta_t^d \frac{y^f \gamma^d \left(\frac{e^{-\beta^d t} - e^{-\beta^d T}}{\beta^d}\right) Z_t^d + y^d (1-\gamma^f) q_0^d \left(\frac{e^{-\beta^f t} - e^{-\beta^f T}}{\beta^f}\right) Z_t^f}{y^f(\gamma^d \alpha_t^d Z_t^d) + y^d((1-\gamma^f) q_0^d \alpha_t^f Z_t^f)}. \tag{G.56}$$

$$= \delta_t^d \frac{\frac{(1-e^{-\beta^d (T-t)})}{\beta^d} K_t^{d,d} Z_t^d + \frac{(1-e^{-\beta^f (T-t)})}{\beta^f} K_t^{d,f} Z_t^f}{K_t^{d,d} Z_t^d + K_t^{d,f} Z_t^f}. \tag{G.57}$$

□

Also, since this model can express S_t^d without complex integral or expectation unlike (56), we can derive the volatility of the stock directly as follows:

$$S_t^d = \delta_t^d \frac{\frac{(1-e^{-\beta^d (T-t)})}{\beta^d} K_t^{d,d} Z_t^d + \frac{(1-e^{-\beta^f (T-t)})}{\beta^f} K_t^{d,f} Z_t^f}{K_t^{d,d} Z_t^d + K_t^{d,f} Z_t^f} \equiv \delta_t^d Z_t^{d,f}, \tag{G.58}$$

with

$$Z_t^{d,f} = \frac{\frac{(1-e^{-\beta^d (T-t)})}{\beta^d} K_t^{d,d} Z_t^d + \frac{(1-e^{-\beta^f (T-t)})}{\beta^f} K_t^{d,f} Z_t^f}{K_t^{d,d} Z_t^d + K_t^{d,f} Z_t^f}. \tag{G.59}$$

Note that the volatility of $Z_t^{d,f}$, denoted by $\sigma_t^{Z^{d,f}}$, is given by

$$\sigma_t^{Z^{d,f}} = \left(\frac{(1-e^{-\beta^d (T-t)})}{\beta^d} - Z_t^{d,f}\right)\left(\frac{K_t^{d,d} Z_t^d}{K_t^{d,d} Z_t^d + K_t^{d,f} Z_t^f}\right)\hat{\lambda}_{d,t}$$
$$+ \left(\frac{(1-e^{-\beta^f (T-t)})}{\beta^f} - Z_t^{d,f}\right)\left(\frac{K_t^{d,f} Z_t^f}{K_t^{d,d} Z_t^d + K_t^{d,f} Z_t^f}\right)\hat{\lambda}_{f,t}. \tag{G.60}$$

Hence, as $S_t^d = \delta_t^d Z_t^{d,f}$, by comparing the diffusion terms of dS_t^d and $d(\delta_t^d Z_t^{d,f})$, we obtain:

$$S_t^d \sigma_{S,t}^{d\top} = \delta_t^d Z_t^{d,f} \sigma_{\delta,t}^d + \delta_t^d \sigma_t^{Z^{d,f}}. \tag{G.61}$$

Dividing both sides by S_t^d, we obtain the stock volatility as follows:

$$\sigma_{S,t}^{d\top} = \sigma_{\delta,t}^d + \frac{1}{Z_t^{d,f}} \sigma_t^{Z^{d,f}}. \tag{G.62}$$

Thus, we can express the volatility of the stock price S_t^d as follows:

$$\sigma_{S,t}^d = \sigma_t^{\delta,d\top} + \frac{1}{Z_t^{d,f}}\left(\frac{(1-e^{-\beta^d (T-t)})}{\beta^d} - Z_t^{d,f}\right)\left(\frac{K_t^{d,d}}{K_t^{d,d} Z_t^d + K_t^{d,f} Z_t^f}\right) Z_t^d \hat{\lambda}_{d,t}^\top$$
$$+ \frac{1}{Z_t^{d,f}}\left(\frac{(1-e^{-\beta^f (T-t)})}{\beta^f} - Z_t^{d,f}\right)\left(\frac{K_t^{d,f}}{K_t^{d,d} Z_t^d + K_t^{d,f} Z_t^f}\right) Z_t^f \hat{\lambda}_{f,t}^\top. \tag{G.63}$$

Similarly, the volatility of S_t^f is as follows:

$$\sigma_{S,t}^f = \sigma_t^{\delta,f\top} + \frac{1}{Z_t^{f,d}} \left(\frac{(1 - e^{-\beta^d (T-t)})}{\beta^d} - Z_t^{f,d} \right) \left(\frac{K_t^{f,d}}{K_t^{f,d} Z_t^d + K_t^{f,f} Z_t^f} \right) Z_t^d \hat{\lambda}_{d,t}^\top$$
$$+ \frac{1}{Z_t^{f,d}} \left(\frac{(1 - e^{-\beta^f (T-t)})}{\beta^f} - Z_t^{f,d} \right) \left(\frac{K_t^{f,f}}{K_t^{f,d} Z_t^d + K_t^{f,f} Z_t^f} \right) Z_t^f \hat{\lambda}_{f,t}^\top. \qquad \text{(G.64)}$$