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Masaaki Fujii
Graduate School of Economics, The University of Tokyo

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Abstract

This work solves the equilibrium price formation problem for the risky stock by combining mean-field game theory with the binomial tree framework, following the classic approach of Cox, Ross & Rubinstein. For agents with exponential and recursive utilities of exponential-type, we prove the existence of a unique mean-field market-clearing equilibrium and derive an explicit analytic formula for equilibrium transition probabilities of the stock price on the binomial lattice. The agents face stochastic terminal liabilities and incremental endowments that depend on unhedgeable common and idiosyncratic factors, in addition to the stock price path. We also incorporate an external order flow. Furthermore, the analytic tractability of the proposed approach allows us to extend the framework in two important directions: First, we incorporate multi-population heterogeneity, allowing agents to differ in functional forms for their liabilities, endowments, and risk coefficients. Second, we relax the rational expectations hypothesis by modeling agents operating under subjective probability measures which induce stochastically biased views on the stock transition probabilities. Our numerical examples illustrate the qualitative effects of these components on the equilibrium price distribution.

Keywords: mean-field game, subjective measure, non-rational agents , multiple populations, recursive utility, market-clearing

1 Introduction

The mean-field game (MFG) theory was pioneered independently by the seminal works of Lasry & Lions [33, 34, 35] and by those of Huang et al. [27, 28, 29] in the mid- to late-2000s. These works are based on analytic methods using coupled Hamilton-Jacobi-Bellman (HJB) and Kolmogorov equations. Subsequently, the probabilistic approach to the MFG theory, based on forward-backward stochastic differential equations (FBSDEs) of McKean-Vlasov type, was established by Carmona & Delarue [9, 10]. The two volumes by the same authors, [11] and [12], provide full details on the probabilistic approach and its applications.

The greatest advantage of MFG theory is its ability to transform a complex problem of stochastic differential games among many agents into a standard optimization and fixed-point problem. A growing body of literature attempts to solve many-agent problems using the MFG technique. The MFG theory requires, in principle, symmetric interactions among the agents. A particularly large number of MFG applications can be found in financial and energy markets because symmetric interactions are standard in these settings. There are also many economic applications of

*mfujii@e.u-tokyo.ac.jp, Graduate School of Economics, The University of Tokyo, Tokyo, Japan

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mean field games, in particular, those focusing on general equilibrium models of growth, inequality and unemployment, dynamic demand response and persuasion problems among others. See, for example, [1, 2, 3, 4, 5, 22] and the references therein.

In recent years, there have also been major advances in MFG theory for applications in the equilibrium price-formation problem, where the asset price process is endogenously constructed to ensure that supply and demand always balance among the heterogeneous but exchangeable agents, rather than being exogenously given. Gomes & Saúde [24] present a deterministic model of electricity price. Its extension with random supply is given by Gomes et al.[23]. Ashrafyan et al. [6] propose a duality approach transforming these problems into variational ones that are numerically more tractable. Shrivats et al. [40] deal with a price formation problem for the solar renewable energy certificate (SREC) by solving FBSDEs of McKean-Vlasov type. Féron et al. [15] develop a tractable equilibrium model for intraday electricity markets. Sarto et al. [37] study cap-and-trade pollution regulation and derive the equilibrium price for the carbon emission. Regarding the price formation of securities, Fujii & Takahashi [19] show that the equilibrium price process can be characterized by FBSDEs of conditional McKean-Vlasov type. Its strong convergence to the mean-field limit from a finite-agent setting is proved in [20], and its extension to the presence of a major player is given in [21] by the same authors. Fujii [16] develops a model that allows the co-presence of cooperative and non-cooperative populations to learn how the price process is formed when the agents in one population act in a coordinated manner.

There remain two mathematical limitations in the above results: firstly, the relevant control of each agent is the trading rate and hence their asset position is constrained to follow an absolutely continuous process with respect to the Lebesgue measure dt; secondly, the cost function of each agent consists of penalties on the trading speed and on the inventory size of the assets. Therefore, the above frameworks cannot deal with the general self-financing trading strategies nor with the utility functions defined directly in terms of the wealth process of the portfolio. A major obstacle in dealing with a utility function of the wealth resides in the difficulty of guaranteeing the convexity of the Hamiltonian associated with the Pontryagin's maximum principle and in the difficulty of getting enough regularity to prove the well-posedness of the associated FBSDEs. These problems are solved by Fujii & Sekine [17, 18] for the agents with exponential utilities by applying the method of Hu, Imkeller & Müller [26] based on the martingale optimality principle. An extension to the setting with partial information is studied by Sekine [38]. They find that a novel quadratic-growth backward SDE (qg-BSDE) of conditional McKean-Vlasov type characterizes the equilibrium risk-premium process. Unfortunately, however, the existence of the solution of this mean-field qg-BSDE has been proved only under rather restrictive conditions. This is because the conditional McKean-Vlasov nature of the equation makes the classical approach of Kobylanski [32] no longer applicable. Moreover, even if the well-posedness of the equation were to be completely solved, its numerical evaluation would remain prohibitively difficult. This is a major hurdle for practical applications, a limitation shared by all existing works of continuous-time framework.

In this work, we study the price-formation problem for the risky stock. To understand how the equilibrium price process changes its behavior due to the market environment and the differences in the distribution of characteristics among the agents, it is necessary to obtain more explicit and numerically tractable solutions than those in the existing literature. To this end, we propose a framework that combines the MFG theory with the classical idea of binomial trees, initiated by Sharpe [39] and formalized in Cox, Ross & Rubinstein [13]. By restricting the stock price trajectories onto a binomial lattice, we are able to search for an appropriate set of transition probabilities that clears the stock market in a straightforward way. This discrete-time, tree-based approach is the key technical device that allows us to incorporate more general market environments without sacrificing the tractability of the equilibrium solution.

The agents in our model—financial and investment firms—have standard exponential utilities or recursive utilities of the exponential type. We allow for general stochastic terminal liabilities and incremental endowments that depend on unhedgeable common factors Y and idiosyncratic factors Z^i, as well as the stock price path. The incremental endowments represent non-tradable income from a firm's other business activities. We also analyze the impact of external order flow from outside groups or a major financial player. The presence of unhedgeable risk factors (Y, Z^i) in these financial items creates an incomplete market for the agents, motivating their demand for the risky stock to hedge their *coupled* exposures.

The explicit solutions obtained through our binomial tree approach provide significant analytical and numerical tractability, allowing us to extend the framework in two important ways that would be a formidable challenge in the continuous-time setting:

- Addressing Market Heterogeneity: We readily extend the model to a multi-population equilibrium (Section 4), capturing the structural diversity of the financial sector by allowing different populations to have distinct functional forms for liabilities, endowments, and risk characteristics.
- Relaxing Rational Expectations: We generalize the model to agents operating under subjective probability measures (Section 5), which induce stochastically biased state-dependent views on the stock price transition probabilities. This extension is in the spirit of the recent development in MFG theory initiated by Moll & Ryzhik [36], and relaxes the standard rational expectation hypothesis implicitly adopted in most of the MFG literature.

Our explicit solutions for the equilibrium transition probabilities enable us to numerically evaluate the marginal and conditional equilibrium price distributions with respect to the common macroeconomic factors Y. Our numerical examples reveal the qualitative behaviors of the equilibrium distributions to the agents' characteristics and the above introduced elements.

The analytic tractability achieved via our binomial tree framework is the key contribution that overcomes the theoretical and numerical difficulties in the continuous-time approaches, which involve either a (coupled systems of) complex McKean-Vlasov FBSDEs (or qg-BSDEs), or a coupled system of HJB and Kolmogorov equations. Beyond theoretical advancement, our methodology offers a readily implementable framework for practical applications. It provides the equilibrium excess return required to compensate agents for the risk in the stock position, necessary to clear the market. The ability to obtain such explicit solutions offers a valuable tool for regulatory bodies performing market stability analysis under various agent-based scenarios.

We structure the rest of the paper as follows: Section 2 investigates the mean-field equilibrium among the agents with exponential utilities subject to terminal liabilities. Section 3 deals with an extension to recursive utilities and also introduces cash spending (i.e., nominal consumption) and incremental endowments. Section 4 studies an extension to the presence of multiple populations. Section 5 extends the previous frameworks to incorporate agents operating under respective subjective measures which induces state-dependent stochastic biases. Section 6 provides several illustrative numerical examples and examines their important implications. Section 7 summarizes our findings and discusses possible directions for future research. [1]

[1] For the sake of gender-neutrality and brevity, the singular pronoun 'they' (and its forms) is used throughout the paper when referring to an anonymous agent or entity (e.g., agent-i).

2 Exponential utility with terminal liability

We begin our investigation into the mean-field price formation problem by considering a simple model with a countably infinite number of agents possessing exponential utility. These agents are optimizing their wealth with terminal liability by carrying out self-financing trading strategies on a deterministic money market account and a single risky stock. Each agent must manage financial risk arising from the common market shocks as well as their own idiosyncratic shocks. Notably, our model incorporates non-tradable macroeconomic and/or environmental shocks that affect all the agents, in addition to the shocks from the stock price process.

Remark 2.1 (On the stock). *In this work, the financial institution's overall equity exposure is simplified to a single asset which serves as a proxy for a broad, diversified market index (e.g., S&P 500) to keep the focus on the mean-field price formation problem.*

2.1 The setup and notation

Let us start from explaining the relevant probability spaces. $(\Omega^0, \mathcal{F}^0, (\mathcal{F}^0_{t_n})_{n=1}^N, \mathbb{P}^0)$ is a complete filtered probability space, where $0 = t_0 < t_1 < \cdots < t_N = T$ is an equally spaced time sequence using a time step $\Delta := T/N$ where $T < \infty$ and $N \in \mathbb{N}$ are given constants. The filtration $(\mathcal{F}^0_{t_n})_{n=0}^N$ is generated by two stochastic processes, one is a strictly positive process $(S_n := S(t_n))_{n=0}^N$ and the other is a d_Y-dimensional process $(Y_n = Y(t_n))_{n=0}^N$, i.e. $\mathcal{F}^0_{t_n} := \sigma\{S_k, Y_k, 0 \le k \le n\}$. In the model below, we shall use S_n to denote the stock price at t_n and Y_n the common shocks affecting all the agents at t_n. $S_0 > 0$ and $Y_0 \in \mathbb{R}^{d_Y}$ are given constants and thus $\mathcal{F}^0_0$ is trivial.

In addition to the above space, we consider a countably infinite number of complete filtered probability spaces $(\Omega^i, \mathcal{F}^i, (\mathcal{F}^i_{t_n})_{n=0}^N, \mathbb{P}^i),\ i \in \mathbb{N}$. For each i, $(\Omega^i, \mathcal{F}^i, (\mathcal{F}^i_{t_n})_{n=0}^N, \mathbb{P}^i)$ is endowed with $\mathcal{F}^i_0$-measurable random variables (ξ_i, γ_i) as well as $(\mathcal{F}^i_{t_n})_{n=0}^N$-adapted stochastic process $(Z^i_n = Z^i(t_n))_{n=0}^N$. Here, ξ_i, γ_i are both $\mathbb{R}$-valued and ξ_i is used to represent the initial wealth and γ_i the size of absolute risk aversion of agent-i. The d_Z-dimensional process $(Z^i_n)_{n=0}^N$ is used to model idiosyncratic shocks to each agent. The fact that (ξ_i, γ_i) are $\mathcal{F}^i_0$-measurable means that the agent-i knows their initial wealth and the size of risk aversion at time zero.

By the standard procedures (see, for example, Klenke [31, Chapter 14]), the complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_{t_n})_{n=0}^N, \mathbb{P})$ is defined as the product of all the above spaces

$$(\Omega, \mathcal{F}, (\mathcal{F}_{t_n})_{n=0}^N, \mathbb{P}) := (\Omega^0, \mathcal{F}^0, (\mathcal{F}^0_{t_n})_{n=0}^N, \mathbb{P}^0) \otimes_{i=1}^\infty (\Omega^i, \mathcal{F}^i, (\mathcal{F}^i_{t_n})_{n=0}^N, \mathbb{P}^i)$$

which denotes the full probability space containing the entire environment of our model. Therefore, by construction, $((S_n), (Y_n))$ and $(\xi_i, \gamma_i, (Z^i_n)),\ i \in \mathbb{N}$ are mutually independent. On the other hand, the relevant probability space for each agent-i is the product probability space defined by

$$(\Omega^{0,i}, \mathcal{F}^{0,i}, (\mathcal{F}^{0,i}_{t_n})_{n=0}^N, \mathbb{P}^{0,i}) := (\Omega^0, \mathcal{F}^0, (\mathcal{F}^0_{t_n})_{n=0}^N, \mathbb{P}^0) \otimes (\Omega^i, \mathcal{F}^i, (\mathcal{F}^i_{t_n})_{n=0}^N, \mathbb{P}^i),$$

which reflects our assumption that common shocks are public knowledge, but the idiosyncratic shocks are private to each agent. We shall use the same symbols, such as $(S_n, Y_n, \gamma_i, \cdots)$, if they are defined as trivial extensions on larger product probability spaces. Expectations with respect to $\mathbb{P}^0$, $\mathbb{P}^i$, $\mathbb{P}^{0,i}$ and $\mathbb{P}$ are denoted by $\mathbb{E}^0[\cdot]$, $\mathbb{E}^i[\cdot]$, $\mathbb{E}^{0,i}[\cdot]$ and $\mathbb{E}[\cdot]$, respectively. We also denote by $\overline{E}[\cdot]$ the expectation with respect to the product measure $\otimes_{i=1}^\infty \mathbb{P}^i$.

In this work, we restrict the trajectories of $(S_n)_{n=0}^N$ onto a recombining binomial tree. For each

$n = 1, \cdots, N$, the random variable $\widetilde{R}_n := S_n/S_{n-1}$ takes only the two possible values, either $\widetilde{u}$ or $\widetilde{d}$. This means that the set of all possible values taken by S_n is given by $\mathcal{S}_n := \{S_0\widetilde{u}^k\widetilde{d}^{n-k}, 0 \le k \le n\}$, which is a finite subset of $(0,\infty)$. Let $\mathcal{S}^n := \mathcal{S}_0 \times \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_n$ be the set of all values taken by the stock price trajectory $(S_k)_{k=0}^n$. Moreover, in order to avoid technicalities regarding the conditional probabilities, we also assume that the process Y takes only a finite number of values at every t_n. We use $\mathcal{Y}_n$, which is a finite subset of $\mathbb{R}^{d_Y}$, to denote the set of all values taken by Y_n. We also use $\mathcal{Y}^n := \mathcal{Y}_0 \times \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n \subset \mathbb{R}^{d_Y \times (n+1)}$ for each $0 \le n \le N$, to represent the set of all values taken by the trajectory $(Y_k)_{k=0}^n$. In a similar manner, we denote the support of the random variable Z_n^i by $\mathcal{Z}_n$ and define $\mathcal{Z}^n := \mathcal{Z}_0 \times \mathcal{Z}_1 \times \cdots \times \mathcal{Z}_n$ as the support of $(Z_k^i)_{k=0}^n$. There is no need to impose the restriction of finite state space on variables other than (S,Y). The time-t_n value of the risk-free money market account is given by $\exp\bigl(rn\Delta\bigr)$ where $r > 0$ is a positive constant denoting the risk-free (nominal) interest rate. We also use the symbol $\beta := \exp(r\Delta)$. For later use, let us also define

$$u := \widetilde{u} - \exp(r\Delta), \quad d := \widetilde{d} - \exp(r\Delta)$$

and

$$R_n := \widetilde{R}_n - \exp(r\Delta),\ 1 \le n \le N.$$

The random variable R_n takes the values either u or d.

In the following, to lighten the notational burden, we use $\mathbb{E}^{0,i}\bigl[\,\cdot\,|s,y,z^i,\gamma_i\bigr]$ to denote $\mathbb{E}^{0,i}\bigl[\,\cdot\,|S_{n-1} = s, Y_{n-1} = y, Z_{n-1}^i = z^i, \gamma_i = \gamma_i\bigr]$ for $(s,y,z^i) \in \mathcal{S}_{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1}$. With a slight abuse of notation, we shall use the same symbols for the realizations of $\mathcal{F}_0^i$-measurable random variables (γ_i in the above example).

Assumption 2.1. (i): *$\widetilde{u}$ and $\widetilde{d}$ are real constants satisfying $0 < \widetilde{d} < \exp(r\Delta) < \widetilde{u} < \infty$.*
(ii): *Every $(\xi_i, \gamma_i, (Z_n^i)_{n=0}^N)$ is identically distributed across all agents $i = 1, 2, \ldots$.*
(iii): *There exist constants $\underline{\xi}, \overline{\xi}$ and $\underline{\gamma}, \overline{\gamma}$ so that for every $i \in \mathbb{N}$,*

$$\xi_i \in [\underline{\xi}, \overline{\xi}] \subset \mathbb{R}, \quad \gamma_i \in \Gamma := [\underline{\gamma}, \overline{\gamma}] \subset (0,\infty).$$

(iv): *For every $0 \le n \le N$, $\mathcal{Z}_n$ is a bounded subset of $\mathbb{R}^{d_Z}$.*
(v): *For each i, $(Z_n^i)_{n=0}^N$ is a Markov process i.e., $\mathbb{E}^i[f(Z_n^i)|\mathcal{F}_{t_k}^i] = \mathbb{E}^i[f(Z_n^i)|Z_k^i]$ for every bounded measurable function f on $\mathcal{Z}_n$ and $k \le n$.*
(vi): *$(Y_n)_{n=0}^N$ is a Markov process i.e., $\mathbb{E}^0[f(Y_n)|\mathcal{F}_{t_k}^0] = \mathbb{E}^0[f(Y_n)|Y_k]$ for every bounded measurable function f on $\mathcal{Y}_n$ and $k \le n$.*
(vii): *The transition probabilities of $(S_n)_{n=0}^N$ satisfy, for every $0 \le n \le N-1$,*

$$\mathbb{P}^0(S_{n+1} = \widetilde{u}S_n|\mathcal{F}_{t_n}^0) = \mathbb{P}^0(S_{n+1} = \widetilde{u}S_n|S_n, Y_n) =: p_n(S_n, Y_n),$$
$$\mathbb{P}^0(S_{n+1} = \widetilde{d}S_n|\mathcal{F}_{t_n}^0) = \mathbb{P}^0(S_{n+1} = \widetilde{d}S_n|S_n, Y_n) =: q_n(S_n, Y_n),$$

where $p_n, q_n\ (= 1 - p_n) : \mathcal{S}_n \times \mathcal{Y}_n \to \mathbb{R},\ 0 \le n \le N-1$ are bounded measurable functions satisfying

$$0 < p_n(s,y), q_n(s,y) < 1$$

for every $(s,y) \in \mathcal{S}_n \times \mathcal{Y}_n$.

Let us give some remarks on the above assumptions. Firstly, by the condition (i), we have $d < 0 < u$. It is well-known that the transition probabilities under the risk-neutral measure $\mathbb{Q}$ for the classical binomial framework are given by $p^{\mathbb{Q}} := (-d)/(u-d) = (\exp(r\Delta) - \widetilde{d})/(\widetilde{u} - \widetilde{d})$ for the up-move and $q^{\mathbb{Q}} := 1 - p^{\mathbb{Q}}$ for the down-move. These probabilities are uniquely determined by the parameters $(\widetilde{u}, \widetilde{d})$ and the risk-free interest rate. In this paper, we fix the relative jump size $(\widetilde{u}, \widetilde{d})$ to be constant across all nodes; however this is done merely for simplicity. The entire discussion of our paper still holds even if $(\widetilde{u}, \widetilde{d})$ varies from node to node. In fact, the method works even for general non-recombining binomial trees. Moreover, thanks to the famous result by Derman & Kani (1994) [14], one can construct so-called "*implied binomial trees*" by adjusting the position $(\widetilde{u}, \widetilde{d})$ node by node to reproduce implied volatility surface in the option market, while keeping the recombining property of binomial trees intact. Therefore, if necessary, our discussion below can be applied to a binomial tree whose risk-neutral distribution is consistent with the option market. The boundedness assumption in (iii) and (iv) is not crucial. One can relax it by adding appropriate integrability conditions.

Our goal in this section is to find a set of transition probabilities of the form $\big(p_n(s,y)\big)_{n=0}^{N-1}$ so that the demand and supply of the stock are balanced among the agents at every node $(s,y) \in \mathcal{S}_n \times \mathcal{Y}_n, 0 \le n \le N-1$. Note that we can assume, without any loss of generality, that the process $(Y_n)_{n=0}^N$ and $(Z_n^i)_{n=0}^N$ are Markov, since, if necessary, we can recover Markovian property by lifting Y, Z^i to higher dimensional processes. However, the condition (vii) is not trivial. In fact, in the next section, we shall study more general situations where the transition probability must be dependent on the past history of the stock price to achieve the market-clearing equilibrium. Under the current condition (vi) and (vii), (S_{n+1}, Y_{n+1}) satisfy the property:

$$\mathbb{E}^0[f(S_{n+1})g(Y_{n+1})|\mathcal{F}^0_{t_n}] = \mathbb{E}^0[f(S_{n+1})|S_n, Y_n]\mathbb{E}^0[g(Y_{n+1})|Y_n], \quad 0 \le n \le N-1, \tag{2.1}$$

for any bounded measurable functions $f : \mathcal{S}_{n+1} \to \mathbb{R}$ and $g : \mathcal{Y}_{n+1} \to \mathbb{R}$. We can interpret that the process $(Y_n)_{n=0}^N$ represents some standalone macroeconomic and/or environmental factors which are not influenced by the agents' trading activities. It may naturally serve as a state process in regime switching models.

Remark 2.2. *Note that the bound for the transition probabilities in (vii) guarantees the equivalence of probability measures $\mathbb{P}^0 \circ S_n^{-1}$ and $\mathbb{Q} \circ S_n^{-1}$. Hence, our system is arbitrage free.*

2.2 The individual optimization problem

To obtain the market-clearing equilibrium of the risky stock among a large number of strategically interacting agents, we adopt the Mean-Field Game (MFG) technique. The standard MFG approach involves two steps: (i) we assume the stock price process (or transition probabilities in our case), which is formed by the collective actions of the agents and adapted to the public information $(\mathcal{F}^0_{t_n})_{n=0}^N$, is given and then solve each agent's optimization problem as a price taker; (ii) we solve the fixed-point problem of the consistency condition, (i.e., the market-clearing), to obtain the equilibrium price distribution. In this subsection, we address the optimization task of step (i).

We now formulate the optimization problem for each agent. Agent-i, for $i \in \mathbb{N}$, with an initial wealth ξ_i, engages in self-financing trading involving the risk-free money market account and a single risky stock. They adopt an $(\mathcal{F}^{0,i}_{t_n})_{n=0}^{N-1}$-adapted trading strategy $(\phi_n^i)_{n=0}^{N-1}$, representing the invested amount of cash in the stock at time t_n. The associated wealth process of agent-i, denoted by $(X^i_n = X^i(t_n))_{n=0}^N$, follows the dynamics

$$\begin{aligned} X^i_{n+1} &= \exp(r\Delta)(X^i_n - \phi^i_n) + \phi^i_n \widetilde{R}_{n+1} \\ &= \beta X^i_n + \phi^i_n R_{n+1}, \end{aligned}$$

where $X^i_0 = \xi_i$ and $\beta = \exp(r\Delta)$. Recall that $R_{n+1} := \widetilde{R}_{n+1} - \exp(r\Delta)$.

Each agent-i is supposed to solve the optimization problem:

$$\sup_{(\phi^i_n)_{n=0}^{N-1} \in \mathbb{A}^i} \mathbb{E}^{0,i}\Big[-\exp\Big(-\gamma_i\big(X^i_N - F(S_N, Y_N, Z^i_N)\big)\Big)|\mathcal{F}^{0,i}_0\Big], \tag{2.2}$$

where

$$\mathbb{A}^i := \big\{(\phi^i_n)_{n=0}^{N-1} : \phi^i_n \text{ is an } \mathcal{F}^{0,i}_{t_n}\text{-measurable real random variable}\big\}$$

denotes the admissible control space. Here, we assume that agent-i has full knowledge of the common market information and their own private idiosyncratic information, but no knowledge of the private idiosyncratic information of the other agents.

Assumption 2.2. (i): *$F : \mathcal{S}_N \times \mathcal{Y}_N \times \mathcal{Z}_N \to \mathbb{R}$ is a bounded measurable function.*
(ii): *Every agent is a price-taker in the sense that they consider the stock price process (and hence its transition probabilities specified in Assumption 2.1 (vii)) to be exogenously given by the collective actions of the others and unaffected by the agent's own trading strategies.*

$F(S_N, Y_N, Z^i_N)$ denotes the stochastic terminal liability (or the negative of the terminal endowment), depending on S_N, Y_N and Z^i_N. Given the exponential utility assumption, the constant shift $F \to F + c$ does not alter the optimization problem; hence, only the dependence on (S_N, Y_N, Z^i_N) in the function F is relevant. The condition (ii) is a plausible assumption since every agent naturally knows their individual trading share is negligible in the market. This negligibility of individual actions is a key assumption for the standard MFG technique, as previously remarked.

Remark 2.3. *Before solving the optimization problem, we provide some economic motivations for including the stochastic terminal liability $F(S_N, Y_N, Z^i_N)$. Since we primarily model various financial firms as our agents, it is natural to suppose that they are subject to stochastic liabilities (such as a portfolio of derivative contracts) dependent on the stock price. It is also plausible that the size of the liability varies from agent to agent based on their idiosyncratic factors (Z^i_N) as well as common macroeconomic/environmental factors (Y_N). This structure would naturally hold for non-financial firms, too.*

We now characterize the optimal trading strategy for each agent. Applying the well-known scheme of backward induction for discrete-time models, but now in the presence of common as well as idiosyncratic shocks, we establish the following result.

Theorem 2.1. *Let Assumptions 2.1 and 2.2 be in force. Then the problem* (2.2) *has a unique optimal solution $(\phi^{i,*}_{n-1})_{n=1}^N$, which is an a.s. bounded process defined by a measurable function*

$\phi^{i,*}_{n-1} : \mathcal{S}_{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma \to \mathbb{R}$ *such that* $\phi^{i,*}_{n-1} := \phi^{i,*}_{n-1}(S_{n-1}, Y_{n-1}, Z^i_{n-1}, \gamma_i)$*, where*

$$\begin{aligned} \phi^{i,*}_{n-1}(s,y,z^i,\gamma_i) &:= \frac{1}{\gamma_i(u-d)}\frac{\beta^n}{\beta^N}\Big\{\log\Big(-\frac{p_{n-1}(s,y)u}{q_{n-1}(s,y)d}\Big) + \log\big(f_{n-1}(s,y,z^i,\gamma_i)\big)\Big\}, \\ f_{n-1}(s,y,z^i,\gamma_i) &:= \frac{\mathbb{E}^{0,i}[V_n(s\widetilde{u}, Y_n, Z^i_n, \gamma_i)|y,z^i,\gamma_i]}{\mathbb{E}^{0,i}[V_n(s\widetilde{d}, Y_n, Z^i_n, \gamma_i)|y,z^i,\gamma_i]}. \end{aligned} \tag{2.3}$$

Here, $f_{n-1} : \mathcal{S}_{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma \to \mathbb{R}$ *and* $V_n : \mathcal{S}_n \times \mathcal{Y}_n \times \mathcal{Z}_n \times \Gamma \to \mathbb{R}$ *are a.s. strictly positive bounded measurable functions defined recursively for* $1 \le n \le N$ *by*

$$V_N(s,y,z^i,\gamma_i) := \exp\big(\gamma_i F(s,y,z^i)\big),$$

and

$$\begin{aligned} V_{n-1}(s,y,z^i,\gamma_i) := p_{n-1}(s,y)\exp\Big(-\gamma_i\frac{\beta^N}{\beta^n}\phi^{i,*}_{n-1}(s,y,z^i,\gamma_i)u\Big)\mathbb{E}^{0,i}[V_n(s\widetilde{u}, Y_n, Z^i_n, \gamma_i)|y,z^i,\gamma_i] \\ + q_{n-1}(s,y)\exp\Big(-\gamma_i\frac{\beta^N}{\beta^n}\phi^{i,*}_{n-1}(s,y,z^i,\gamma_i)d\Big)\mathbb{E}^{0,i}[V_n(s\widetilde{d}, Y_n, Z^i_n, \gamma_i)|y,z^i,\gamma_i]. \end{aligned} \tag{2.4}$$

The function $\log V_n$ can be interpreted as the continuation value of the liability at time t_n.

Proof. With $V_N : \mathcal{S}_N \times \mathcal{Y}_N \times \mathcal{Z}_N \times \Gamma \to \mathbb{R}$ defined by $V_N(s,y,z^i,\gamma_i) = \exp\big(\gamma_i F(s,y,z^i)\big)$, we temporarily suppose that the problem of agent-i at $t = t_{n-1}$, $1 \le n \le N$, is defined by

$$\sup_{\phi^i}\mathbb{E}^{0,i}\Big[-\exp\Big(-\gamma_i\frac{\beta^N}{\beta^n}X^i_n\Big)V_n(S_n, Y_n, Z^i_n, \gamma_i)|\mathcal{F}^{0,i}_{t_{n-1}}\Big], \tag{2.5}$$

with some a.s. strictly positive bounded measurable function $V_n : \mathcal{S}_n \times \mathcal{Y}_n \times \mathcal{Z}_n \times \Gamma \to \mathbb{R}$. Here, the supremum is taken over $\mathcal{F}^{0,i}_{n-1}$-measurable real random variables. Let us solve the above problem on the set $\{\omega^{0,i} \in \Omega^{0,i} : (X^i_{n-1}, S_{n-1}, Y_{n-1}, Z^i_{n-1}, \gamma_i) = (x^i, s, y, z^i, \gamma_i)\}$[2]. Then the above problem is equivalent to

$$\begin{aligned} &\inf_{\phi^i\in\mathbb{R}}\mathbb{E}^{0,i}\Big[\exp\Big(-\gamma_i\frac{\beta^N}{\beta^n}(\beta x^i + \phi^i R_n)\Big)V_n(s\widetilde{R}_n, Y_n, Z^i_n, \gamma_i)\big|x^i, y, z^i, \gamma_i\Big] \\ &= \exp\Big(-\gamma_i\frac{\beta^N}{\beta^{n-1}}x^i\Big)\inf_{\phi^i\in\mathbb{R}}\Big\{p_{n-1}(s,y)\exp\Big(-\gamma_i\frac{\beta^N}{\beta^n}\phi^i u\Big)\mathbb{E}^{0,i}[V_n(s\widetilde{u}, Y_n, Z^i_n, \gamma_i)|y,z^i,\gamma_i] \\ &\qquad + q_{n-1}(s,y)\exp\Big(-\gamma_i\frac{\beta^N}{\beta^n}\phi^i d\Big)\mathbb{E}^{0,i}[V_n(s\widetilde{d}, Y_n, Z^i_n, \gamma_i)|y,z^i,\gamma_i]\Big\}, \end{aligned}$$

where we have used the property given in Assumption 2.1 (vii). Since $d < 0 < u$, the optimal position $\phi^{i,*}$ is a.s. uniquely characterized by

$$\begin{aligned} &p_{n-1}(s,y)u\exp\Big(-\gamma_i\frac{\beta^N}{\beta^n}\phi^{i,*}u\Big)\mathbb{E}^{0,i}[V_n(s\widetilde{u}, Y_n, Z^i_n, \gamma_i)|y,z^i,\gamma_i] \\ &\qquad + q_{n-1}(s,y)d\exp\Big(-\gamma_i\frac{\beta^N}{\beta^n}\phi^{i,*}d\Big)\mathbb{E}^{0,i}[V_n(s\widetilde{d}, Y_n, Z^i_n, \gamma_i)|y,z^i,\gamma_i] = 0. \end{aligned}$$

[2]With a slight abuse of notation, we use the same symbols for the realizations of $\mathcal{F}^i_0$-measurable random variables.

This gives the results given in (2.3), which is well-defined and bounded since V_n is a strictly positive and bounded, and $0 < p_{n-1}, q_{n-1} < 1$ by our assumption.

It follows that the function V_{n-1} defined by (2.4) becomes once again a.s. strictly positive and bounded on $\mathcal{S}_{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma$. The value function at the next step t_{n-1} is now defined by

$$-\exp\Big(-\gamma_i \frac{\beta^N}{\beta^{n-1}} X^i_{n-1}\Big) V_{n-1}(S_{n-1}, Y_{n-1}, Z^i_{n-1}, \gamma_i)$$

and we have recovered the problem of the same form as in (2.5). Thus the repeat of the above procedures N times yields the desired conclusion. $\square$

2.3 Mean-field equilibrium under stochastic order flow

Our goal is to find a set of transition probability functions $p_n(s,y), (s,y) \in \mathcal{S}_n \times \mathcal{Y}_n, 0 \le n \le N-1$, that clears the market by matching demand and supply at every node. To ensure generality, we also incorporate an external stochastic order flow, $L_{n-1}(S_{n-1}, Y_{n-1})$, which represents the aggregate net stock supply per capita at each time t_{n-1}, for $1 \le n \le N$. The external order flow serves to model the aggregate contribution from other populations. In particular, it can be used to represent the aggregate net supply from individual investors, which is often difficult to be modeled by rigorous optimizations, or the supply from a major financial institution such as a central bank.

Assumption 2.3. *For every $1 \le n \le N$, $L_{n-1} : \mathcal{S}_{n-1} \times \mathcal{Y}_{n-1} \to \mathbb{R}$ is a bounded measurable function.*

Definition 2.1. *We say that the system is in the mean-field equilibrium if*

$$\lim_{N_p \to \infty} \frac{1}{N_p} \sum_{i=1}^{N_p} \phi^{i,*}_{n-1}(S_{n-1}, Y_{n-1}, Z^i_{n-1}, \gamma_i) = L_{n-1}(S_{n-1}, Y_{n-1}),$$

$\mathbb{P}$-a.s. for every $1 \le n \le N$ with $\phi^{i,*}_{n-1}$ defined by (2.3).

Since the idiosyncratic factors $(Z^i, \gamma_i), i \in \mathbb{N}$ are independent and identically distributed (i.i.d.), and also independent of the process (S, Y), the above condition for the mean-field equilibrium is equivalently expressed as

$$\mathbb{E}^1\big[\phi^{1,*}_{n-1}(s, y, Z^1_{n-1}, \gamma_1)\big] = L_{n-1}(s,y) \tag{2.6}$$

for every $(s,y) \in \mathcal{S}_{n-1} \times \mathcal{Y}_{n-1}$, $1 \le n \le N$. Under the mean-field equilibrium, the excess demand/supply per capita converges to zero as the population size tends to infinity. Our first main result is then established as follows. One can observe a conditional McKean-Vlasov nature, where the transition probabilities depend on the distribution of idiosyncratic factors conditioned on the realization of common shocks (S, Y), as expected from the result in [19, 17].

Theorem 2.2. *Let Assumptions 2.1, 2.2 and 2.3 be in force. Then there exists a unique mean-field equilibrium and the associated transition probabilities of the stock price are given by*

$$\begin{aligned} p_{n-1}(s,y) &:= \mathbb{P}^0\Big(S_n = \widetilde{u} S_{n-1} | (S_{n-1}, Y_{n-1}) = (s,y)\Big) \\ &= (-d)\Big/\Big\{u \exp\Big(\frac{1}{\mathbb{E}^1[1/\gamma_1]}\Big[\mathbb{E}^1\Big(\frac{\log(f_{n-1}(s,y,Z^1_{n-1},\gamma_1))}{\gamma_1}\Big) - (u-d)\frac{\beta^N}{\beta^n} L_{n-1}(s,y)\Big]\Big) - d\Big\}, \end{aligned} \tag{2.7}$$

for every $(s,y)\in\mathcal{S}_{n-1}\times\mathcal{Y}_{n-1}$, $1\le n\le N$. Under the above transition probabilities, the optimal strategy of agent-i is given by

$$\begin{aligned}\phi^{i,*}_{n-1}(s,y,Z^i_{n-1},\gamma_i)=\frac{1}{(u-d)}\frac{\beta^n}{\beta^N}\Big\{&\frac{\log f_{n-1}(s,y,Z^i_{n-1},\gamma_i)}{\gamma_i}-\frac{1/\gamma_i}{\mathbb{E}^1[1/\gamma_1]}\mathbb{E}^1\Big[\frac{\log f_{n-1}(s,y,Z^1_{n-1},\gamma_1)}{\gamma_1}\Big]\Big\}\\&+\frac{1/\gamma_i}{\mathbb{E}^1[1/\gamma_1]}L_{n-1}(s,y).\end{aligned}\tag{2.8}$$

Moreover, there exists some positive constant C_{n-1} such that

$$\mathbb{E}\Big|\frac{1}{N_p}\sum_{i=1}^{N_p}\phi^{i,*}_{n-1}(S_{n-1},Y_{n-1},Z^i_{n-1},\gamma_i)-L_{n-1}(S_{n-1},Y_{n-1})\Big|^2\le\frac{C_{n-1}}{N_p}\tag{2.9}$$

for every $1\le n\le N$, which gives the convergence rate in the large population limit.

Proof. The first claim (2.7) is a direct consequence of (2.3). The form of $p_{n-1}(s,y)$ is uniquely determined by the condition (2.6). By substituting the resultant expression for $p_{n-1}(s,y)$ (and $q_{n-1}(s,y)$) into (2.3), we obtain (2.8).

We only need to verify that the set of $(p_{n-1}(s,y))$ satisfies the condition (vii) in Assumption 2.1. At $t=t_{N-1}$, f_{N-1} is an a.s. strictly positive and bounded function on $\mathcal{S}_{N-1}\times\mathcal{Y}_{N-1}\times\mathcal{Z}_{N-1}\times\Gamma$ by the corresponding assumption on F. Since $d<0<u$, it is easy to see $0<p_{N-1}(s,y),q_{N-1}(s,y)<1$ for every $(s,y)\in\mathcal{S}_{N-1}\times\mathcal{Y}_{N-1}$, and hence consistent with the condition. This then implies that $\phi^{i,*}_{N-1}$ is an a.s. bounded function on $\mathcal{S}_{N-1}\times\mathcal{Y}_{N-1}\times\mathcal{Z}_{N-1}\times\Gamma$, which results in a.s. strictly positive and bounded V_{N-1} given by (2.4) on $\mathcal{S}_{N-1}\times\mathcal{Y}_{N-1}\times\mathcal{Z}_{N-1}\times\Gamma$. This in turn ensures that f_{N-2} satisfies the desired properties, and so do $(p_{N-2}(s,y),q_{N-2}(s,y))$, $(s,y)\in\mathcal{S}_{N-2}\times\mathcal{Y}_{N-2}$. In this way, by simple induction, we get the desired consistency for every time step.

For the second claim, it suffices to show that there is some constant C_{n-1} such that an inequality

$$\overline{\mathbb{E}}\Big|\frac{1}{N_p}\sum_{i=1}^{N_p}\phi^{i,*}_{n-1}(s,y,Z^i_{n-1},\gamma_i)-L_{n-1}(s,y)\Big|^2\le\frac{C_{n-1}}{N_p}$$

holds for every $(s,y)\in\mathcal{S}_{n-1}\times\mathcal{Y}_{n-1}$, $1\le n\le N$. Using the i.i.d. property of (Z^i_{n-1},γ_i) and the boundedness of f_{n-1},L_{n-1} and $1/\gamma_i$, we can arrange the expression (2.8) to show that there exists some constant C_{n-1} such that

$$\begin{aligned}&\overline{\mathbb{E}}\Big|\frac{1}{N_p}\sum_{i=1}^{N_p}\phi^{i,*}_{n-1}(s,y,Z^i_{n-1},\gamma_i)-L_{n-1}(s,y)\Big|^2\\&\le\frac{C_{n-1}}{N_p^2}\overline{\mathbb{E}}\Big[\Big|\sum_{i=1}^{N_p}\Big(\frac{1}{\gamma_i}-\mathbb{E}^1\Big[\frac{1}{\gamma_1}\Big]\Big)\Big|^2+\Big|\sum_{i=1}^{N_p}\Big(\frac{\log f_{n-1}(s,y,Z^i_{n-1},\gamma_i)}{\gamma_i}-\mathbb{E}^1\Big[\frac{\log f_{n-1}(s,y,Z^1_{n-1},\gamma_1)}{\gamma_1}\Big]\Big)\Big|^2\Big]\\&\le\frac{C_{n-1}}{N_p}\mathbb{E}^1\Big[\Big|\frac{1}{\gamma_1}-\mathbb{E}^1\Big(\frac{1}{\gamma_1}\Big)\Big|^2+\Big|\frac{\log f_{n-1}(s,y,Z^1_{n-1},\gamma_1)}{\gamma_1}-\mathbb{E}^1\Big[\frac{\log f_{n-1}(s,y,Z^1_{n-1},\gamma_1)}{\gamma_1}\Big]\Big|^2\Big]\end{aligned}$$

which establishes the desired result. Note that the cross terms appearing in the second line vanish due to the independence of $(Z^i, \gamma_i)_{i\in\mathbb{N}}$. The variances in the righ-hand side of the third line are finite uniformly in $(s,y) \in \mathcal{S}_{n-1} \times \mathcal{Y}_{n-1}$, owing to the boundedness of functions f_{n-1} and $1/\gamma_i$. $\square$

Remark 2.4 (On the choice of exponential utility)**.** *Before considering the implications of these findings, we offer a brief remark on our choice of exponential utility. The most important characteristic of the exponential utility is that the optimal trading strategy $\phi^{i,*}$ is independent of the wealth process X^i as shown in* (2.3)*. This property is crucial for solving the mean-field condition in Theorem 2.2. Indeed, if other utilities were adopted, the optimal position $\phi^{i,*}_n$ would generally depend on the agent's wealth X^i_n. The equilibrium condition* (2.6) *would then yield a complex fixed-point problem involving the forward X^i and backward $\phi^{i,*}$ processes. The exponential utility also allows the agents to hold negative net wealth. This is quite advantageous for modeling financial institutions who naturally hold substantial portfolios of derivatives, the values of which can be negative with nonzero probabilities. Although the constant absolute risk aversion (CARA) is a clear limitation, given that most financial institutions routinely revise their risk management objectives, the restriction of CARA for a relatively short-term horizon T seems to be a reasonable approximation. The same observations hold for the recursive utility of exponential-type, which we discuss in Section 3.*

2.4 Some implications

Let us discuss some implications of the results obtained in this section. A key advantage of our closed-form solution is the explicit dependence of the transition probabilities $(p_{n-1}(s,y))$ on the macroeconomic factor Y. This allows us to clearly analyze how changes in the macroeconomic conditions affect the equilibrium price distribution and consequently the excess return.

In order to understand the qualitative behavior of our model, assume first that there is no external order flow, i.e., $L \equiv 0$ for simplicity. Recalling that the risk-neutral probability of the up-move at each node is $p^{\mathbb{Q}} = (-d)/(u-d)$, one can see from (2.7) that $p_{n-1}(s,y) > p^{\mathbb{Q}}$ (i.e. a positive excess return at this node) occurs if and only if

$$\mathbb{E}^1[\log(f_{n-1}(s,y,Z^1_{n-1},\gamma_1))/\gamma_1] < 0.$$

This happens if V_n is a decreasing function of the stock price S_n, where V_n represents the continuation liability derived from the terminal liability F. In Section 6, we shall confirm this point with numerical examples. The corresponding situation occurs when the agents' liability (or, the negative of their endowment) decreases as the stock price increases. In this case, adding to the long position in the stock increases the risk in the same direction as the liability's exposure, and hence the agents require a higher risk premium. Therefore, for a liability whose size varies countercyclically with the stock price, the higher the leverage of the financial and investment firms, the higher the risk premium demanded. Suppose, on the other hand, that the agents' liability increases when the stock price goes up. For example, imagine that agents have a net short position in call options on the stock. Then, the agents have a strong incentive to increase their long position in the same stock (to hedge the options), and hence may accept even a negative risk premium.

As one can see from (2.7), it is not necessary to add idiosyncratic shocks to significantly influence the size of excess return, which is mainly determined by the sensitivity of the liability to the stock price. However, the absence of idiosyncratic shocks gives rise to a very unrealistic market structure where there is no trade among the agents. From the expression of the optimal position in (2.8), we can observe that the trading volume per capita $\mathbb{E}[|\phi_n^{1,*}|^2]^{\frac{1}{2}}$ in the market is primarily governed by the variation of the idiosyncratic factors. Note that, by the definition of the mean-field equilibrium, $\mathbb{E}[\phi_n^{1,*}] = 0, \forall i \in \mathbb{N}$ when $L_n = 0$. Therefore, $\mathbb{E}[|\phi_n^{1,*}|^2]^{\frac{1}{2}}$ gives the standard deviation of the stock position among the agents at t_n.

In addition to the condition $L \equiv 0$, let us now suppose that the function F is independent of the stock price. We then have $f_{N-1} = 1$ a.s. since V_N is S_N-independent and thus $\phi_{N-1}^{i,*} = 0$ a.s. by (2.8). This in turn makes V_{N-1} independent from S_{N-1}. In this way, a simple induction shows that $f_{n-1} = 1$ a.s. for every $1 \leq n \leq N$ and the equilibrium price distribution becomes equal to the one in the risk-neutral measure. In this case, there is no trade in the market although each agent has different risk aversion, which corresponds to the classical (but uninteresting) example of the representative agent with CARA utility.

Finally, let us turn on the external order flow. It is clearly seen from (2.7) that the positive inflow $L > 0$ to the stock market increases the equilibrium risk premium. This may sound slightly counter-intuitive since we think a big sell-off in the stock should lead to a sharp decline in the stock price. In order to understand that there is no contradiction, it is important to recall that what we have found above is the transition probabilities so that there exists equilibrium. As a common characteristic of price-formation frameworks, our model does not argue the performance of a stock price given the business performance of its issuer. Rather, it provides the excess return (and more generally price distribution) required by the agents to support the existence of the market equilibrium. If there is positive supply of the stock, the agents must accept larger long position (and hence larger risk) in the stock to maintain the balance of demand and supply; consequently, the agents require a higher risk premium to compensate for this additional risk. If the risk premium is not high enough, there would be no equilibrium and thus the stock market might crash. If the required risk premium turns out to be unrealistically high, then one can infer that the stock market cannot support such a large external supply. The same analysis can be done for the sustainability of the stock equilibrium under highly levered financial firms.

3 Recursive utility with path-dependent financial items

In the previous section, we obtained mean-field equilibrium by choosing an appropriate set of transition probabilities in the form of $p_n(s, y)$. Suppose now that the stochastic liability (or the negative of the endowment) F is dependent not only on the terminal stock price S_N but also on the stock-price history $(S_n)_{n=0}^N$, which is just as plausible. In this case, a quick inspection of the proofs for Theorems 2.1 and 2.2 shows that the transition probabilities of the simple form $p_n(s, y)$ cannot clear the market anymore. This strongly suggests that we need path-dependence also in the transition probabilities. We also want to examine whether we can include cash spending (i.e. nominal consumption) and to analyze its impact on the excess return. In this section, we shall thus adopt recursive utility that incorporates standard time-separable utility over nominal consumptions as its special case. We include a path-dependent terminal liability as well as path-dependent incremental endowments in the model for generality.

3.1 The setup and notation

In this section, for each $i \in \mathbb{N}$, we enlarge the probability space $(\Omega^i, \mathcal{F}^i, (\mathcal{F}^i_{t_n})_{n=0}^N, \mathbb{P}^i)$ so that it supports $(\xi_i, \gamma_i, \zeta_i, \psi_i, \delta_i)$ as $\mathcal{F}^i_0$-measurable random variables, in addition to $(\mathcal{F}^i_{t_n})_{n=0}^N$-adapted stochastic process $(Z_n^i = Z^i(t_n))_{n=0}^N$. Here, ζ_i is the coefficient of absolute risk aversion for cash spending and the parameter ψ_i is used to control the importance of the continuation utility relative to the current spending. δ_i denotes the coefficient of time preference. We introduce an $\mathcal{F}_0^i$-measurable 4-tuple $\varrho_i := (\gamma_i, \zeta_i, \psi_i, \delta_i)$ for simpler notation. Let us also introduce the symbol $\mathbf{S}^n := (S_0, S_1, \ldots, S_n)$ to denote a stock-price trajectory and $\mathbf{s}^n = (s_0, \ldots, s_n) \in \mathcal{S}^n$ as its specific realization. For $\mathbf{s} \in \mathcal{S}^{n-1}$, we also use the symbols $(\mathbf{s}\widetilde{u})^n := (\mathbf{s}^{n-1}, s_{n-1}\widetilde{u}) \in \mathcal{S}^n$ and $(\mathbf{s}\widetilde{d})^n := (\mathbf{s}^{n-1}, s_{n-1}\widetilde{d}) \in \mathcal{S}^n$. As in the last section, we shall use the expressions such as $\mathbb{E}^{0,i}[\,\cdot\,|\mathbf{s}, y, z^i, \varrho_i]$ for $(\mathbf{s}, y, z^i) \in \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1}$ to denote the conditional expectation $\mathbb{E}^{0,i}[\,\cdot\,|\mathbf{S}^{n-1} = \mathbf{s}, Y_{n-1} = y, Z_{n-1}^i = z_i, \varrho_i = \varrho_i]$, where, with a slight abuse of notation, the same symbol is used for a realization of the $\mathcal{F}_0^i$-measurable random variable ϱ_i. Except these points, we will use the same setup and notation given in Section 2.1. In particular, we impose the finite state space condition only on (S_n) and (Y_n). Now, let us update Assumption 2.1 for this section.

Assumption 3.1. (i): *$\widetilde{u}$ and $\widetilde{d}$ are real constants satisfying*

$$0 < \widetilde{d} < \exp(r\Delta) < \widetilde{u} < \infty.$$

(ii): *Every $(\xi_i, \gamma_i, \zeta_i, \psi_i, \delta_i, (Z_n^i)_{n=0}^N)$ is identically distributed across all agents $i = 1, 2, \ldots$.*
(iii): *There exist real constants $\underline{\xi}, \overline{\xi}$, $\underline{\gamma}, \overline{\gamma}$, $\underline{\zeta}, \overline{\zeta}$, $\underline{\psi}, \overline{\psi}$ and $\underline{\delta}, \overline{\delta}$ so that for every $i \in \mathbb{N}$,*

$$\begin{aligned}
&\xi_i \in [\underline{\xi}, \overline{\xi}] \subset \mathbb{R}, \\
&\varrho_i := (\gamma_i, \zeta_i, \psi_i, \delta_i) \in \Gamma := [\underline{\gamma}, \overline{\gamma}] \times [\underline{\zeta}, \overline{\zeta}] \times [\underline{\psi}, \overline{\psi}] \times [\underline{\delta}, \overline{\delta}] \subset (0, \infty)^4.
\end{aligned}$$

(iv): *For every $0 \le n \le N$, $\mathcal{Z}_n$ is a bounded subset of $\mathbb{R}^{d_Z}$.*
(v): *For each i, $(Z_n^i)_{n=0}^N$ is a Markov process i.e. $\mathbb{E}^i[f(Z_n^i)|\mathcal{F}_{t_k}^i] = \mathbb{E}^i[f(Z_n^i)|Z_k^i]$ for every bounded measurable function f on $\mathcal{Z}_n$ and $k \le n$.*
(vi): *$(Y_n)_{n=0}^N$ is a Markov process i.e. $\mathbb{E}^0[f(Y_n)|\mathcal{F}_{t_k}^0] = \mathbb{E}^0[f(Y_n)|Y_k]$ for every bounded measurable function f on $\mathcal{Y}_n$ and $k \le n$.*
(vii): *The transition probabilities of $(S_n)_{n=0}^N$ satisfy, for every $0 \le n \le N-1$,*

$$\begin{aligned}
\mathbb{P}^0(S_{n+1} = \widetilde{u}S_n|\mathcal{F}_{t_n}^0) = \mathbb{P}^0(S_{n+1} = \widetilde{u}S_n|\mathbf{S}^n, Y_n) =: p_n(\mathbf{S}^n, Y_n), \\
\mathbb{P}^0(S_{n+1} = \widetilde{d}S_n|\mathcal{F}_{t_n}^0) = \mathbb{P}^0(S_{n+1} = \widetilde{d}S_n|\mathbf{S}^n, Y_n) =: q_n(\mathbf{S}^n, Y_n),
\end{aligned}$$

where $p_n, q_n\ (= 1 - p_n) : \mathcal{S}^n \times \mathcal{Y}_n \to \mathbb{R},\ 0 \le n \le N-1$ are bounded measurable functions satisfying

$$0 < p_n(\mathbf{s}, y), q_n(\mathbf{s}, y) < 1$$

for every $(\mathbf{s}, y) \in \mathcal{S}^n \times \mathcal{Y}_n$.

Under the above assumptions, we have, instead of (2.1), the relation

$$\mathbb{E}^0\big[f(S_{n+1})g(Y_{n+1})|\mathcal{F}_{t_n}^0\big] = \mathbb{E}^0[f(S_{n+1})|\mathbf{S}^n, Y_n]\mathbb{E}^0[g(Y_{n+1})|Y_n], \quad 0 \le n \le N-1, \tag{3.1}$$

for any bounded measurable functions $f : \mathcal{S}_{n+1} \to \mathbb{R}$ and $g : \mathcal{Y}_{n+1} \to \mathbb{R}$.

Remark 3.1. *As in the last section, the condition $0 < p_n(\mathbf{s}, y), q_n(\mathbf{s}, y) < 1, \forall(\mathbf{s}, y) \in \mathcal{S}^n \times \mathcal{Y}_n, 0 \le n \le N-1$ guarantees the equivalence of $\mathbb{P}^0 \circ S_n^{-1}$ and $\mathbb{Q} \circ S_n^{-1}$. Hence the system is arbitrage free.*

3.2 The individual optimization problem

In this section, as previously mentioned, we assume that each agent-i engages in self-financing trading using the money-market account and the risky stock, while also spending some cash at the beginning of each period. Moreover, they receive a stochastic endowment at each time $t_n, 1 \le n \le N$. Thus the wealth of the agent-i, $(X_n^i := X^i(t_n))_{n=0}^N$, follows the dynamics:

$$\begin{aligned} X_{n+1}^i &= \exp(r\Delta)(X_n^i - c_n^i\Delta - \phi_n^i) + \phi_n^i \widetilde{R}_{n+1} + g_{n+1}(\mathbf{S}^{n+1}, Y_{n+1}, Z_{n+1}^i) \\ &= \beta(X_n^i - c_n^i\Delta) + \phi_n^i R_{n+1} + g_{n+1}(\mathbf{S}^{n+1}, Y_{n+1}, Z_{n+1}^i), \end{aligned} \tag{3.2}$$

where $X_0^i = \xi_i$. Recall that $R_n := \widetilde{R}_n - \exp(r\Delta)$. Here, $c_n^i, 0 \le n \le N-1$ denotes the cash spending at t_n. For the scaling purposes in the discrete-time model, this value is defined such that the actual cash used is $c_n^i\Delta$. $g_n(\mathbf{S}^n, Y_n, Z_n^i), 1 \le n \le N$ is the stochastic endowment (i.e., income originating from the agent's other business lines) paid at t_n, which is dependent on the stock-price trajectory $\mathbf{S}^n$ in addition to the common and the idiosyncratic shocks (Y_n, Z_n^i). Since we do not restrict g_n to be positive, it can also represent incremental liability incurred at t_n when negative. As discussed in Remark 2.3 concerning the terminal liability, the ability of our framework to incorporate such a term is crucial for developing realistic models of financial firms.

We suppose that the $(\mathcal{F}_{t_n}^{0,i})_{n=0}^N$-adapted process of utilities $(U_n^i)_{n=0}^N$ is defined recursively by

$$\begin{aligned} U_n^i &:= -\frac{1}{\zeta_i} \log\Big\{\exp\big(-\zeta_i c_n^i\big)\Delta + \delta_i \exp\Big(\frac{\psi_i}{\gamma_i} \log\big(\mathbb{E}^{0,i}[e^{-\gamma_i U_{n+1}^i} | \mathcal{F}_{t_n}^{0,i}]\big)\Big)\Big\}, \\ &\leftrightarrow \exp(-\zeta_i U_n^i) = \exp(-\zeta_i c_n^i)\Delta + \delta_i \exp\Big(\frac{\psi_i}{\gamma_i} \log\big(\mathbb{E}^{0,i}[e^{-\gamma_i U_{n+1}^i} | \mathcal{F}_{t_n}^{0,i}]\big)\Big), \end{aligned} \tag{3.3}$$

with the terminal condition

$$U_N^i := X_N^i - F(\mathbf{S}^N, Y_N, Z_N^i).$$

Here, $F(\mathbf{S}^N, Y_N, Z_N^i)$ denotes the terminal liability as in the last section, but now dependent on the price trajectory $\mathbf{S}^N$. Each agent-i is supposed to solve the optimization problem

$$\sup_{(\phi_n^i, c_n^i)_{n=0}^{N-1} \in \mathbb{A}^i} U_0^i, \tag{3.4}$$

over the admissible space defined by

$$\mathbb{A}^i := \big\{(\phi_n^i, c_n^i)_{n=0}^{N-1} : (\phi_n^i, c_n^i) \text{ is an } \mathcal{F}_n^{0,i}\text{-measurable } \mathbb{R}^2\text{-valued random variable}\big\}.$$

For simplicity, we do not restrict (c_n^i) to non-negative values. One may interpret negative spending as positive income from costly labor for the corresponding period.

Assumption 3.2. (i): *The function $F : \mathcal{S}^N \times \mathcal{Y}_N \times \mathcal{Z}_N \to \mathbb{R}$ is measurable and bounded.*
(ii): *For every $1 \le n \le N$, the function $g_n : \mathcal{S}^n \times \mathcal{Y}_n \times \mathcal{Z}_n \to \mathbb{R}$ is measurable and bounded.*
(iii): *Every agent is a price-taker in the sense that they consider the stock price process (and hence its transition probabilities specified in Assumption 3.1 (vii)) to be exogenously given by the collective actions of the others and unaffected by the agent's own trading strategies.*

Before going into the details, let us consider the special case: $\zeta_i = \gamma_i = \psi_i$. Then we have

$$\begin{aligned}\exp(-\zeta_i U_n^i) &= \exp(-\zeta_i c_n^i)\Delta + \delta_i \mathbb{E}^{0,i}[\exp(-\zeta_i U_{n+1}^i)|\mathcal{F}_{t_n}^{0,i}] \\ &= \exp(-\zeta_i c_n^i)\Delta + \mathbb{E}^{0,i}\big[\delta_i \exp(-\zeta_i c_{n+1}^i)\Delta|\mathcal{F}_{t_n}^{0,i}\big] + \delta_i^2 \mathbb{E}^{0,i}[\exp(-\zeta_i U_{n+2}^i)|\mathcal{F}_{t_n}^{0,i}] \\ &= \cdots = \exp(-\zeta_i c_n^i)\Delta + \mathbb{E}^{0,i}\Big[\sum_{k=n+1}^{N-1} \delta_i^{k-n}\exp(-\zeta_i c_k^i)\Delta + \delta_i^{N-n}\exp(-\zeta_i U_N^i)|\mathcal{F}_{t_n}^{0,i}\Big],\end{aligned}$$

which thus corresponds to the standard time-separable utility over nominal consumptions with the terminal liability F. One can see that the parameter ψ_i determines the relative importance of the continuation utility. We now derive the optimal strategy for each agent with respect to the above-defined recursive utility.

Theorem 3.1. *Let Assumptions 3.1 and 3.2 be in force. Then the problem* (3.4) *has a unique optimal solution* $(\phi_{n-1}^{i,*}, c_{n-1}^{i,*})_{n=1}^N$, *where* $(\phi_{n-1}^{i,*})_{n=1}^N$ *and* $(c_{n-1}^{i,*})_{n=1}^N$ *are a.s. bounded processes defined by measurable functions* $\phi_{n-1}^{i,*} : \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma \to \mathbb{R}$ *and* $c_{n-1}^{i,*} : \mathbb{R} \times \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma \to \mathbb{R}$ *such that* $\phi_{n-1}^{i,*} := \phi_{n-1}^{i,*}(\mathbf{S}^{n-1}, Y_{n-1}, Z_{n-1}^i, \varrho_i)$ *and* $c_{n-1}^{i,*} := c_{n-1}^{i,*}(X_{n-1}^i, \mathbf{S}^{n-1}, Y_{n-1}, Z_{n-1}^i, \varrho_i)$ *respectively, where, for each* $(x^i, \mathbf{s}, y, z^i, \varrho_i) \in \mathbb{R} \times \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma$,

$$\phi_{n-1}^{i,*}(\mathbf{s}, y, z^i, \varrho_i) := \frac{1}{\gamma_i \eta_n^i (u-d)}\Big\{\log\Big(-\frac{p_{n-1}(\mathbf{s},y)u}{q_{n-1}(\mathbf{s},y)d}\Big) + \log\big(f_{n-1}(\mathbf{s}, y, z^i, \varrho_i)\big)\Big\}, \tag{3.5}$$

$$c_{n-1}^{i,*}(x^i, \mathbf{s}, y, z^i, \varrho_i) := \frac{\psi_i \eta_n^i \beta}{\zeta_i + \Delta\psi_i\eta_n^i\beta} x^i - \frac{1}{\zeta_i + \Delta\psi_i\eta_n^i\beta}\log\Big\{\frac{\delta_i\psi_i\eta_n^i\beta}{\zeta_i}\exp\Big(\frac{\psi_i}{\gamma_i}\log\widetilde{V}_{n-1}(\mathbf{s}, y, z^i, \varrho_i)\Big)\Big\}. \tag{3.6}$$

Here, $f_{n-1} : \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma \to \mathbb{R}$ *is an a.s. strictly positive bounded measurable function:*

$$f_{n-1}(\mathbf{s}, y, z^i, \varrho_i) := \frac{\mathbb{E}^{0,i}\big[\exp\big(\gamma_i[V_n((\mathbf{s}\widetilde{u})^n, Y_n, Z_n^i, \varrho_i) - \eta_n^i g_n((\mathbf{s}\widetilde{u})^n, Y_n, Z_n^i)]\big)|y, z^i, \varrho_i\big]}{\mathbb{E}^{0,i}\big[\exp\big(\gamma_i[V_n((\mathbf{s}\widetilde{d})^n, Y_n, Z_n^i, \varrho_i) - \eta_n^i g_n((\mathbf{s}\widetilde{d})^n, Y_n, Z_n^i)]\big)|y, z^i, \varrho_i\big]}, \tag{3.7}$$

and $V_n : \mathcal{S}^n \times \mathcal{Y}_n \times \mathcal{Z}_n \times \Gamma \to \mathbb{R}$ *is an a.s. bounded measurable function defined recursively by*

$$V_{n-1}(\mathbf{s}, y, z^i, \varrho_i) := \frac{\eta_{n-1}^i}{\eta_n^i\gamma_i\beta}\log\widetilde{V}_{n-1}(\mathbf{s}, y, z^i, \varrho_i) + \frac{1}{\zeta_i + \Delta\psi_i\eta_n^i\beta}\log\Big(\frac{\delta_i\psi_i\eta_n^i\beta}{\zeta_i}\Big) - \frac{1}{\zeta_i}\log(\eta_{n-1}^i), \tag{3.8}$$

with

$$\begin{aligned}&\widetilde{V}_{n-1}(\mathbf{s}, y, z^i, \varrho_i) \\ &:= p_{n-1}(\mathbf{s}, y)e^{-\gamma_i\eta_n^i\phi_{n-1}^{i,*}u}\mathbb{E}^{0,i}\big[\exp(\gamma_i[V_n((\mathbf{s}\widetilde{u})^n, Y_n, Z_n^i, \varrho_i) - \eta_n^i g_n((\mathbf{s}\widetilde{u})^n, Y_n, Z_n^i)])|y, z^i, \varrho_i\big] \\ &+ q_{n-1}(\mathbf{s}, y)e^{-\gamma_i\eta_n^i\phi_{n-1}^{i,*}d}\mathbb{E}^{0,i}\big[\exp(\gamma_i[V_n((\mathbf{s}\widetilde{d})^n, Y_n, Z_n^i, \varrho_i) - \eta_n^i g_n((\mathbf{s}\widetilde{d})^n, Y_n, Z_n^i)])|y, z^i, \varrho_i\big],\end{aligned} \tag{3.9}$$

starting from the terminal condition $V_N(\mathbf{S}^N, Y_N, Z_N^i, \varrho_i) := F(\mathbf{S}^N, Y_N, Z_N^i)$. $(\eta_n^i)_{n=0}^N$ *are strictly positive and bounded* $\mathcal{F}_0^i$*-measurable random variables given by the recursive relation:*

$$\eta_{n-1}^i := \frac{\psi_i\eta_n^i\beta}{\zeta_i + \Delta\psi_i\eta_n^i\beta}, \quad \eta_N^i \equiv 1. \tag{3.10}$$

Similar to Theorem 2.1, V_n can be interpreted as the continuation value of liability at t_n.

Proof. We first hypothesize that the utility U_n^i at $t = t_n$ is given by the following form:

$$U_n^i(X_n^i, \mathbf{S}^n, Y_n, Z_n^i, \varrho_i) = \eta_n^i X_n^i - V_n(\mathbf{S}^n, Y_n, Z_n^i, \varrho_i),$$

where $V_n : \mathcal{S}^n \times \mathcal{Y}_n \times \mathcal{Z}_n \times \Gamma \to \mathbb{R}$ is a measurable a.s. bounded function and η_n^i is $\mathcal{F}_0^i$-measurable strictly positive and bounded random variable. The hypothesis obviously holds at the terminal point with $V_N(\mathbf{S}^N, Y_N, Z_N^i, \varrho_i) = F(\mathbf{S}^N, Y_N, Z_N^i)$ and $\eta_N^i \equiv 1$. We shall show by induction that our hypothesis holds in every period. Under the hypothesis, the problem for the agent-i at t_{n-1} becomes to find $\mathcal{F}_{n-1}^{0,i}$-measurable strategy (ϕ^i, c^i) that solves

$$\inf_{(\phi^i,c^i)} \Big\{ \exp(-\zeta_i c^i)\Delta + \delta_i \exp\Big(\frac{\psi_i}{\gamma_i} \log\Big(\mathbb{E}^{0,i}\big[\exp(-\gamma_i U_n^i(X_n^i, \mathbf{S}^n, Y_n, Z_n^i, \varrho_i)) | \mathcal{F}_{n-1}^{0,i} \big] \Big) \Big) \Big\}. \tag{3.11}$$

We consider the problem on the set $\{\omega^{0,i} \in \Omega^{0,i} : (X_{n-1}^i, \mathbf{S}^{n-1}, Y_{n-1}, Z_{n-1}^i, \varrho_i) = (x^i, \mathbf{s}, y, z^i, \varrho_i)\}$. By Assumption 3.1 (vii), (3.1), and the above hypothesis, we have

$$\begin{aligned}
&\mathbb{E}^{0,i}\Big[\exp\big(-\gamma_i U_n^i(X_n^i, \mathbf{S}^n, Y_n, Z_n^i, \varrho_i)\big) | x^i, \mathbf{s}, y^i, z^i, \varrho_i \Big] \\
&= \mathbb{E}^{0,i}\Big[\exp\Big(-\gamma_i \eta_n^i \big(\beta(x^i - c^i\Delta) + \phi^i R_n + g_n(\mathbf{S}^n, Y_n, Z_n^i) \big) + \gamma_i V_n(\mathbf{S}^n, Y_n, Z_n^i, \varrho_i) \Big) | \mathbf{s}, y, z^i, \varrho_i \Big] \\
&= e^{-\gamma_i \eta_n^i \beta(x^i - c^i\Delta)} \Big\{ p_{n-1}(\mathbf{s}, y) e^{-\gamma_i \eta_n^i \phi^i u} \mathbb{E}^{0,i}\big[\exp(\gamma_i [V_n((\mathbf{s}\widetilde{u})^n, Y_n, Z_n^i, \varrho_i) - \eta_n^i g_n((\mathbf{s}\widetilde{u})^n, Y_n, Z_n^i)]) | y, z^i, \varrho_i \big] \\
&\qquad + q_{n-1}(\mathbf{s}, y) e^{-\gamma_i \eta_n^i \phi^i d} \mathbb{E}^{0,i}\big[\exp(\gamma_i [V_n((\mathbf{s}\widetilde{d})^n, Y_n, Z_n^i, \varrho_i) - \eta_n^i g_n((\mathbf{s}\widetilde{d})^n, Y_n, Z_n^i)]) | y, z^i, \varrho_i \big] \Big\}.
\end{aligned}$$

Thus the problem (3.11) can be restated as

$$\begin{aligned}
&\inf_{(\phi^i,c^i)} \Big\{ \exp(-\zeta_i c^i)\Delta + \delta_i \exp(-\psi_i \eta_n^i \beta(x^i - c^i\Delta)) \\
&\times \exp\Big(\frac{\psi_i}{\gamma_i} \log\Big\{ p_{n-1}(\mathbf{s}, y) e^{-\gamma_i \eta_n^i \phi^i u} \mathbb{E}^{0,i}\big[\exp(\gamma_i [V_n((\mathbf{s}\widetilde{u})^n, Y_n, Z_n^i, \varrho_i) - \eta_n^i g_n((\mathbf{s}\widetilde{u})^n, Y_n, Z_n^i)]) | y, z^i, \varrho_i \big] \\
&\qquad + q_{n-1}(\mathbf{s}, y) e^{-\gamma_i \eta_n^i \phi^i d} \mathbb{E}^{0,i}\big[\exp(\gamma_i [V_n((\mathbf{s}\widetilde{d})^n, Y_n, Z_n^i, \varrho_i) - \eta_n^i g_n((\mathbf{s}\widetilde{d})^n, Y_n, Z_n^i)]) | y, z^i, \varrho_i \big] \Big\} \Big) \Big\}.
\end{aligned}$$

The optimization over (ϕ^i, c^i) can now be solved separately. Since $d < 0 < u$, the optimal $\phi^{i,*}$ is characterized uniquely by

$$\begin{aligned}
0 &= p_{n-1}(\mathbf{s}, y) u e^{-\gamma_i \eta_n^i \phi^i u} \mathbb{E}^{0,i}\big[\exp(\gamma_i [V_n((\mathbf{s}\widetilde{u})^n, Y_n, Z_n^i, \varrho_i) - \eta_n^i g_n((\mathbf{s}\widetilde{u})^n, Y_n, Z_n^i)]) | y, z^i, \varrho_i \big] \\
&\quad + q_{n-1}(\mathbf{s}, y) d e^{-\gamma_i \eta_n^i \phi^i d} \mathbb{E}^{0,i}\big[\exp(\gamma_i [V_n((\mathbf{s}\widetilde{d})^n, Y_n, Z_n^i, \varrho_i) - \eta_n^i g_n((\mathbf{s}\widetilde{d})^n, Y_n, Z_n^i)]) | y, z^i, \varrho_i \big],
\end{aligned}$$

which gives the desired solution (3.5) for $\phi_{n-1}^{i,*}$ with f_{n-1} defined as in (3.7). Thanks to the boundedness of g_n and our hypothesis on V_n, f_{n-1} is proved to be an a.s. strictly positive and bounded function on $\mathcal{S}^{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma$. Combined with the assumption on (p_{n-1}, q_{n-1}) and our hypothesis on η_n^i, $\phi_{n-1}^{i,*}$ is also a.s. bounded on $\mathcal{S}^{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma$.

With $\widetilde{V}_{n-1}$ defined as in (3.9), the optimization with respect to c^i reduces to

$$\inf_{c^i}\Big\{\exp(-\zeta_i c^i)\Delta+\delta_i\exp\big(-\psi_i\eta_n^i\beta(x^i-c^i\Delta)\big)\exp\Big(\frac{\psi_i}{\gamma_i}\log\widetilde{V}_{n-1}(\mathbf{s},y,z^i,\varrho_i)\Big)\Big\}.$$

Thus the optimal solution is characterized by the equation:

$$0=-\zeta_i\exp(-\zeta_i c^i)+\delta_i\psi_i\eta_n^i\beta\exp\big(-\psi_i\eta_n^i\beta(x^i-c^i\Delta)\big)\exp\Big(\frac{\psi_i}{\gamma_i}\log\widetilde{V}_{n-1}(\mathbf{s},y,z^i,\varrho_i)\Big), \tag{3.12}$$

which gives the unique solution $c_{n-1}^{i,*}$ in (3.6) as desired. Note that it is a.s. bounded if the wealth x^i at t_{n-1} is bounded. Therefore, once our induction is complete, the spending process is shown to be a.s. bounded since ξ_i takes values in a bounded interval $[\underline{\xi},\overline{\xi}]$.

In order to complete the induction argument, we need to obtain the utility U_{n-1}^i for the next period. By (3.3), it satisfies

$$\exp(-\zeta_i U_{n-1}^i)=\exp(-\zeta_i c_{n-1}^{i,*})\Delta+\delta_i\exp\big(-\psi_i\eta_n^i\beta(x^i-c_{n-1}^{i,*}\Delta)\big)\exp\Big(\frac{\psi_i}{\gamma_i}\log\widetilde{V}_{n-1}(\mathbf{s},y,z^i,\varrho_i)\Big).$$

The right-hand side of the above equality can be evaluated by using (3.12) as

$$\begin{aligned}
&\exp(-\zeta_i c_{n-1}^{i,*})\Delta+\frac{1}{\psi_i\eta_n^i\beta}\zeta_i\exp(-\zeta_i c_{n-1}^{i,*})=\frac{\zeta_i+\Delta\psi_i\eta_n^i\beta}{\psi_i\eta_n^i\beta}\exp(-\zeta_i c_{n-1}^{i,*})\\
&=\frac{\zeta_i+\Delta\psi_i\eta_n^i\beta}{\psi_i\eta_n^i\beta}\exp\Big\{-\frac{\zeta_i\psi_i\eta_n^i\beta}{\zeta_i+\Delta\psi_i\eta_n^i\beta}x^i+\frac{\zeta_i}{\zeta_i+\Delta\psi_i\eta_n^i\beta}\log\Big[\frac{\delta_i\psi_i\eta_n^i\beta}{\zeta_i}\exp\Big(\frac{\psi_i}{\gamma_i}\log\widetilde{V}_{n-1}(\mathbf{s},y,z^i,\varrho_i)\Big)\Big]\Big\}.
\end{aligned}$$

It follows that the utility U_{n-1}^i is given by

$$\begin{aligned}
U_{n-1}^i(x^i,\mathbf{s},y,z^i,\varrho_i)=&\frac{\psi_i\eta_n^i\beta}{\zeta_i+\Delta\psi_i\eta_n^i\beta}x^i-\frac{1}{\zeta_i+\Delta\psi_i\eta_n^i\beta}\log\Big\{\frac{\delta_i\psi_i\eta_n^i\beta}{\zeta_i}\exp\Big(\frac{\psi_i}{\gamma_i}\log\widetilde{V}_{n-1}(\mathbf{s},y,z^i,\varrho_i)\Big)\Big\}\\
&-\frac{1}{\zeta_i}\log\Big(\frac{\zeta_i+\Delta\psi_i\eta_n^i\beta}{\psi_i\eta_n^i\beta}\Big).
\end{aligned}$$

on the set $\{\omega^{0,i}\in\Omega^{0,i}:(X_{n-1}^i,\mathbf{S}^{n-1},Y_{n-1},Z_{n-1}^i,\varrho_i)=(x^i,\mathbf{s},y,z^i,\varrho_i)\}$. By setting the right-hand side equal to $\eta_{n-1}^i x^i-V_{n-1}(\mathbf{s},y,z^i,\varrho_i)$, we obtained the desired recursive relation for η_n^i and V_n. It is now clear that $(\eta_n^i)_{n=1}^N$ are $\mathcal{F}_0^i$-measurable, strictly positive and bounded, and that V_n is a bounded function on $\mathcal{S}^n\times\mathcal{Y}_n\times\mathcal{Z}_n\times\Gamma\to\mathbb{R}$ for every $0\le n\le N$. □

Remark 3.2 (On time inconsistency). *It is well known that the optimization problem for utilities becomes, in general, time-inconsistent if the associated coefficients are time-dependent, such as* $(\gamma_i(t),\zeta_i(t),\delta_i(t),\psi_i(t))$*. However, even in this case, the solution* $(\phi_n^{i,*},c_n^{i,*})_{n=0}^{N-1}$ *produces the sub-game perfect Nash equilibrium (SPNE), provided it is derived via backward induction, as we have done in this section. If we suppose that there exists an independent agent responsible for the optimization in each interval* $[t_n,t_{n+1}]$*, and they do not commit to past decisions made by other agents and believes future agents act similarly, then the backward induction solution* $(\phi_n^{i,*},c_n^{i,*})_{n=0}^{N-1}$ *yields a Nash equilibrium among them. This is an important characteristic of any games defined on trees over finite intervals. Considering most of the financial firms revise their risk and budgetary targets periodically, the concept of SPNE fits well the real situation. Therefore, even if we* *deal with time-inconsistent problem with time-dependent coefficients $(\gamma_i(t), \zeta_i(t), \delta_i(t), \psi_i(t))$, our scheme still provides a meaningful result. This is an additional advantage of adopting the binomial tree approach, which sidesteps the complexities often encountered in continuous-time settings. Unfortunately, however, if the coefficients follow general stochastic processes not measurable by $\mathcal{F}_0^i$, explicit solvability would be lost.*

3.3 Mean-field equilibrium among the agents with recursive utilities

Finally, as a main goal of this section, we shall derive a set of transition probabilities of the stock price so that the mean-field equilibrium holds among the agents with recursive utilities. As before, we incorporate the existence of stochastic external order flow L_n at each t_n, but it is now allowed to be path-dependent on the stock price:

Assumption 3.3. *For every $1 \le n \le N$, $L_{n-1} : \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1} \to \mathbb{R}$ is a bounded measurable function.*

Definition 3.1. *We say that the system is in the mean-field equilibrium if*

$$\lim_{N_p\to\infty} \frac{1}{N_p}\sum_{i=1}^{N_p} \phi_{n-1}^{i,*}(\mathbf{S}^{n-1}, Y_{n-1}, Z_{n-1}^i, \varrho_i) = L_{n-1}(\mathbf{S}^{n-1}, Y_{n-1}),$$

$\mathbb{P}$-*a.s. for every $1 \le n \le N$ with $\phi_{n-1}^{i,*}$ defined by* (3.5).

Since $(Z^i, \varrho_i), i \in \mathbb{N}$ are independent, identically distributed, and also independent of the process (S, Y), the above condition for the mean-field equilibrium can be represented by

$$\mathbb{E}^1\left[\phi_{n-1}^{1,*}(\mathbf{s}, y, Z_{n-1}^1, \varrho_1)\right] = L_{n-1}(\mathbf{s}, y) \tag{3.13}$$

for every $(\mathbf{s}, y) \in \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1}$, $1 \le n \le N$. It is now straightforward to derive the counterpart of Theorem 2.2.

Theorem 3.2. *Let Assumptions 3.1, 3.2 and 3.3 be in force. Then there exists a unique mean-field equilibrium and the associated transition probabilities of the stock price are given by*

$$\begin{aligned} p_{n-1}(\mathbf{s}, y) &:= \mathbb{P}^0\Big(S_n = \widetilde{u} S_{n-1} | (\mathbf{S}^{n-1}, Y_{n-1}) = (\mathbf{s}, y)\Big) \\ &= (-d)\Big/\Big\{u\exp\Big(\frac{1}{\mathbb{E}^1[1/(\gamma_1\eta_n^1)]}\Big[\mathbb{E}^1\Big(\frac{\log(f_{n-1}(\mathbf{s}, y, Z_{n-1}^1, \varrho_1))}{\gamma_1\eta_n^1}\Big) - (u-d)L_{n-1}(\mathbf{s}, y)\Big]\Big) - d\Big\} \end{aligned}$$

for every $(\mathbf{s}, y) \in \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1}$, $1 \le n \le N$. Under the above transition probabilities, the optimal strategy of agent-i is given by

$$\begin{aligned} \phi_{n-1}^{i,*}(\mathbf{s}, y, z^i, \varrho_i) = \frac{1}{(u-d)}\Big\{\frac{\log f_{n-1}(\mathbf{s}, y, z^i, \varrho_i)}{\gamma_i\eta_n^i} - \frac{1/(\gamma_i\eta_n^i)}{\mathbb{E}^1[1/(\gamma_1\eta_n^1)]}\mathbb{E}^1\Big(\frac{\log f_{n-1}(\mathbf{s}, y, Z_{n-1}^1, \varrho_1)}{\gamma_1\eta_n^1}\Big)\Big\} \\ + \frac{1/(\gamma_i\eta_n^i)}{\mathbb{E}^1[1/(\gamma_1\eta_n^1)]}L_{n-1}(\mathbf{s}, y). \end{aligned} \tag{3.14}$$

Moreover, there exists some positive constant C_{n-1} such that

$$\mathbb{E}\Big|\frac{1}{N_p}\sum_{i=1}^{N_p}\phi_{n-1}^{i,*}(\mathbf{S}^{n-1},Y_{n-1},Z_{n-1}^i,\varrho_i)-L_{n-1}(\mathbf{S}^{n-1},Y_{n-1})\Big|^2\leq\frac{C_{n-1}}{N_p}$$

for every $1\leq n\leq N$, which gives the convergence rate in the large population limit.

Proof. The expression for the transition probabilities $p_{n-1}(\mathbf{s},y)$ is a direct consequence of (3.5) and (3.13). By substituting this expression for $p_{n-1}(\mathbf{s},y)$ (and $q_{n-1}(\mathbf{s},y)$) into (3.5), we obtain (3.14). The rest of the proof is analogous to that of Theorem 2.2. Assumptions 3.2 and 3.3 guarantees that f_{N-1} is a.s. strictly positive and bounded and hence the condition (vii) in Assumption 3.1 is satisfied for (p_{N-1},q_{N-1}). It then makes $\phi_{N-1}^{i,*}$ a.s. bounded and thus $\widetilde{V}_{N-1}$ becomes a.s. strictly positive and bounded. It then follows that V_{N-1} and hence f_{N-2} are a.s. bounded functions, which shows that (p_{N-2},q_{N-2}) satisfies the condition (vii). In this way, a simple induction shows that the transition probabilities satisfy Assumption 3.1 (vii) for every period. The second claim can be proved from (3.14) and the fact that all the relevant components are bounded. □

Remark 3.3. *The path-dependent extension for the model in Section 2, which replaces $F(S_N,Y_N,Z_N^i)$ and $(L_{n-1}(S_{n-1},Y_{n-1}))_{n=1}^N$ with the path-dependent forms $F(\mathbf{S}^N,Y_N,Z_N^i)$ and $(L_{n-1}(\mathbf{S}^{n-1},Y_{n-1}))_{n=1}^N$, can be done in exactly the same way. Specifically, the corresponding results are obtained by substituting the path-dependent states $(\mathbf{s},(\mathbf{s}\widetilde{u})^n,(\mathbf{s}\widetilde{d})^n)\in\mathcal{S}^{n-1}\times\mathcal{S}^n\times\mathcal{S}^n$ for one-time states $(s,s\widetilde{u},s\widetilde{d})\in\mathcal{S}_{n-1}\times\mathcal{S}_n\times\mathcal{S}_n$ in the statements of Theorems 2.1 and 2.2.*

More generally, it is not difficult to confirm that the path dependence horizon required for the transition probabilities matches that for the liability and the incremental endowments. In particular, if F and g_n in the current recursive utility model depend only on S_N and S_n respectively, as in Section 2, then the mean-field equilibrium is achieved using transition probabilities of the form $(p_{n-1}(s,y),q_{n-1}(s,y))$ with $s\in\mathcal{S}_{n-1}$; by simply replacing $\mathbf{s}\in\mathcal{S}^{n-1}$ with $s\in\mathcal{S}_{n-1}$, we obtain the corresponding results for Theorems 3.1 and 3.2.

3.4 Some implications

Let us discuss some implications of the results in this section for the recursive utility. The key advantage of the explicit dependence of the transition probabilities $(p_{n-1}(s,y))$ on the macroeconomic factor Y still holds in the current case. Regarding the relation between the mean-field price distribution and the one in the risk-neutral measure, most of the discussions given in Section 2.4 still hold. Indeed, the situation $p_{n-1}(\mathbf{s},y)>p^{\mathbb{Q}}$ (i.e., a positive excess return at the node $(\mathbf{s},y)$) occurs if and only if

$$\mathbb{E}^1\big[\log f_{n-1}(\mathbf{s},y,Z_{n-1}^1,\varrho_1)/(\gamma_1\eta_n^1)\big]<0,$$

when there is no external order flow $L_{n-1}\equiv 0$.

As one can see from (3.7), f_{n-1} now receives contributions from the incremental endowment g_n in addition to the liability V_n. For example, although the investment funds typically do not have substantial liabilities, their endowments (fees from their customers) are naturally expected to grow as the stock price goes higher, since these fees are generally proportional to the Assets Under Management (AUM), which pushes the required excess return to higher values. If the liability size decreases and the endowment size simultaneously increases as the stock price grows, both effects will amplify the deviations from the risk-neutral distribution, leading to a higher excess return.

Since the equilibrium price distribution of the stock is determined by the need for risk hedging, the relative importance of the continuation utility with respect to the current nominal consumption is a crucial factor in controlling the size of the risk premium. From the expressions in (3.8) and (3.9), one can expect that the ratio

$$\eta^i_{n-1}/\eta^i_n$$

is the key value. Using $\beta \simeq 1$ and $\Delta \ll 1$, we have $\eta^i_{n-1}/\eta^i_n \simeq \psi_i/\zeta_i$ from (3.10). Therefore, if $\psi_i < \zeta_i$ holds for the majority of agents, we expect that the relative importance of the continuation utility quickly decays and we would see only a small impact from it in the earlier periods. In this case, significant deviations from the risk-neutral price distribution can be observed only in the later periods, near the maturity. On the other hand, in the case of $\psi_i \geq \zeta_i$, we can expect to see significant deviations throughout the interval. We shall confirm this behavior by numerical examples in Section 6.

As for the expected trading volume $\mathbb{E}[|\phi^{1,*}_n|^2]^{\frac{1}{2}}$, which gives the standard deviation of the stock position among the agents at $t = t_n$, the result is consistent with that in Section 2.4. The expression for $\phi^{i,*}_{n-1}$ in (3.14) shows that its size is governed by the variation of idiosyncratic factors defined on the space $(\Omega^i, \mathcal{F}^i, \mathbb{P}^i)$. It is also quite consistent with our intuition that the agents' heterogeneity in idiosyncratic factors is the origin of the trading activity in the market. Moreover, we can make use of the degrees of freedom in the process (Z^i_n), in particular its volatility, to obtain the desired trading volume. We shall demonstrate this property by numerical examples in Section 6.2.

Finally, we comment on the fact that the constant shifts in F and g_n, i.e. $F \mapsto F + c$ and $g_n \mapsto g_n + c'$ with some constants $c, c' \in \mathbb{R}$ do not affect the equilibrium price distribution. This property can be checked by a simple induction as follows: By (3.7), the value of f_{N-1} remains unchanged and so are $p_{N-1}(\mathbf{s}, y)$ and $\phi^{i,*}_{N-1}$; the value of $\widetilde{V}_{N-1}$ is changed only by an $\mathcal{F}^i_0$-measurable multiplicative factor; the value of V_{N-1} is simply shifted by an $\mathcal{F}^i_0$-measurable term; thus f_{N-2} remains once again unchanged, and so are $p_{N-2}(\mathbf{s}, y)$ and $\phi^{i,*}_{N-2}$, and so on. Therefore, the signs of F and g_n can be altered without affecting the equilibrium price distribution. Note however that the cash spending is affected by these shifts.

4 Mean-field equilibrium of multiple populations

A primary drawback of the previous frameworks lies in their restriction to a single homogeneous population, where all agents share identical functional forms for terminal liabilities F and incremental endowments $(g_n)^N_{n=1}$. That is, their heterogeneity is limited to the realizations of the idiosyncratic factor process Z^i. Beyond this structural constraint, risk aversion coefficients and time preferences are expected to have substantially different distributions across diverse financial entities, such as investment banks, commercial banks, insurance firms, pension, and other investment funds. To address this fundamental market heterogeneity, we propose a multi-population extension of the mean-field equilibrium studied in the preceding sections.

Let us consider the case where there are $m \in \mathbb{N}$ populations. For simplicity, we assume that all the agents have recursive utilities of exponential-type as in Section 3. We adopt the same model setup and notation as given in Sections 2.1 and 3.1, in particular those for $(S_n)^N_{n=0}$ and $(Y_n)^N_{n=0}$. However to incorporate the heterogeneity across the m populations, we need to introduce population-specific probability spaces for the idiosyncratic factors. We now introduce a countably infinite number of complete filtered probability spaces $(\Omega^{i,p}, \mathcal{F}^{i,p}, (\mathcal{F}^{i,p}_{t_n})^N_{n=0}, \mathbb{P}^{i,p})$, $i \in \mathbb{N}$ for each population p, $p = 1, \dots, m$. We introduce $(\xi_i^p, \gamma_i^p, \zeta_i^p, \psi_i^p, \delta_i^p)$ as $\mathcal{F}_0^{i,p}$-measurable $\mathbb{R}$-valued random variables and $(Z_n^{i,p} = Z^{i,p}(t_n))_{n=0}^N$ as $(\mathcal{F}_{t_n}^{i,p})_{n=0}^N$-adapted d_Z-dimensional stochastic process. Since m is finite, we can take d_Z to be the same for every population by using a sufficiently large integer. As before, we denote the support of the random variable $Z_n^{i,p}$ by $\mathcal{Z}_n^p$ and define $\mathcal{Z}^{n,p} := \mathcal{Z}_0^p \times \mathcal{Z}_1^p \times \cdots \times \mathcal{Z}_n^p$ as the support of $(Z_k^{i,p})_{k=0}^n$. Importantly, while these variables are assumed to be i.i.d. copies within each population p, their distributions can vary across the populations.

With the same probability space $(\Omega^0, \mathcal{F}^0, (\mathcal{F}_{t_n}^0)_{n=0}^N, \mathbb{P}^0)$ as in Section 2.1, we define

$$(\Omega, \mathcal{F}, (\mathcal{F}_{t_n})_{n=0}^N, \mathbb{P}) := (\Omega^0, \mathcal{F}^0, (\mathcal{F}_{t_n}^0)_{n=0}^N, \mathbb{P}^0) \otimes_{p=1}^m \otimes_{i=1}^\infty (\Omega^{i,p}, \mathcal{F}^{i,p}, (\mathcal{F}_{t_n}^{i,p})_{n=0}^N, \mathbb{P}^{i,p})$$

as the full probability space containing the entire environment of the m-population model. For agent-i in the population p (it will be denoted by agent-(i,p) hereafter), the relevant probability space is given by

$$(\Omega^{0,(i,p)}, \mathcal{F}^{0,(i,p)}, (\mathcal{F}_{t_n}^{0,(i,p)})_{n=0}^N, \mathbb{P}^{0,(i,p)}) := (\Omega^0, \mathcal{F}^0, (\mathcal{F}_{t_n}^0)_{n=0}^N, \mathbb{P}^0) \otimes (\Omega^{i,p}, \mathcal{F}^{i,p}, (\mathcal{F}_{t_n}^{i,p})_{n=0}^N, \mathbb{P}^{i,p}).$$

Expectations with respect to $\mathbb{P}^{i,p}$ and $\mathbb{P}^{0,(i,p)}$ are denoted by $\mathbb{E}^{i,p}[\cdot]$ and $\mathbb{E}^{0,(i,p)}[\cdot]$, respectively. We assume the following for the multi-population model:

Assumption 4.1. (i): *$\widetilde{u}$ and $\widetilde{d}$ are real constants satisfying $0 < \widetilde{d} < \exp(r\Delta) < \widetilde{u} < \infty$*
(ii): *Every $(\xi_i^p, \gamma_i^p, \zeta_i^p, \psi_i^p, \delta_i^p, (Z_n^{i,p})_{n=0}^N)$ is identically distributed across all agents $i = 1, 2, \dots$ within each population $1 \le p \le m$.*
(iii): *There exist real constants $\underline{\xi}, \overline{\xi}$, $\underline{\gamma}, \overline{\gamma}$, $\underline{\zeta}, \overline{\zeta}$, $\underline{\psi}, \overline{\psi}$ and $\underline{\delta}, \overline{\delta}$ so that for every $1 \le p \le m, i \in \mathbb{N}$,*

$$\begin{aligned}
&\xi_i^p \in [\underline{\xi}, \overline{\xi}] \subset \mathbb{R}, \\
&\varrho_i^p := (\gamma_i^p, \zeta_i^p, \psi_i^p, \delta_i^p) \in \Gamma := [\underline{\gamma}, \overline{\gamma}] \times [\underline{\zeta}, \overline{\zeta}] \times [\underline{\psi}, \overline{\psi}] \times [\underline{\delta}, \overline{\delta}] \subset (0, \infty)^4.
\end{aligned}$$

(iv): *For every $1 \le p \le m$ and $0 \le n \le N$, $\mathcal{Z}_n^p$ is a bounded subset of $\mathbb{R}^{d_Z}$.*
(v): *For each (i,p), $(Z_n^{i,p})_{n=0}^N$ is a Markov process i.e., $\mathbb{E}^{i,p}[f(Z_n^{i,p})|\mathcal{F}_{t_k}^{i,p}] = \mathbb{E}^{i,p}[f(Z_n^{i,p})|Z_k^{i,p}]$ for every bounded measurable function f on $\mathcal{Z}_n^p$ and $k \le n$.*
(vi): *$(Y_n)_{n=0}^N$ is a Markov process i.e., $\mathbb{E}^0[f(Y_n)|\mathcal{F}_{t_k}^0] = \mathbb{E}^0[f(Y_n)|Y_k]$ for every bounded measurable function f on $\mathcal{Y}_n$ and $k \le n$.*
(vii): *The transition probabilities of $(S_n)_{n=0}^N$ satisfy, for every $0 \le n \le N-1$,*

$$\begin{aligned}
\mathbb{P}^0(S_{n+1} = \widetilde{u} S_n | \mathcal{F}_{t_n}^0) &= \mathbb{P}^0(S_{n+1} = \widetilde{u} S_n | \mathbf{S}^n, Y_n) =: p_n(\mathbf{S}^n, Y_n), \\
\mathbb{P}^0(S_{n+1} = \widetilde{d} S_n | \mathcal{F}_{t_n}^0) &= \mathbb{P}^0(S_{n+1} = \widetilde{d} S_n | \mathbf{S}^n, Y_n) =: q_n(\mathbf{S}^n, Y_n),
\end{aligned}$$

where $p_n, q_n\ (= 1 - p_n) : \mathcal{S}^n \times \mathcal{Y}_n \to \mathbb{R},\ 0 \le n \le N-1$ are bounded measurable functions satisfying

$$0 < p_n(\mathbf{s}, y), q_n(\mathbf{s}, y) < 1$$

for every $(\mathbf{s}, y) \in \mathcal{S}^n \times \mathcal{Y}_n$.

We also introduce the terminal liabilities F^p and the incremental endowments $(g_n^p)_{n=1}^N$, which may have different functional forms across the populations $p = 1, \dots, m$ to capture the diversity of financial entities.

Assumption 4.2. *For each $1 \le p \le m$,*
(i): *the function $F^p : \mathcal{S}^N \times \mathcal{Y}_N \times \mathcal{Z}^p_N \to \mathbb{R}$ is measurable and bounded.*
(ii): *for every $1 \le n \le N$, the function $g^p_n : \mathcal{S}^n \times \mathcal{Y}_n \times \mathcal{Z}^p_n \to \mathbb{R}$ is measurable and bounded.*
(iii): *every agent is a price-taker in the sense that they consider the stock price process (and hence its transition probabilities specified in Assumption 4.1 (vii)) to be exogenously given by the collective actions of the others and unaffected by the agent's own trading strategies.*

We suppose that each agent-(i,p) solves the optimization problem

$$\sup_{(\phi^{i,p}_n, c^{i,p}_n)_{n=0}^{N-1} \in \mathbb{A}^{i,p}} U^{i,p}_0,$$

over the admissible space defined by

$$\mathbb{A}^{i,p} := \big\{(\phi^{i,p}_n, c^{i,p}_n)_{n=0}^{N-1} : (\phi^i_n, c^i_n) \text{ is an } \mathcal{F}^{0,(i,p)}_n\text{-measurable } \mathbb{R}^2\text{-valued random variable}\big\}.$$

The $(\mathcal{F}^{0,(i,p)}_{t_n})_{n=0}^N$-adapted process of the recursive utilities $(U^{i,p}_n)_{n=0}^N$ is defined by

$$U^{i,p}_n := -\frac{1}{\zeta^p_i} \log\Big\{\exp(-\zeta^p_i c^{i,p}_n)\Delta + \delta^p_i \exp\Big(\frac{\psi^p_i}{\gamma^p_i} \log\big(\mathbb{E}^{0,(i,p)}[e^{-\gamma^p_i U^{i,p}_{n+1}} | \mathcal{F}^{0,(i,p)}_{t_n}]\big)\Big)\Big\}$$

with the terminal condition

$$U^{i,p}_N := X^{i,p}_N - F^p(\mathbf{S}^N, Y_N, Z^{i,p}_N).$$

Here, $(X^{i,p}_n)_{n=0}^N$ denotes the wealth process of agent-(i,p) which follows

$$X^{i,p}_{n+1} = \beta(X^{i,p}_n - c^{i,p}_n\Delta) + \phi^{i,p}_n R_{n+1} + g^p_{n+1}(\mathbf{S}^{n+1}, Y_{n+1}, Z^{i,p}_{n+1}),$$

where $X^{i,p}_0 = \xi^p_i$. The interpretations of the coefficients are the same as Section 3.

Under Assumptions 4.1 and 4.2, a direct application of Theorem 3.1 for each population $p = 1, \ldots, m$ provides the optimal solution $(\phi^{i,p,*}_{n-1}, c^{i,p,*}_{n-1})_{n=1}^N$. In particular, we have $\phi^{i,p,*}_{n-1} := \phi^{i,p,*}_{n-1}(\mathbf{S}^{n-1}, Y_{n-1}, Z^{i,p}_{n-1}, \varrho^p_i)$, where for each $(\mathbf{s}, y, z^{i,p}, \varrho^p_i) \in \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}^p_{n-1} \times \Gamma$,

$$\phi^{i,p,*}_{n-1}(\mathbf{s}, y, z^{i,p}, \varrho^p_i) := \frac{1}{\gamma^p_i \eta^{i,p}_n (u-d)} \Big\{\log\Big(-\frac{p_{n-1}(\mathbf{s},y)u}{q_{n-1}(\mathbf{s},y)d}\Big) + \log\big(f^p_{n-1}(\mathbf{s}, y, z^{i,p}, \varrho^p_i)\big)\Big\}. \tag{4.1}$$

Here, f^p_{n-1} is given by (3.7) by replacing F with F^p and g_n with g^p_n in its recursive definition. Similarly, $\eta^{i,p}_n$ is given by (3.10) with $\psi_i \to \psi^p_i$ and $\zeta_i \to \zeta^p_i$.

We now consider the large population limit and the associated market-clearing condition. As discussed in [19, 16], we need to keep track of the ratio of population size. Let us denote the number of agents in population p by N_p and set $\mathcal{N} := N_1 + \cdots + N_m$. We use $w_p := N_p/\mathcal{N}$ to denote the relative size of population p. We obviously have the following decomposition:

$$\frac{1}{\mathcal{N}} \sum_{p=1}^m \sum_{i=1}^{N_p} \phi^{i,p,*}_{n-1} = \sum_{p=1}^m w_p \Big(\frac{1}{N_p} \sum_{i=1}^{N_p} \phi^{i,p,*}_{n-1}\Big). \tag{4.2}$$

We thus consider the limit $\mathcal{N} \to \infty$ while keeping $w_p, 1 \le p \le m$ constant, and then introduce the following concept of mean-field equilibrium in the presence of an external order flow:

Definition 4.1. *We say that the system is in the mean-field equilibrium if*

$$\lim_{\mathcal{N}\to\infty} \frac{1}{\mathcal{N}} \sum_{p=1}^{m} \sum_{i=1}^{N_p} \phi_{n-1}^{i,p,*}(\mathbf{S}^{n-1}, Y_{n-1}, Z_{n-1}^{i,p}, \varrho_i^p) = L_{n-1}(\mathbf{S}^{n-1}, Y_{n-1}),$$

$\mathbb{P}$*-a.s. for every* $1 \le n \le N$ *with* $\phi_{n-1}^{i,p,*}$ *defined by* (4.1)*, where the large population limit is taken with the population ratios* $(w_p)_{p=1}^m$ *kept constant.*

We can now derive the equilibrium transition probabilities among the m populations.

Theorem 4.1. *Let Assumptions 4.1, 4.2 and 3.3 be in force. Then there exists a unique mean-field equilibrium and the associated transition probabilities of the stock price are given by*

$$\begin{aligned} p_{n-1}(\mathbf{s}, y) &:= \mathbb{P}^0\Big(S_n = \widetilde{u} S_{n-1} | (\mathbf{S}^{n-1}, Y_{n-1}) = (\mathbf{s}, y)\Big) \\ &= (-d) \Big/ \left\{ u \exp\left(\frac{\sum_{p=1}^m w_p \mathbb{E}^{1,p}\Big[\frac{\log(f_{n-1}^p(\mathbf{s}, y, Z_{n-1}^{1,p}, \varrho_1^p))}{\gamma_1^p \eta_n^{1,p}}\Big] - (u-d) L_{n-1}(\mathbf{s}, y)}{\sum_{p=1}^m w_p \mathbb{E}^{1,p}\big[1/(\gamma_1^p \eta_n^{1,p})\big]} \right) - d \right\} \end{aligned}$$

for every $(\mathbf{s}, y) \in \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1}$*,* $1 \le n \le N$*. Moreover, with the above transition probabilities, there exists some positive constant* C_{n-1} *such that*

$$\mathbb{E}\Big| \frac{1}{\mathcal{N}} \sum_{p=1}^{m} \sum_{i=1}^{N_p} \phi_{n-1}^{i,p,*}(\mathbf{S}^{n-1}, Y_{n-1}, Z_{n-1}^{i,p}, \varrho_i^p) - L_{n-1}(\mathbf{S}^{n-1}, Y_{n-1}) \Big|^2 \le \frac{C_{n-1}}{\mathcal{N}}$$

for every $1 \le n \le N$*, which gives the convergence rate in the large population limit.*

Proof. By the i.i.d. property within each population and the decomposition (4.2), the condition for the mean-field equilibrium given by Definition 4.1 can be rewritten as

$$\sum_{p=1}^{m} w_p \mathbb{E}^{1,p}\big[\phi_{n-1}^{1,p,*}(\mathbf{s}, y, Z_{n-1}^{1,p}, \varrho_1^p)\big] = L_{n-1}(\mathbf{s}, y) \tag{4.3}$$

for every $(\mathbf{s}, y) \in \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1}$, $1 \le n \le N$. Using (4.1), the proof can be done exactly in the same way as in Theorems 2.2 and 3.2. □

As discussed in Remark 3.3, if we turn off the path-dependence in F^p and (g_n^p), then the simpler form of transition probabilities, $(p_{n-1}(s, y), q_{n-1}(s, y))$, $(s, y) \in \mathcal{S}_{n-1} \times \mathcal{Y}_{n-1}$, is sufficient to clear the market. In this case, at least, numerical costs would not be so high and one can calculate the equilibrium price distribution in the same way as the next Section 6. We can even mix the populations with standard exponential utilities and the recursive utilities in a similar manner.

It should be noted that the analytical tractability achieved in this section is quite remarkable when compared with the situation in continuous-time settings. In fact, attempting to solve the corresponding problem in the formulation of Fujii & Sekine [17] would lead to a coupled system of mean-field qg-BSDEs. Also in the optimal storage-type modeling adopted in many of the existing literature such as [19, 37, 40], the multi-population extension would produce a coupled system of FBSDEs of McKean-Vlasov type. The well-posedness of these equations is substantially more challenging than in the single population case, to say nothing of its numerical evaluation.

5 Subjective Measures: Stochastic Bias and Equilibrium

This section addresses the question of whether every agent must adopt the correct objective probability measure $\mathbb{P}^0$ for the dynamics of the stock price to achieve the market-clearing equilibrium. All existing literature on MFG implicitly adopts the rational expectation hypothesis, assuming that every agent shares correct knowledge of the objective probability distributions governing the relevant state processes or outcomes. A recent paper by Moll & Ryzhik [36] (2025) studies MFGs without rational expectations. They show that, in some cases, departing from rational expectations completely sidesteps the Master equation, allowing for the solution of much simpler finite-dimensional HJB equations instead.

In the preceding sections, we adopt Assumptions 2.2 (ii), 3.2 (iii), and 4.2 (iii), all of which assume that every agent-i possesses true knowledge of the stock dynamics by sharing the objective measure $\mathbb{P}^0$. Specifically, they use the objective probability measure $\mathbb{P}^{0,i} := \mathbb{P}^0 \otimes \mathbb{P}^i$, which incorporates the correct objective distributions for (S, Y) and the agents' idiosyncratic shocks. As emphasized in [36], it is more natural to suppose that each agent-i adopts a subjective measure, which is estimated, for example, by a statistical technique or based on a plausible economic model. Then the resultant dynamics of the stock price under the subjective measure is likely to exhibit a bias relative to the true dynamics. Moreover, this bias is expected to change stochastically dependent on the agents' risk characteristics as well as the market environment. The ultimate goal of this section is to determine the objective transition probability of the stock price under the measure $\mathbb{P}^0$ that clears the market among these agents operating under their respective subjective measures.

5.1 The setup and notation

In this section, we directly extend the results in Section 3 to incorporate agents' subjective measures on the stock dynamics. We adopt the same setup and notation as in Section 3.1. The key novelty in this section is captured by the following assumption:

Assumption 5.1. *Each agent-i adopts a subjective probability measure $\mathcal{P}^{0,i}$ on $(\Omega^{0,i}, \mathcal{F}^{0,i}, (\mathcal{F}^{0,i}_{t_n})_{n=0}^N)$ which may differ from the objective measure $\mathbb{P}^{0,i}$. The two measures $\mathcal{P}^{0,i}$ and $\mathbb{P}^{0,i}$ coincide for the dynamics of the variables $(Y, Z^i, \xi_i, \varrho_i)$ but differ only in the stock dynamics in the following way:*

$$\mathcal{P}^{0,i}(S_{n+1} = \widetilde{u}S_n|\mathcal{F}^{0,i}_{t_n}) = \mathcal{P}^{0,i}(S_{n+1} = \widetilde{u}S_n|\mathbf{S}^n, Y_n, Z^i_n, \varrho_i) =: \mathfrak{p}_n(\mathbf{S}^n, Y_n, Z^i_n, \varrho_i),$$
$$\mathcal{P}^{0,i}(S_{n+1} = \widetilde{d}S_n|\mathcal{F}^{0,i}_{t_n}) = \mathcal{P}^{0,i}(S_{n+1} = \widetilde{d}S_n|\mathbf{S}^n, Y_n, Z^i_n, \varrho_i) =: \mathfrak{q}_n(\mathbf{S}^n, Y_n, Z^i_n, \varrho_i),$$

where $\mathfrak{p}_n$ and $\mathfrak{q}_n = 1 - \mathfrak{p}_n$ are measurable functions defined on $\mathcal{S}^n \times \mathcal{Y}_n \times \mathcal{Z}_n \times \Gamma$ and satisfy, for each $n = 0, \ldots, N-1$,

$$\frac{\mathfrak{p}_n(\mathbf{s}, y, z^i, \varrho_i)}{\mathfrak{q}_n(\mathbf{s}, y, z^i, \varrho_i)} = \varpi_n(\mathbf{s}, y, z^i, \varrho_i)\Big(\frac{p_n(\mathbf{s}, y)}{q_n(\mathbf{s}, y)}\Big),$$

where p_n, q_n are the transition probabilities under the objective measure given in Assumption 3.1 (vii) and $\varpi_n : \mathcal{S}^n \times \mathcal{Y}_n \times \mathcal{Z}_n \times \Gamma \to \mathbb{R}$ is a measurable function that expresses the agents' biased *view on the transition probabilities. There exist some positive constants* $0 < \underline{\varpi} < \overline{\varpi} < \infty$ *such that*

$$\underline{\varpi} \leq \varpi_n(\mathbf{s}, y, z^i, \varrho_i) \leq \overline{\varpi}$$

for every $(\mathbf{s}, y, z^i, \varrho_i) \in \mathcal{S}^n \times \mathcal{Y}_n \times \mathcal{Z}_n \times \Gamma$*,* $0 \leq n \leq N-1$*.*

We denote by $\mathbb{E}_{\mathcal{P}}^{0,i}[\cdot]$ the expectation with respect to $\mathcal{P}^{0,i}$. Note that $\mathcal{P}^{0,i}$ is equivalent to $\mathbb{P}^{0,i}$ due to the strict positivity and boundedness of $(\varpi_n)_n$. This equivalence guarantees that each agent's subjective view remains arbitrage-free, which is a fundamental requirement for the well-posedness and economic consistency of the individual optimization problem.

In this model, the individual bias on the transition probabilities at t_n is captured by the function ϖ_n. Specifically, $\varpi_n > 1$ (resp., $\varpi_n < 1$) implies a positive (resp., negative) view on the stock performance for the period $[t_n, t_{n+1}]$, which is conditional on the time t_n realizations of the stock price history, macroeconomic/environmental factors Y, idiosyncratic shocks Z^i, as well as the agent's risk coefficients. For example, if agent-i is relatively more cautious (resp., optimistic), they might choose $\varpi_n < 1$ (resp., $\varpi_n > 1$) when the stock price and the macroeconomic factor take large (i.e., good) values.

5.2 The individual optimization problem

As in Section 3.2, the $(\mathcal{F}_{t_n}^{0,i})_{n=0}^N$-adapted process of utilities $(U_n^i)_{n=0}^N$ is recursively defined by

$$U_n^i := -\frac{1}{\zeta_i} \log\Big\{\exp(-\zeta_i c_n^i)\Delta + \delta_i \exp\Big(\frac{\psi_i}{\gamma_i} \log\big(\mathbb{E}_{\mathcal{P}}^{0,i}[e^{-\gamma_i U_{n+1}^i} | \mathcal{F}_{t_n}^{0,i}]\big)\Big), \tag{5.1}$$

with the terminal condition $U_N^i := X_N^i - F(\mathbf{S}^N, Y_N, Z_N^i)$. The sole difference from (3.3) is the use of the expectation $\mathbb{E}_{\mathcal{P}}^{0,i}[\cdot]$ with respect to the subjective measure $\mathcal{P}^{0,i}$. The wealth process of agent-i, $(X_n^i)_{n=0}^N$, follows the same dynamics as in (3.2), and the agent problem is also given by (3.4) with the identical admissible space $\mathbb{A}^i$.

Theorem 5.1. *Let Assumptions 3.1, 3.2* (i), (ii)*, and 5.1 be in force. Then the problem* (3.4) *with utility defined by* (5.1) *has a unique optimal solution* $(\phi_{n-1}^{i,*}, c_{n-1}^{i,*})_{n=1}^N$*, where* $(\phi_{n-1}^{i,*})_{n=1}^N$ *and* $(c_{n-1}^{i,*})_{n=1}^N$ *are a.s. bounded processes defined by measurable functions* $\phi_{n-1}^{i,*} : \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma \to \mathbb{R}$ *and* $c_{n-1}^{i,*} : \mathbb{R} \times \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma \to \mathbb{R}$ *such that* $\phi_{n-1}^{i,*} := \phi_{n-1}^{i,*}(\mathbf{S}^{n-1}, Y_{n-1}, Z_{n-1}^i, \varrho_i)$ *and* $c_{n-1}^{i,*} := c_{n-1}^{i,*}(X_{n-1}^i, \mathbf{S}^{n-1}, Y_{n-1}, Z_{n-1}^i, \varrho_i)$ *respectively. Furthermore, all the expressions* (3.5)*-*(3.10) *in Theorem 3.1 hold, provided that the objective transition probabilities* $(p_{n-1}(\mathbf{s}, y), q_{n-1}(\mathbf{s}, y))$ *in* (3.5) *and* (3.9) *are replaced by the subjective probabilities* $(\mathfrak{p}_{n-1}(\mathbf{s}, y, z^i, \varrho_i), \mathfrak{q}_{n-1}(\mathbf{s}, y, z^i, \varrho_i))$ *for each* $(\mathbf{s}, y, z^i, \varrho_i) \in \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma$*.*

Proof. The proof of this theorem follows exactly the same arguments as that of Theorem 3.1, with the sole modification that the objective transition probabilities $(p_{n-1}(\mathbf{s}, y), q_{n-1}(\mathbf{s}, y))$ are replaced by the subjective probabilities $(\mathfrak{p}_{n-1}(\mathbf{s}, y, z^i, \varrho_i), \mathfrak{q}_{n-1}(\mathbf{s}, y, z^i, \varrho_i))$. Note, in particular, that on the set $\{\omega^{0,i} \in \Omega^{0,i} : (X_{n-1}^i, \mathbf{S}^{n-1}, Y_{n-1}, Z_{n-1}^i, \varrho_i) = (x^i, \mathbf{s}, y, z^i, \varrho_i)\}$, we have the following equality:

$$\begin{aligned}
&\mathbb{E}_{\mathcal{P}}^{0,i}\Big[\exp\big(-\gamma_i U_n^i(X_n^i, \mathbf{S}^n, Y_n, Z_n^i, \varrho_i)\big) | x^i, \mathbf{s}, y^i, z^i, \varrho_i\Big] \\
&= \mathbb{E}_{\mathcal{P}}^{0,i}\Big[\exp\Big(-\gamma_i \eta_n^i\big(\beta(x^i - c^i\Delta) + \phi^i R_n + g_n(\mathbf{S}^n, Y_n, Z_n^i)\big) + \gamma_i V_n(\mathbf{S}^n, Y_n, Z_n^i, \varrho_i)\Big) | \mathbf{s}, y, z^i, \varrho_i\Big]
\end{aligned} \tag{5.2}$$

$$
\begin{aligned}
&= \exp(-\gamma_i \eta_n^i \beta(x^i - c^i\Delta)) \\
&\quad \times \Big\{ \mathfrak{p}_{n-1}(\mathbf{s}, y, z^i, \varrho_i) e^{-\gamma_i \eta_n^i \phi^i u} \mathbb{E}^{0,i}\big[\exp(\gamma_i [V_n((\mathbf{s}\widetilde{u})^n, Y_n, Z_n^i, \varrho_i) - \eta_n^i g_n((\mathbf{s}\widetilde{u})^n, Y_n, Z_n^i)]) | y, z^i, \varrho_i\big] \\
&\qquad + \mathfrak{q}_{n-1}(\mathbf{s}, y, z^i, \varrho_i) e^{-\gamma_i \eta_n^i \phi^i d} \mathbb{E}^{0,i}\big[\exp(\gamma_i [V_n((\mathbf{s}\widetilde{d})^n, Y_n, Z_n^i, \varrho_i) - \eta_n^i g_n((\mathbf{s}\widetilde{d})^n, Y_n, Z_n^i)]) | y, z^i, \varrho_i\big] \Big\}.
\end{aligned}
$$

Here, we have used Assumption 5.1 and the coincidence of $\mathbb{E}^{0,i}$ and $\mathbb{E}^{0,i}_{\mathcal{P}}$ on $\sigma(\{Y, Z^i, \varrho_i\})$. Using the strict positivity and boundedness of ϖ_n, the remainder of the arguments are identical to those in the proof of Theorem 3.1. $\square$

From Theorem 5.1, we obtain, for each $(\mathbf{s}, y, z^i, \varrho_i) \in \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma$,

$$
\phi_{n-1}^{i,*}(\mathbf{s}, y, z^i, \varrho_i) = \frac{1}{\gamma_i \eta_n^i (u-d)} \Big\{ \log\Big(-\frac{\mathfrak{p}_{n-1}(\mathbf{s}, y, z^i, \varrho_i) u}{\mathfrak{q}_{n-1}(\mathbf{s}, y, z^i, \varrho_i) d} \Big) + \log\big(f_{n-1}(\mathbf{s}, y, z^i, \varrho_i) \big) \Big\},
$$

which is the optimal strategy of agent-i. Using Assumption 5.1, we have

$$
\phi_{n-1}^{i,*}(\mathbf{s}, y, z^i, \varrho_i) = \frac{1}{\gamma_i \eta_n^i (u-d)} \Big\{ \log\Big(-\frac{p_{n-1}(\mathbf{s}, y) u}{q_{n-1}(\mathbf{s}, y) d} \Big) + \log\big(f^{\pi}_{n-1}(\mathbf{s}, y, z^i, \varrho_i) \big) \Big\}, \tag{5.3}
$$

where $f^{\pi}_{n-1} : \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1} \times \mathcal{Z}_{n-1} \times \Gamma \to \mathbb{R}$ is a strictly positive, bounded, measurable function defined by

$$
f^{\pi}_{n-1}(\mathbf{s}, y, z^i, \varrho_i) := \varpi_{n-1}(\mathbf{s}, y, z, \varrho_i) f_{n-1}(\mathbf{s}, y, z^i, \varrho_i). \tag{5.4}
$$

This suggests that the stochastically biased views manifest effectively as a modification to the continuation liabilities and/or incremental endowments. [3]

5.3 Mean-field equilibrium among the agents with subjective measures

As a main goal of this section, we shall now derive a set of transition probabilities (p_{n-1}, q_{n-1}) of the stock price under the objective measure $\mathbb{P}^0$ so that the mean-field equilibrium holds among the agents operating under respective subjective measures with stochastically biased estimates on the stock transition probabilities. The mean-field market-clearing condition is defined in Definition 3.1 with $(\phi_{n-1}^{i,*})$ given by Theorem 5.1. This is still characterized by equation (3.13). Using the expression (5.3), it is straightforward to modify Theorem 3.2 for the current setting:

Theorem 5.2. *Let Assumptions 3.1, 3.2* (i), (ii)*, 3.3 and 5.1 be in force. Then there exists a unique mean-field equilibrium, and the associated objective transition probabilities of the stock price are given by*

$$
\begin{aligned}
&p_{n-1}(\mathbf{s}, y) := \mathbb{P}^0\Big(S_n = \widetilde{u} S_{n-1} | (\mathbf{S}^{n-1}, Y_{n-1}) = (\mathbf{s}, y) \Big) \\
&= (-d) \Big/ \Big\{ u \exp\Big(\frac{1}{\mathbb{E}^1[1/(\gamma_1 \eta_n^1)]} \Big[\mathbb{E}^1\Big(\frac{\log(f^{\pi}_{n-1}(\mathbf{s}, y, Z^1_{n-1}, \varrho_1))}{\gamma_1 \eta_n^1} \Big) - (u-d) L_{n-1}(\mathbf{s}, y) \Big] \Big) - d \Big\}
\end{aligned} \tag{5.5}
$$

for every $(\mathbf{s}, y) \in \mathcal{S}^{n-1} \times \mathcal{Y}_{n-1}$, $1 \le n \le N$. Under the above transition probabilities, the optimal

[3] In addition to the multiplicative factors $(\varpi_n)_n$, the subjective transition probabilities $(\mathfrak{p}_n, \mathfrak{q}_n)_n$ also change the values of $(V_n, \widetilde{V}_n)_n$ and hence also of $(f_n)_n$ through their updates in the recursive formula.

strategy of agent-i is given by

$$\phi_{n-1}^{i,*}(\mathbf{s},y,z^i,\varrho_i) = \frac{1}{(u-d)}\Big\{\frac{\log f_{n-1}^{\pi}(\mathbf{s},y,z^i,\varrho_i)}{\gamma_i\eta_n^i} - \frac{1/(\gamma_i\eta_n^i)}{\mathbb{E}^1[1/(\gamma_1\eta_n^1)]}\mathbb{E}^1\Big(\frac{\log f_{n-1}^{\pi}(\mathbf{s},y,Z_{n-1}^1,\varrho_1)}{\gamma_1\eta_n^1}\Big)\Big\} + \frac{1/(\gamma_i\eta_n^i)}{\mathbb{E}^1[1/(\gamma_1\eta_n^1)]}L_{n-1}(\mathbf{s},y).$$

Here, f^π is defined by (5.4) *and the other variables are as defined in Theorem 5.1. Moreover, there exists some positive constant C_{n-1} such that*

$$\mathbb{E}\Big|\frac{1}{N_p}\sum_{i=1}^{N_p}\phi_{n-1}^{i,*}(\mathbf{S}^{n-1},Y_{n-1},Z_{n-1}^i,\varrho_i) - L_{n-1}(\mathbf{S}^{n-1},Y_{n-1})\Big|^2 \le \frac{C_{n-1}}{N_p}$$

for every $1 \le n \le N$, which gives the convergence rate in the large population limit.

Proof. The proof of this theorem follows exactly the same arguments as that of Theorem 3.2 by using (f_{n-1}^{π}) instead of (f_{n-1}). Since (ϖ_n) is strictly positive and bounded, every argument for Theorem 3.2 is still valid. $\square$

It is instructive to decompose f^π in the expression (5.5) in the following way:

$$\begin{aligned} p_{n-1}(\mathbf{s},y) &:= \mathbb{P}^0\Big(S_n = \widetilde{u}S_{n-1}|(\mathbf{S}^{n-1},Y_{n-1}) = (\mathbf{s},y)\Big) \\ &= (-d)\Big/\Big\{u\exp\Big(\frac{1}{\mathbb{E}^1[1/(\gamma_1\eta_n^1)]}\Big[\mathbb{E}^1\Big(\frac{\log f_{n-1}(\mathbf{s},y,Z_{n-1}^1,\varrho_1)}{\gamma_1\eta_n^1}\Big) \\ &\qquad -(u-d)L_{n-1}(\mathbf{s},y) + \mathbb{E}^1\Big(\frac{\log \varpi_{n-1}(\mathbf{s},y,Z_{n-1}^1,\varrho_1)}{\gamma_1\eta_n^1}\Big)\Big]\Big) - d\Big\}. \end{aligned}$$

This result implies that, apart from the subtle effects induced by the changes in f_{n-1}, the aggregated effect of the biased views on the stock price transition probabilities can also manifest as an external order flow term L_{n-1} in the standard rational expectation setting. Specifically, it is clear that the negative view ($\varpi_{n-1} < 1$) adds to the external supply L_{n-1}, and the positive one ($\varpi_{n-1} > 1$) does the opposite. Combined with the discussions in Section 2.4, the above result implies that the existence of a large portion of agents whose bias reflects a counter-cyclical, contrarian view—i.e., cautious (pessimistic) view ($\varpi_{n-1} < 1$) when the market-wide economic factor and stock price are at high levels, and, conversely, exhibits an optimistic view ($\varpi_{n-1} > 1$) when they are at low levels—tends to make the equilibrium price distributions more fat-tailed.

Remark 5.1 (Path Dependence and Extension to Multiple Populations)**.** *We remark that the discussions in Remark 3.3 regarding the necessary length of the price trajectory in the transition probabilities remain valid for the results in this section. Moreover, the framework of multi-population equilibrium presented in Section 4 can be readily extended to the current model for agents with subjective measures. Specifically, we only need to replace, for each $p = 1,\ldots,m$, the set of functions (f_n^p) by $(f_n^{\pi,p})$ where $f_n^{\pi,p} = \varpi_n^p \times f_n^p$ with population-specific bias functions (ϖ_n^p).*

6 Numerical examples and implications

In this section, we provide some numerical examples for the models introduced in Sections 2 and 3, focusing on cases without path dependence. We also provide an example with stochastically biased agents developed in Section 5. Since the models are too flexible for a thorough analysis, we focus only on a few simple setups to illustrate characteristic behaviors of the mean-field price distributions. In order to reduce numerical cost, we assume that Y and Z^i are one-dimensional discrete processes taking values on binomial trees, and that all $\mathcal{F}_0^i$-measurable random variables are uniformly distributed over finite sets.

For our analytical formulation, the finite state space for (S, Y) has played an important role. However, there is no requirement for Y to follow a binomial process. Moreover, the coefficients $\varrho_i := (\gamma_i, \zeta_i, \delta_i, \psi_i)$ and the process (Z_n^i) can have continuous distributions. The specific assumptions made in this section are solely for numerical convenience.

6.1 Utility with terminal liability

We first consider the model discussed in Section 2. γ_i is assumed to be uniformly distributed over the $(N_\gamma + 1)$ discrete values given by

$$\gamma_i(k_\gamma) := \underline{\gamma} + (\overline{\gamma} - \underline{\gamma})k_\gamma/N_\gamma, \quad k_\gamma = 0, \cdots, N_\gamma.$$

The process $(Z_n^i)_{n=0}^N$ is supposed to follow a one-dimensional binomial process modeled by

$$Z_{n+1}^i = Z_n^i R_{n+1}^i,$$

where (R_n^i) is an $(\mathcal{F}_{t_n}^i)$-adapted process taking values either u_z or d_z. Specifically, $R_n^i = u_z$ occurs with probability p_z and $R_n^i = d_z$ with $q_z := 1 - p_z$. We take $u_z = (d_z)^{-1} = \exp(\sigma_z\sqrt{\Delta})$. We also assume $Z_0^i = z_0 \in (0, \infty)$ is common for all the agents in order to reduce numerical costs. We model the process $(Y_n)_{n=0}^N$ similarly but it is assumed to follow an approximate Gaussian process:

$$Y_{n+1} = Y_n + R_{n+1}^y,$$

where (R_n^y) is an $(\mathcal{F}_{t_n}^0)$-adapted process taking values either u_y or d_y, where $R_n^y = u_y$ with probability p_y and $R_n^y = d_y$ with $q_y := 1 - p_y$. We take $u_y = (-d_y) = \sigma_y\sqrt{\Delta}$. Finally, for the stock-price process (S_n), we set $\widetilde{u} = (\widetilde{d})^{-1} = \exp(\sigma\sqrt{\Delta})$ and $S_0 = 1.0$.

The parameter values to be used throughout this subsection are summarized in Table 1 below. We recall that the initial wealth ξ_i is irrelevant for our analysis.

parameter	$\underline{\gamma}$	$\overline{\gamma}$	N_γ	z_0	σ_z	p_z	Y_0	σ_y	p_y	S_0	σ	r	T	N
value	0.5	1.5	4	1.0	12%	0.5	1.0	12%	0.5	1.0	15%	3.3%	3yr	48

Table 1: parameter values

Let us first assume that there is no external oder flow $L_n \equiv 0$, and that the terminal liability F is given by

$$F(S_N, Y_N, Z_N^i) := C - 3S_N Y_N Z_N^i, \tag{6.1}$$

where $C \in \mathbb{R}$ is an arbitrary real constant. Since the result is invariant to a constant shift, one may adjust the constant C, if necessary, to make the liability positive. From the discussion in Section 2.4, we expect that the equilibrium price distribution for this liability will yield a positive excess return. We can verify this expectation by observing Figure 1, which presents the comparison of the marginal price distributions under the equilibrium measure ($\mathbb{P}$) and the risk-neutral measure ($\mathbb{Q}$) at 1-year and 3-year points.

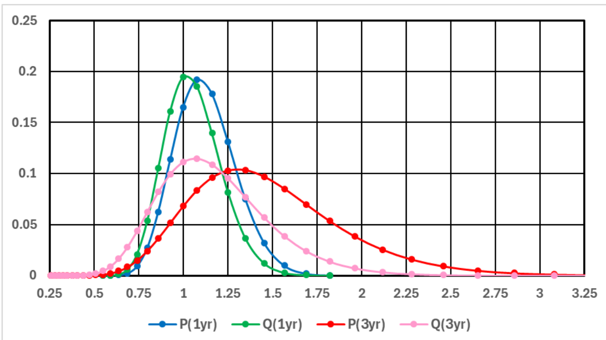

Figure 1: Comparison of the marginal price distributions under the equilibrium measure ($\mathbb{P}$) and the risk-neutral measures ($\mathbb{Q}$) at 1-year and 3-year points for (6.1).

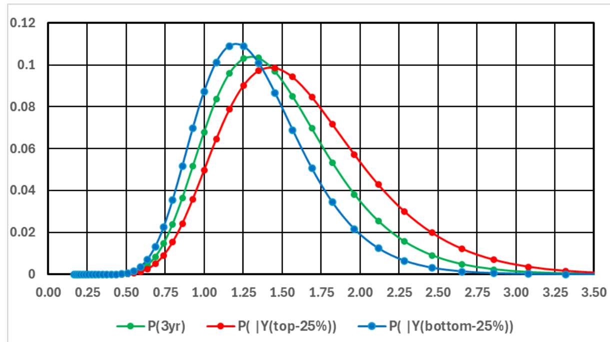

Figure 2: Comparison of the marginal price distribution $\mathbb{P}$ and the conditional price distributions $\mathbb{P}(\cdot|Y^{\text{top}-25\%})$ and $\mathbb{P}(\cdot|Y^{\text{bottom}-25\%})$ at 3-year point for (6.1).

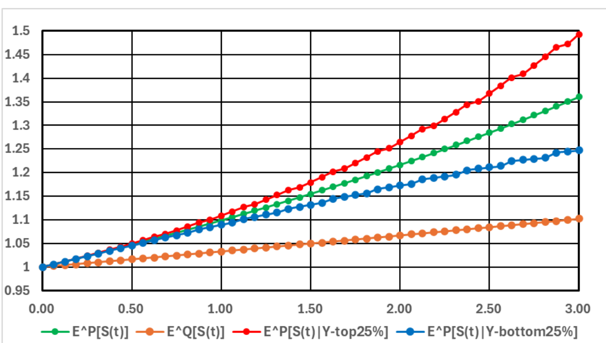

Figure 3: Comparison of the expected values of $S(t_n)$ under $\mathbb{Q}$, $\mathbb{P}$, $\mathbb{P}(\cdot|Y^{\text{top}-25\%})$, and $\mathbb{P}(\cdot|Y^{\text{bottom}-25\%})$ for (6.1).

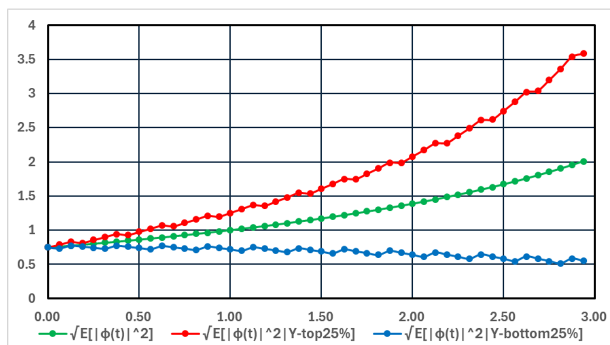

Figure 4: The time evolution of the expected trading volume $\mathbb{E}[|\phi^{1,*}(t)|^2]^{\frac{1}{2}}$, $\mathbb{E}[|\phi^{1,*}(t)|^2|Y^{\text{top}-25\%}]^{\frac{1}{2}}$, and $\mathbb{E}[|\phi^{1,*}(t)|^2|Y^{\text{bottom}-25\%}]^{\frac{1}{2}}$ for (6.1).

We can also provide the conditional price distribution $\mathbb{P}(S_n \in A|Y_n = y)$, $\forall A \subset \mathcal{S}_n$ for each $y \in \mathcal{Y}_n$. At the three-year point ($n = 48$), the value of Y marking the 75th percentile ($Y^{\text{top}-25\%}$) is equivalent to a total of 36 up moves, while the 25th percentile ($Y^{\text{bottom}-25\%}$) is equivalent to 12 up moves. Figure 2 compares the conditional distributions $\{\mathbb{P}(S_n = s|Y^{\text{top}-25\%}), \mathbb{P}(S_n = s|Y^{\text{bottom}-25\%}), \forall s \in \mathcal{S}_n\}$ with the marginal distribution $\mathbb{P}(\cdot)$ at the three-year point. As expected from the functional form in (6.1), the deviations from the risk-neutral distribution are positive and become larger for the larger value of Y. If Y is a stochastic process possessing large jumps (instead of the diffusion-like process in this section), which is the case, for example, for regime-switching models, we expect a substantial sudden shift of the transition probabilities and hence the equilibrium excess return.

In Figure 3, we provide the time-evolution of the expected value of the stock price under the equilibrium ($\mathbb{P}$) and the risk-neutral ($\mathbb{Q}$) measures ($\mathbb{E}^{\mathbb{P}}[S(t)], \mathbb{E}^{\mathbb{Q}}[S(t)]$). We also include the conditional expectation values ($\mathbb{E}^{\mathbb{P}}[S(t)|Y^{\text{top}-25\%}], \mathbb{E}^{\mathbb{P}}[S(t)|Y^{\text{bottom}-25\%}]$) for Y in the 75th ($Y^{\text{top}-25\%}$) and 25th percentiles ($Y^{\text{bottom}-25\%}$), choosing the nearest nodes at each time. Since the risk-free rate is $r = 3.3\%$ per annum, one can observe from Figure 3 that the excess return is roughly 8% for the equilibrium distribution $\mathbb{P}$, and roughly 13% and 5% per annum, conditional on $Y^{\text{top}-25\%}$ and $Y^{\text{bottom}-25\%}$, respectively. Figure 4 gives the time-evolution of the expected trading volume $\mathbb{E}^{\mathbb{P}}[|\phi^{1,*}(t)|^2]^{\frac{1}{2}}$ under the equilibrium distribution, as well as the conditional expectations $(\mathbb{E}^{\mathbb{P}}[|\phi^{1,*}(t)|^2|Y^{\text{top}-25\%}]^{\frac{1}{2}}, \mathbb{E}^{\mathbb{P}}[|\phi^{1,*}(t)|^2|Y^{\text{bottom}-25\%}]^{\frac{1}{2}})$. The slightly irregular zigzag behavior observed in Figures 3 and 4 is an artifact of our selection of the nearest nodes when determining the top and bottom 25th percentiles for Y at each time step.

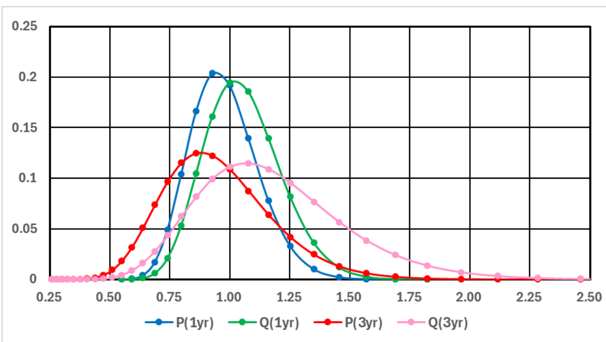

Figure 5: Comparison of the marginal price distributions under the equilibrium measure ($\mathbb{P}$) and the risk-neutral measures ($\mathbb{Q}$) at 1-year and 3-year points for (6.2).

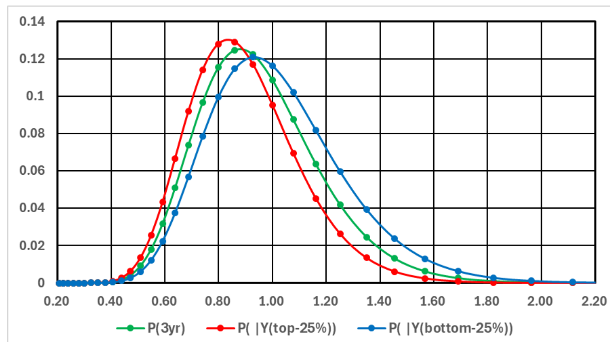

Figure 6: Comparison of the marginal price distribution $\mathbb{P}$ and the conditional price distributions $\mathbb{P}(\cdot|Y^{\text{top}-25\%})$ and $\mathbb{P}(\cdot|Y^{\text{bottom}-25\%})$ at 3-year point for (6.2).

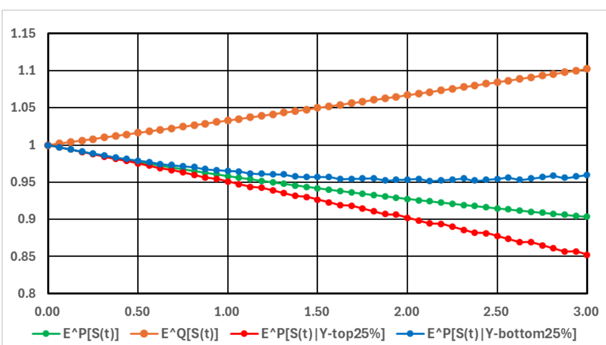

Figure 7: Comparison of the expected values of $S(t_n)$ under $\mathbb{Q}$, $\mathbb{P}$, $\mathbb{P}(\cdot|Y^{\text{top}-25\%})$, and $\mathbb{P}(\cdot|Y^{\text{bottom}-25\%})$ for (6.2).

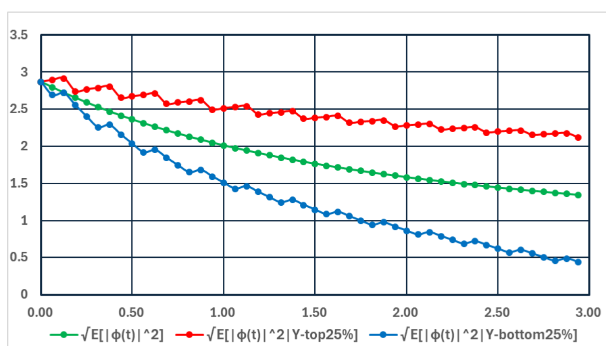

Figure 8: The time evolution of the expected trading volume $\mathbb{E}[|\phi^{1,*}(t)|^2]^{\frac{1}{2}}$, $\mathbb{E}[|\phi^{1,*}(t)|^2|Y^{\text{top}-25\%}]^{\frac{1}{2}}$, and $\mathbb{E}[|\phi^{1,*}(t)|^2|Y^{\text{bottom}-25\%}]^{\frac{1}{2}}$ for (6.2).

In Figures 5, 6, 7 and 8, we present the corresponding results for the different liability function

$$F(S_N, Y_N, Z_N^i) := C + 3S_N Y_N Z_N^i \tag{6.2}$$

which exhibits the opposite sign of sensitivity to the stock price. As expected, we observe that the excess return becomes negative in this case.

Now, we introduce the external order flow. We continue to use the same parameter values from Table 1 and the liability function defined by (6.1), but now including an external order flow (without y-dependence):

$$L_n(s) := a \max(s - c, 0). \tag{6.3}$$

We set $c = 1.75$ and consider two scenarios, one is $a = 7$ and the other is $a = -7$. In the former case, there is positive supply of the stock when the stock price is very large $s > 1.75$, and in the latter case there is positive demand (i.e., negative supply) when $s > 1.75$. In Figure 9, we compare the marginal as well as conditional price distribution as in Figure 2 with the positive external order flow in the left panel and the negative one in the right panel. We can clearly observe that the positive supply generates a heavy right tail in the equilibrium distributions and that the positive demand causes the opposite.

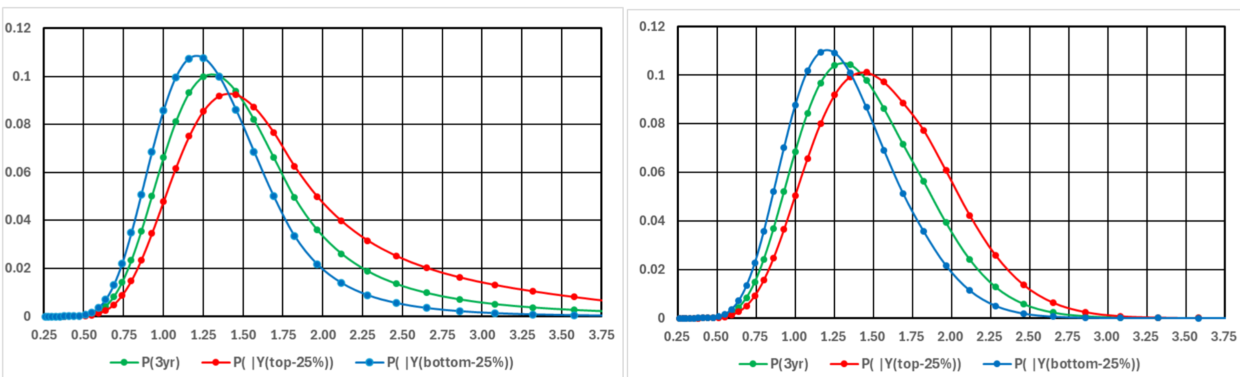

Figure 9: Comparison of the marginal price distribution $\mathbb{P}$ and the conditional price distributions $\mathbb{P}(\cdot|Y^{\text{top}-25\%})$ and $\mathbb{P}(\cdot|Y^{\text{bottom}-25\%})$ at 3-year point. F is given by (6.1) and the external order flow is equal to $L_n(s) = 7\max(s-1.75, 0)$ (i.e., positive supply) in the left panel and $L_n(s) = -7\max(s - 1.75, 0)$ (i.e., positive demand) in the right panel.

6.2 Recursive utility

We now consider the recursive utility model discussed in Section 3. The model of stochastically biased agents in Section 5 is treated only in Figure 13. For numerical ease, we assume no path-dependence. The liability F is thus assumed to depend solely on the terminal stock price S_N, while the incremental endowments g_n depend solely on the current price S_n. See Remark 3.3 for the corresponding analytic solutions. We use the same models for $(S_n, Y_n, Z_n^i)_{n=0}^N$ and γ_i as in Section 6.1. ψ_i is assumed to be uniformly distributed over the $(N_\psi + 1)$ discrete values given by

$$\psi_i(k_\psi) := \underline{\psi} + (\overline{\psi} - \underline{\psi})k_\psi/N_\psi, \quad k_\psi = 0, \cdots, N_\psi.$$

For simplicity, we assume that the time preference coefficient $\delta_i := \exp(-\rho\Delta)$ takes a common value across the agents. Moreover, we assume that ζ_i and ψ_i are related by

$$\psi_i/\zeta_i = a_\zeta,$$

where a_ζ is a positive constant common across the agents. We shall use the parameter a_ζ to control the ratio

$$\eta^i_{n-1}/\eta^i_n \simeq \psi_i/\zeta_i.$$

See the discussion in Section 3.4 for the role of this ratio. The parameter values to be used throughout this subsection (except the last example) are summarized in Table 2 below:

parameter	$\underline{\gamma}$	$\overline{\gamma}$	N_γ	$\underline{\psi}$	$\overline{\psi}$	N_ψ	ρ	z_0	σ_z	p_z	Y_0	σ_y	p_y	S_0	σ	r	T	N
value	0.4	1.6	3	0.5	1.5	2	5.0%	1.0	12%	0.5	1.0	12%	0.5	1.0	15%	3.3%	3yr	48

Table 2: parameter values

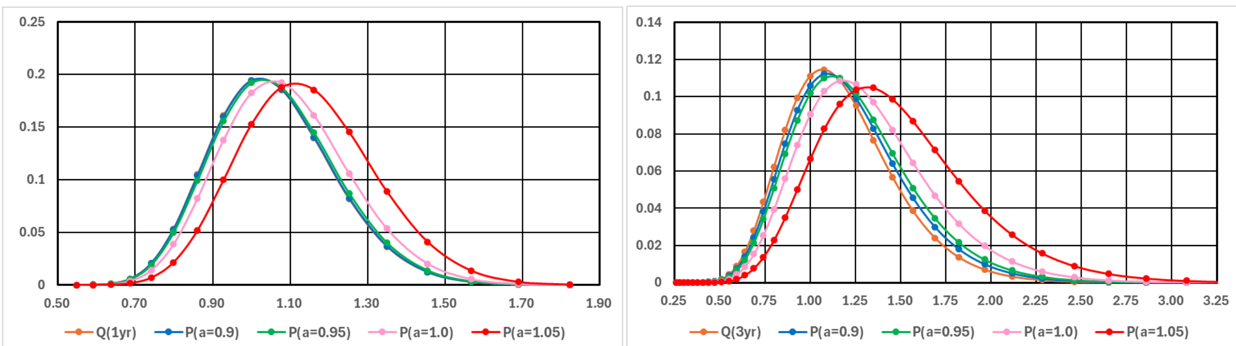

Figure 10: Comparison of the risk-neutral as well as the equilibrium marginal price distributions with $a_\zeta = 0.9, 0.95, 1.0, 1.05$ at 1-year (left panel) and 3-year (right panel) points.

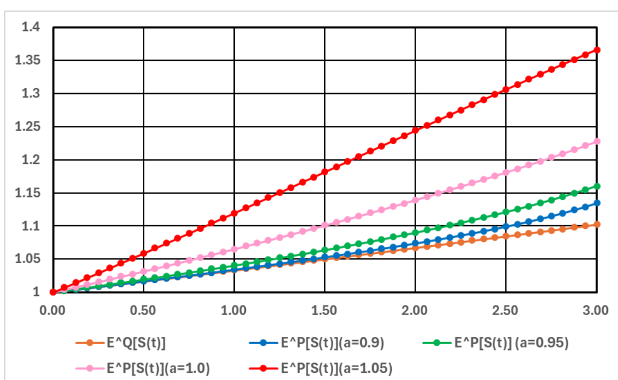

Figure 11: Comparison of the expected values of $S(t_n)$ under the risk-neutral as well as the equilibrium price distributions with $a_\zeta = 0.9, 0.95, 1.0, 1.05$.

Since the effects of the stochastic liability and incremental endowments on the equilibrium price distributions are as expected from the results in the previous subsection, let us first concentrate on the effect of the ratio $a_\zeta = \psi_i/\zeta_i$. We set $L_n \equiv 0$, $\forall n$ and define the liability function and the incremental endowments in the following way:

$$F(S_N, Y_N, Z_N^i) := C - 2S_N Y_N Z_N^i, \tag{6.4}$$
$$g_n(S_n, Y_n, Z_n^i) := C' + 1.5\Delta S_n Y_n Z_n^i, \quad 1 \le n \le N. \tag{6.5}$$

Here, C, C' are arbitrary constants irrelevant for the equilibrium distributions. In Figure 10, we plot the risk-neutral as well as the equilibrium marginal price distributions with 4 different values of the ratio $a_\zeta = 0.9, 0.95, 1.0, 1.05$ at 1-year (left panel) and 3-year (right panel) points. Figure 11 provides the time evolution of the expected value of the stock price $S(t_n)$ for each case. As inferred from the discussion in Section 3.4, the deviations from the risk-neutral distribution become smaller as a_ζ decreases. This effect (smaller deviations) is more pronounced in earlier periods. We can observe that the value of $a_\zeta = \psi_i/\zeta_i$ can efficiently control the level of the excess return without changing the other parameters. More specifically, higher values of a_ζ put more weight on the continuation utilities relative to current consumptions, thus increasing the agents' hedge needs, and leads to higher excess returns.

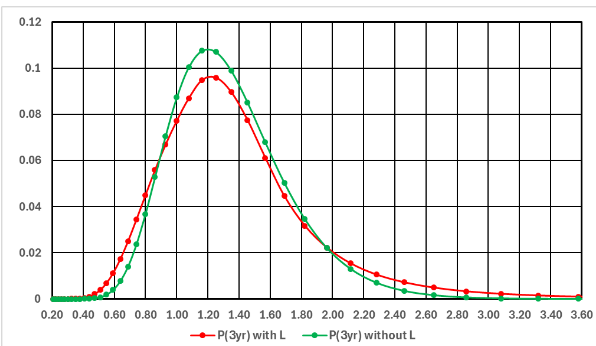

Figure 12: Comparison of the stock price distributions at 3-year point with L_n given by (6.6) and $L_n \equiv 0$

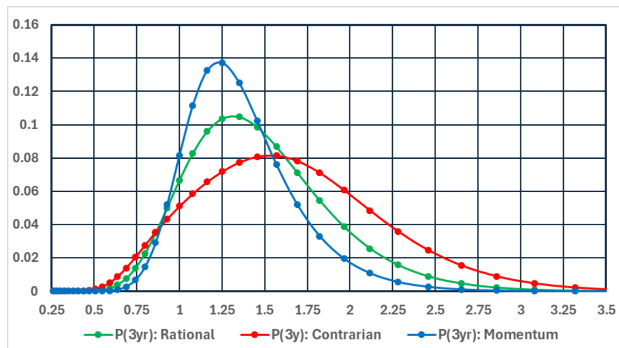

Figure 13: Comparison of the marginal price distribution at 3-year point with the rational agents, those with contrarian-bias (ϖ_n^c), and with momentum-bias (ϖ_n^m) defined in (6.7).

Next, we set the incremental endowments $g_n \equiv 0$ while keeping the liability function the same as in (6.4). We are now going to study the effects of an external order flow defined by

$$L(S_n) := 8\max(S_n - 1.6, 0) - 8\max(1.1 - S_n, 0), \quad 0 \le n \le N-1. \tag{6.6}$$

This demonstration aims to show how flexibly the shape of equilibrium distributions can change. The definition in (6.6) implies a positive supply (i.e., sell orders from other groups) of the stock when the price S_n is high and a positive demand (i.e., buy orders from other groups) when the price is low. In Figure 12, with $a_\zeta = 1.07$, we compare the equilibrium price distribution at 3-year point with L_n given by (6.6) and that with $L_n \equiv 0$. It shows that the existence of the external order flow (6.6) makes the equilibrium distribution fat-tailed in both directions, which is as expected by the analysis made in the last paragraph of Section 2.4. This suggests that the existence of individual investors behaving as a **contrarian** makes the equilibrium price distribution fat-tailed, which is an interesting example to demonstrate that microstructures of the market impact on the tail distributions. Corresponding modifications in the terminal liability F and/or the incremental endowments g_n would yield similar results.

As a related analysis that explores the impact of non-rational agents on tails of the equilibrium price distributions, we utilize the subjective measure framework developed in Section 5. We investigate how agents' biased estimates on the transition probabilities impact the equilibrium distribution. We define the contrarian-bias (ϖ_n^c) and momentum-bias (ϖ_n^m) as follows:

$$\begin{aligned} \varpi_n^c(S_n, Z_n^i) &= 0.8 \vee \frac{S_0\beta^n}{S_n}\frac{Z_0}{Z_n^i} \wedge 1.2, \\ \varpi_n^m(S_n, Z_n^i) &= 0.8 \vee \frac{S_n}{S_0\beta^n}\frac{Z_n^i}{Z_0} \wedge 1.2, \end{aligned} \tag{6.7}$$

We use parameter values taken from Table 2, $a_\zeta = 1.05$, and (6.4) and (6.5) for the terminal liability and the incremental endowments. Figure 13 compares the equilibrium distributions for the three agent types: rational, contrarian-biased, and momentum-biased agents. As discussed in Section 5.3, the agents with contrarian-bias produce a fat-tailed equilibrium price distribution and those with momentum-bias a thin-tailed one.

In the last numerical example, we examine the effect of σ_z, the volatility of the process (Z_n^i), on trading volume. This volume is quantified by the standard deviation of the stock position among the agents, which is $\mathbb{E}^1[|\phi_t^{i,*}|^2]^{\frac{1}{2}}$, as discussed in Section 2.4. To highlight the effect of σ_z, we reduce the variation in $(\gamma_i, \psi_i, \zeta_i)$. The parameter values we use are summarized in Table 3 below:

parameter	$\underline{\gamma}$	$\overline{\gamma}$	N_γ	$\underline{\psi}$	$\overline{\psi}$	N_ψ	ρ	z_0	a_ζ	p_z	Y_0	σ_y	p_y	S_0	σ	r	T	N
value	0.95	1.05	2	0.95	1.05	2	5.0%	1.0	1.02	0.5	1.0	12%	0.5	1.0	15%	3.3%	3yr	48

Table 3: value of parameters for Figure 14.

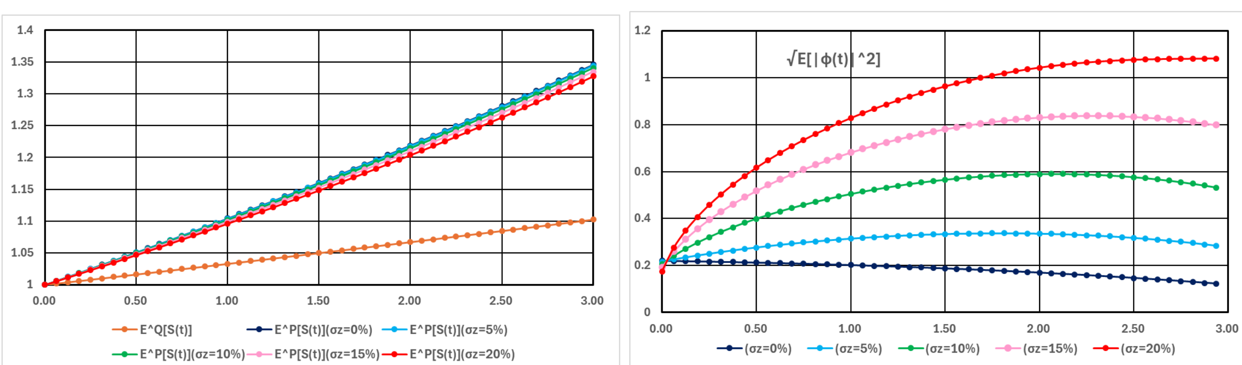

Figure 14: Left panel: Comparison of the expected value of $S(t_n)$ under the risk-neutral as well as the equilibrium price distributions with $\sigma_z = 0\%, 5\%, 10\%, 15\%, 20\%$. Right panel: Comparison of the trading volumes $\mathbb{E}^{\mathbb{P}}[|\phi^{1,*}(t)|^2]^{\frac{1}{2}}$ with $\sigma_z = 0\%, 5\%, 10\%, 15\%, 20\%$.

We put $L_n \equiv 0$ and use the terminal liability and the incremental endowments defined by (6.4) and (6.5). In the right panel of Figure 14, we have plotted the evolution of the trading volume $\mathbb{E}^{\mathbb{P}}[|\phi^{1,*}(t)|^2]^{\frac{1}{2}}$ for 5 different volatilities of the process (Z_n^i): $\sigma_z = 0\%,\ 5\%,\ 10\%,\ 15\%,\ 20\%$. We observe that the trading volume increases with the volatility σ_z. The non-zero trading volume, even when $\sigma_z = 0$, stems from the non-zero variation in the risk-aversion coefficients. The near-identical trading volume in the earliest period is a consequence of our assumption that the agents have the common initial value $Z_0^i \equiv z_0 = 1, \ \forall i \in \mathbb{N}$. This assumption is made solely for numerical convenience.

In the left panel of Figure 14, we have plotted the evolution of the expected value of $S(t)$ for each case of σ_z. (The result for the risk-neutral measure is also plotted for reference.) From this result, we see that the size of excess return is almost unaffected by the volatility σ_z. This stems from the functional form of (6.4) and (6.5), as well as the fact that the expectation value of Z_n^i remains nearly identical across all cases. These results suggest that we can control trading volume by changing σ_z without significantly affecting the excess return. Although trading volume is also significantly influenced by the variation in $\mathcal{F}_0^i$-measurable random variables such as γ_i, ζ_i, and ψ_i, these variables may simultaneously cause a large shift in the excess return.

6.3 Limits of our framework and practical implications

As our examples demonstrated, the equilibrium price distributions and the resultant excess returns can substantially deviate from the risk-neutral measure, depending on the agents' characteristics. This deviation can possibly lead to an odd situation where the required equilibrium excess return (or more generally transition probabilities) becomes totally unrealistic or economically unfeasible. As is common for price-formation frameworks, our model does not argue how we can justify such a stock price performance based on the issuer's business activities and the surrounding business and macroeconomic environment. Our framework only provides the necessary transition probabilities (and thus excess return) required by the agents to compensate their risk for achieving the market-clearing equilibrium. Therefore, it does not tell if the equilibrium can be achieved nor what happens when it fails. However, our framework can still offer regulatory bodies valuable information on potential market instability by providing the necessary transition probabilities, marginal/conditional price distributions, and in particular the excess return required to maintain the equilibrium, which would form a flexible platform for market stability analysis under various agent-based scenarios.

7 Concluding remarks and future research directions

In this work, we have successfully developed a numerically tractable and readily implementable framework for the mean-field equilibrium (MFE) price formation problem by combining MFG theory with the binomial tree structure. This classical simplifying device allows us to bypass the substantial mathematical and computational challenges inherent in continuous-time models, such as the well-posedness and numerical evaluation of FBSDEs of McKean-Vlasov type or coupled non-linear PDEs. We proved the existence of the unique MFE for agents with exponential utilities and recursive utilities of exponential-type and derived an explicit analytic formula for the stock price transition probabilities.

Our framework successfully incorporates highly general model features, including stochastic terminal liabilities and incremental endowments dependent on unhedgeable common and idiosyncratic factors, as well as external order flow. Moreover, the analytic tractability of our approach enabled us to achieve two significant extensions that would be formidable problems in the continuous-time setting:

- We introduced multi-population heterogeneity, allowing populations to differ fundamentally in their risk characteristics, liabilities, and endowments.
- We relaxed the standard rational expectations hypothesis by modeling agents operating under subjective probability measures.

Our results clearly show that the equilibrium distributions can substantially change their shapes in response to these inputs. In particular, we found that countercyclical liabilities (or cyclical endowments) increase the excess return required by the agents, and that the equilibrium price distributions can become fat-tailed due to either contrarian external order flow or contrarian biases arising from agents' subjective measures. Furthermore, trading volume per capita is crucially dependent on the variation in idiosyncratic factors. The explicit and tractable solution offers regulatory bodies a flexible platform for market stability analysis under various agent-based scenarios. Empirical analysis regarding these findings would constitute an important research topic.

Our method may also be applied to other asset classes, such as commodities and foreign exchanges, provided they can be modeled by binomial trees. Specifically, frameworks like the Black-Derman-Toy model (BDT) [8] could be adapted to analyze a mean-field equilibrium for risk-free interest rates. Furthermore, various practitioner techniques, such as implied binomial trees, can now serve as valuable tools for investigating the MFE in our framework, advancing beyond their initial purpose. (For a comprehensive overview of general Markov processes in finance, see, e.g., [7] and [25].)

Nevertheless, there remain several important challenges waiting for further research. First, extensions to general multinomial trees and multi-asset frameworks constitute interesting future research directions. Although our framework remains conceptually the same, there appear several hurdles to be overcome.

- Although it is not difficult to put appropriate assumptions so that there exists a unique optimal solution, its explicit form is generally unavailable.
- There are more degrees of freedom in the transition probabilities than are imposed by the market-clearing conditions. This remaining freedom must be fixed by imposing an appropriate dependence structure among the assets.

Due to these issues, while the second point might be beneficial for flexibility, numerical costs would be significantly higher than the single asset case, in particular, in the presence of common noises. The fact that the market-clearing condition alone does not uniquely determine the price processes in the presence of multiple stocks is already well known. (See, Karatzas & Shreve [30, Chapter 4].) This is because that one can build a equivalent set of mutual funds from the original stocks without affecting the market-clearing condition.

Second, constructing mean-field equilibrium among agents with other utilities, such as power-type utilities, remains one of the most challenging problems. This is a common issue, mirroring the challenge in the continuous-time setting. For utilities other than the exponential-type, the optimal trade position $\phi_n^{i,*}$ is, in general, dependent on the size of wealth at t_n. Since the wealth of each agent X_n^i at t_n depends on the trading strategy up to t_n, the mean-field equilibrium condition leads to a complex fixed-point problem involving the backward $\phi_n^{i,*}$ and forward X_n^i discrete processes. Although we can decouple the wealth process by deliberately constructing the model so that $\phi_n^{i,*} \equiv 0$, the resulting model allows no trading activity in the market and is thus clearly unrealistic.

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